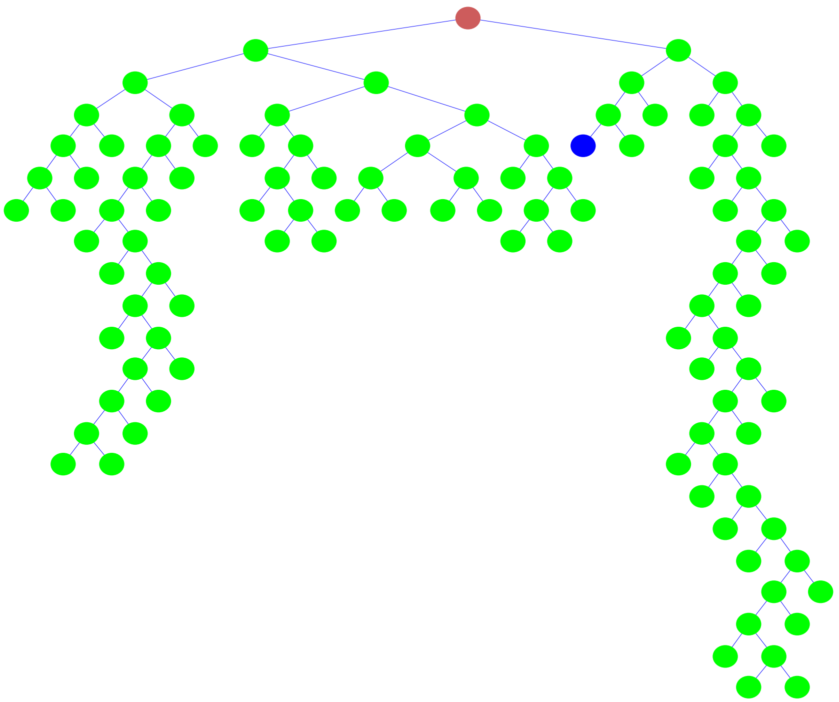
Implementing Branch, Cut, and Price



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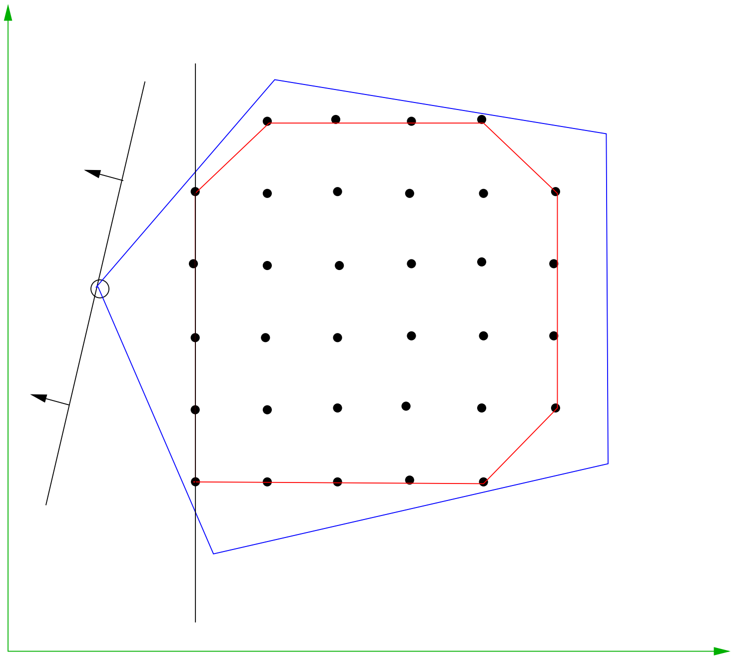
Outline of Talk

- Introduction to Branch, Cut, and Price (BCP)
- Frameworks for BCP
- Implementing BCP
- Description of SYMPHONY and COIN/BCP
- Lessons Learned



LP-based Branch and Bound

- Consider problem P:

$$\begin{aligned} & min\ c^T x \\ s.t. \quad & Ax \leq b \\ & x_i \in \mathbf{Z}\ \forall\ i \in I \end{aligned}$$

  where $(A, b) \in \mathbf{R}^{m \times n+1}, c \in \mathbf{R}^n$.

- Let $\mathcal{P} = conv\{x \in \mathbf{R}^n : Ax \leq b, x_i \in \mathbf{Z}\ \forall\ i \in I\}$.
- Basic Algorithmic Approach
  - Use LP relaxations to produce lower bounds.
  - Branch using hyperplanes.
- Basic Algorithmic Elements
  - A method for producing and tightening the LP relaxations.
  - A method for branching.

Branch, Cut, and Price

- Weyl-Minkowski
  - $\exists(\bar{A}, \bar{b}) \in \mathbf{R}^{\bar{m} \times n+1}$ s.t. $\mathcal{P} = \{x \in \mathbf{R}^n : \bar{A}x \leq \bar{b}\}$
  - We want the solution to $min\{c^T x : \bar{A}x \leq \bar{b}\}$.
  - Solving this LP isn't practical (or necessary).

- BCP Approach
  - Form LP relaxations using submatrices of $\bar{A}$.
  - The submatrices are defined by sets $\mathcal{V} \subseteq [1..n]$ and $\mathcal{C} \subseteq [1..\bar{m}]$.
  - *Forming/managing these relaxations efficiently is one of the primary challenge of BCP*.

The Challenge of BCP

- The efficiency of BCP depends heavily on the size (number of rows and columns) and tightness of the LP relaxations.
- Tradeoff
  - Small LP relaxations ⇒ faster LP solution.
  - Big LP relaxations ⇒ better bounds.
- The goal is to keep relaxations small while not sacrificing bound quality.
- We must be able to easily move constraints and variables in and out of the *active* set.
- This means dynamic generation and deletion.

An Object-oriented Approach

- The rows/columns of a static LP are called *constraints* and *variables*.
- What do these terms mean in a dynamic context?
- Conceptual Definitions
  - Constraint: A mapping

$$f_i^c(\mathcal{C}) : 2^{[1..n]} \rightarrow \mathbf{R}^{|\mathcal{C}|}$$

    generating coefficients for the submatrix C.
  - Variable: A mapping

$$f_j^v(\mathcal{V}) : 2^{[1..\bar{m}]} \rightarrow \mathbf{R}^{|\mathcal{V}|}$$

    generating coefficients for the submatrix C.
- To construct a relaxation, an initial *core* is needed.
- From the core, we can build up the relaxation.

Software Frameworks

- Concept: Provide a *framework* in which the user has only to define constraints, variables, and a core.
  - Branch and bound ⇒ core only
  - Branch and cut ⇒ core plus constraints
  - Branch and price ⇒ core plus variables
  - Branch, cut, and price ⇒ the whole caboodle
- BCP frameworks
  - SYMPHONY (parallel)
  - COIN/BCP (parallel)
  - ABACUS (sequential)
- Other frameworks
  - PICO, PUBB, BoB, PPBB-Lib (branch and bound)
  - MINTO (MIP), BARON (NLP)

Generating the Objects

- We will generically call the constraints and variables *objects*.
- We need to define methods for generating these objects.
- For constraints, such a method is a mapping

$$g^c(x) : \mathbf{R}^n \rightarrow 2^{[1..\bar{m}]}$$

  where x is a primal solution vector.

- For variables, we have

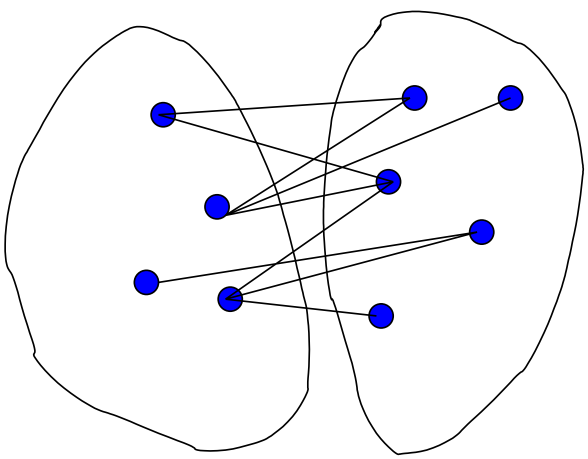
$$g^v(y) : \mathbf{R}^m \rightarrow 2^{[1..n]}$$

  where y is a dual solution vector.

- We can also use *object pools* to help with generation.

Object Representation

- In practice, we may not know the cardinality of the object set.
- We may not easily be able to assign indices to the objects.
- Instead, we must define abstract representations of these objects.
- In C, this means defining a function which takes the abstract data structure as an argument and returns a row/column.
- In C++, this means deriving a new class.
- Example: Subtour elimination constraints.



Example: Traveling Salesman Problem

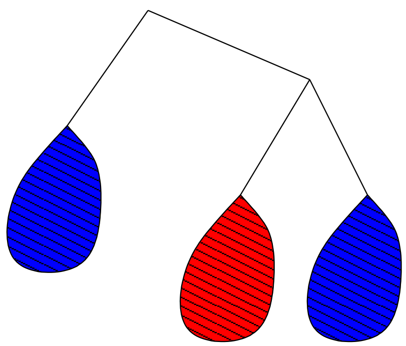
Feasible solutions are those incidence vectors satisfying:

$$\begin{aligned} \sum_{j=1}^{n} x_{ij} &= 2 \quad \forall i \in V \\ \sum_{\substack{i \in S \\ j \notin S}} x_{ij} &\geq 2 \quad \forall S \subset V,\ |S| > 1. \end{aligned}$$

- The variables correspond to the edges of a graph (easy to index).
- The number of facets (constraints) is astronomical.
- The core
  - The k shortest edges adjacent to each node.
  - The degree constraints.
- Generate subtour elimination constraints and other variables dynamically.

Object Pools

- One or more object pools maintain a list of the most "effective" objects found so far.
- Each pool services a subtree – pools are dynamically allocated.
- The use of multiple pools allows locally valid cuts to be generated.
- With multiple pools, pools are smaller and contain cuts that were generated "closer" in the tree ⇒ more likely to be violated.
- The size of each pool is controlled through the purging of "ineffective" objects.



BCP Modules

There are six module types:

Master Maintains problem instance data, spawns other processes, performs I/O, fault tolerance.

Tree Manager Controls overall execution by tracking growth of the tree and dispatching subproblems to the LP solvers.

LP Solvers Perform processing and branching operations.

Object Generators Generate objects.

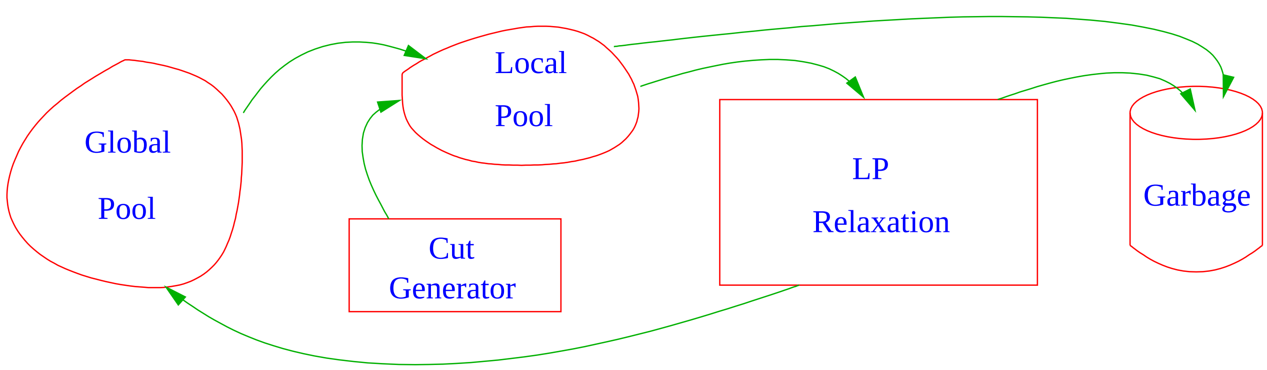
Object Pools Act as auxiliary object generators by maintaining a list of the "most effective" objects found so far.

GUI Allows graphical display of fractional and integer solutions.

Managing the LP Relaxation

Constraints

- Cuts are generated by the cut generators and using cut pools.
- Violated cuts are received and processed by the LP modules.
- Each LP module maintains a small local cut pool.
- A limited number of cuts are added to the LP relaxations each iteration to prevent "saturation."
- Ineffective (non-core) cuts are aggressively removed.
- Cuts are only sent to a global pool if they prove effective locally.



Managing the LP Relaxation

Variables

- Reduced cost/logical fixing are used to remove (non-core) variables.

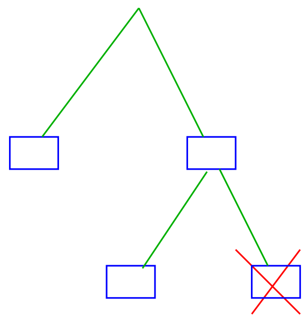
- Variable generation is needed only for very large problems.
  - Unlike cuts, adding variables “loosens” the formulation.
  - Variable generation may be inefficient in cases where it is not needed.
  - Exact generation must take place before fathoming!

- Two-phase algorithm
  - BCP is run to completion on the core variables before generating new ones.
  - Using the upper bound and cuts from the first phase, all variables are priced out in the root node and are then propagated down into the leaves as required.
  - The tree is trimmed by aggregating children back into their parent.
  - Afterwards, each leaf is processed again.

Managing the Search Tree

Fathoming

- Fathoming occurs when
  - the lower bound for the subproblem is provably greater than or equal to the known upper bound, or
  - The subproblem is proven infeasible.
- It is necessary to have every variable either
  - Present in the subproblem, or
  - Fixed by reduced cost.
- This means exact column generation.



Managing the Search Tree

Branching

- If we fail to fathom or locate any more objects that should be included in the current relaxation, we must branch.
- Branching can be done on any object or set of objects.
- All that is needed is to specify object bounds in each branch.
- No matter what objects are used, *strong branching* is a critical tool.
  - Select several branching candidates.
  - "Presolve" each candidate.
  - Choose the "best" for branching.
- *Fractional branching* is also an option.

Managing the Search Tree

Storage and Search Strategy

- Data Storage
  - Efficient data storage is essential.
  - The state of the entire tree is stored, including warm-start info—*important*!
  - The description of each node can be stored explicitly or with respect to its parent, whichever is smaller.
- Tree Management
  - The search algorithm should be an adaptable hybrid of depth-first and best-first.
  - Best-first theoretically minimizes the size of the tree.
  - Depth-first avoids node set-up costs.

Conclusions

Key Points:

- Keep relaxations small!
  - Be conservative when adding cuts and liberal when deleting them.
  - Use object pools wisely.
- Choose the core properly.
- Maintain warm-start information.
- Be careful with variable generation.
- Use strong branching.
- Use an adaptable search strategy.

Shameless Plug

SYMPHONY and COIN/BCP

- Designed to run in a parallel environment.
  - Serial, fully distributed (using PVM), or shared-memory (using threads) modes.
  - No knowledge of parallelism required.
  - Runs in heterogeneous Unix environments.
- User supplies:
  - data structures for objects,
  - object generation subroutines,
  - description of the core,
  - feasibility checker, and
  - other optional subroutines.
- The framework takes care of everything else.
- `www.BranchAndCut.org`