On the Value Function of a Mixed Integer Linear Program

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Outline

Motivation

- The goal of this work is to study the structure of the value function of a general mixed integer linear program (MILP).
- We hope this will lead to methods for approximation useful for
  - Sensitivity analysis
  - Warm starting
  - Multi-level/hierarchical mathematical programming
  - Other methods that require dual information
- Constructing the value function (or even an approximation to it) is difficult, even in a small neighborhood.
- We begin by considering the value functions of single-row relaxations.

Definitions

- We consider the MILP

$$\min_{x \in \mathcal{S}} cx, \tag{P}$$

$c \in \mathbb{R}^n$, $S = \{x \in \mathbb{Z}^r_+ \times \mathbb{R}^{n-r}_+ \mid a'x = b\}$ with $a \in \mathbb{Q}^n$, $b \in \mathbb{R}$.

- The value function of (P) is

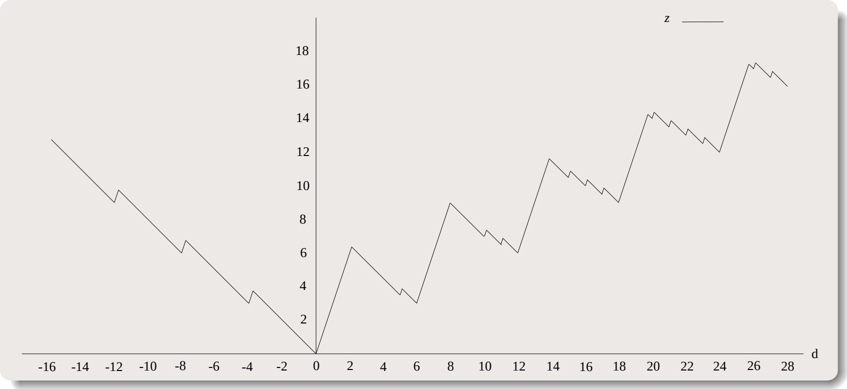
$$z(d) = \min_{x \in \mathcal{S}(d)} cx,$$

where for a given $d \in \mathbb{R}$, $\mathcal{S}(d) = \{x \in \mathbb{Z}^r_+ \times \mathbb{R}^{n-r}_+ \mid a'x = d\}$.

- Assumptions: Let $I = \{1, \ldots, r\}$, $C = \{r+1, \ldots, n\}$, $N = I \cup C$.
  - $z(0) = 0 \implies z : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$,
  - $N^+ = \{i \in N \mid a_i > 0\} \neq \emptyset$ and $N^- = \{i \in N \mid a_i < 0\} \neq \emptyset$,
  - $r < n$, that is, $|C| \geq 1 \implies z : \mathbb{R} \to \mathbb{R}$.

Example

$$\begin{array}{rl} \min & 3x_1 + \frac{7}{2}x_2 + 3x_3 + 6x_4 + 7x_5 + 5x_6 \\ \text{s.t} & 6x_1 + 5x_2 - 4x_3 + 2x_4 - 7x_5 + x_6 = b \quad \text{and} \\ & x_1, x_2, x_3 \in \mathbb{Z}_+, x_4, x_5, x_6 \in \mathbb{R}_+. \end{array} \qquad \text{(SP)}$$



Simple Bounding Functions

- F_L: LP Relaxation $\rightarrow$ Lower Bounding function
- F_U: Continuous Relaxation $\rightarrow$ Upper Bounding function

$$F_L(d) = \begin{cases} \eta d & \text{if } d > 0, \\ 0 & \text{if } d = 0, \\ \zeta d & \text{if } d < 0. \end{cases} \qquad F_U(d) = \begin{cases} \eta^C d & \text{if } d > 0 \\ 0 & \text{if } d = 0 \\ \zeta^C d & \text{if } d < 0 \end{cases}$$

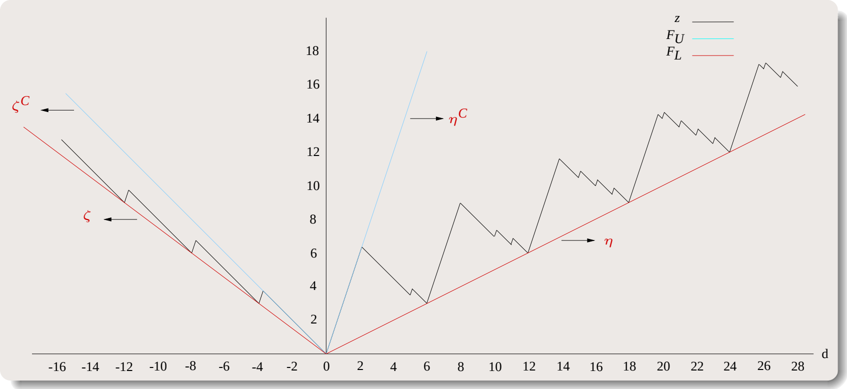
where, setting $C^+ = \{i \in C \mid a_i > 0\}$ and $C^- = \{i \in C \mid a_i < 0\}$,

$$\eta = \min\{\tfrac{c_i}{a_i} \mid i \in N^+\} \quad \text{and} \quad \zeta = \max\{\tfrac{c_i}{a_i} \mid i \in N^-\}$$
$$\eta^C = \min\{\tfrac{c_i}{a_i} \mid i \in C^+\} \quad \text{and} \quad \zeta^C = \max\{\tfrac{c_i}{a_i} \mid i \in C^-\}.$$

- By convention: $C^+ \equiv \emptyset \rightarrow \eta^C = \infty$ and $C^- \equiv \emptyset \rightarrow \zeta^C = -\infty$.
- $F_U \geq z \geq F_L$

Example (cont'd)

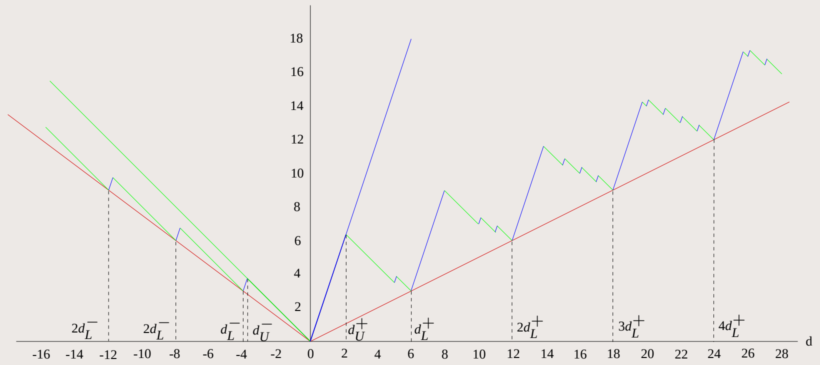
$\eta = \frac{1}{2}$, $\zeta = -\frac{3}{4}$, $\eta^C = 3$ and $\zeta^C = -1$:



- $\{\eta = \eta^C\} \Longleftrightarrow \{z(d) = F_U(d) = F_L(d)\ \forall d \in \mathbb{R}_+\}$
- $\{\zeta = \zeta^C\} \Longleftrightarrow \{z(d) = F_U(d) = F_L(d)\ \forall d \in \mathbb{R}_-\}$

Observations

Consider $d_U^+, d_U^-, d_L^+, d_L^-$:



The relation between F_U and the linear segments of z: $\{\eta^C, \zeta^C\}$

Redundant Variables

Let $T \subseteq C$ be such that

- $t^+ \in T$ if and only if $\eta^C < \infty$ and $\eta^C = \frac{c_{t^+}}{a_{t^+}}$ and similarly,
- $t^- \in T$ if and only if $\zeta^C > -\infty$ and $\zeta^C = \frac{c_{t^-}}{a_{t^-}}$.

and define

$$
\begin{aligned}
\nu(d) = \min \quad & c_I x_I + c_T x_T \\
s.t. \quad & a_I x_I + a_T x_T = d \\
& x_I \in \mathbb{Z}_+^I, \quad x_T \in \mathbb{R}_+^T
\end{aligned}
$$

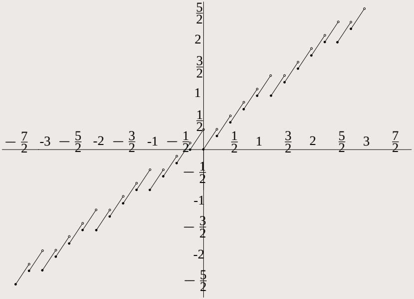
Then

- $\nu(d) = z(d)$ for all $d \in \mathbb{R}$.
- The variables in $C \backslash T$ are redundant.
- z can be represented with at most 2 continuous variables.

Example

$$\begin{array}{rl} \min & x_1 - 3/4x_2 + 3/4x_3 \\ \text{s.t} & 5/4x_1 - x_2 + 1/2x_3 = b, x_1, x_2 \in \mathbb{Z}_+, x_3 \in \mathbb{R}_+. \end{array}$$

$\eta^c = 3/2$, $\zeta^C = -\infty$.



For each discontinuous point d_i, we have $d_i - (5/4y_1^i - y_2^i) = 0$ and each linear segment has the slope of $\eta^C = 3/2$.

Jeroslow Formula

- Let $M \in \mathbb{Z}_+$ be such that for any $t \in T$, $\frac{Ma_j}{a_t} \in \mathbb{Z}$ for all $j \in I$.
- Then there is a Gomory function g such that

$$z(d) = \min_{t \in T}\{g(\lfloor d \rfloor_t) + \frac{c_t}{a_t}(d - \lfloor d \rfloor_t)\}, \quad \lfloor d \rfloor_t = \frac{a_t}{M}\left\lfloor \frac{Md}{a_t} \right\rfloor, \quad \forall d \in \mathbb{R}$$

- Such a Gomory function can be obtained from the value function of a related PILP.
- For $t \in T$, setting

$$\omega_t(d) = g(\lfloor d \rfloor_t) + \frac{c_t}{a_t}(d - \lfloor d \rfloor_t) \ \forall d \in \mathbb{R},$$

we can write

$$z(d) = \min_{t \in T} \omega_t(d) \ \forall d \in \mathbb{R}$$

Piecewise Linearity and Continuity

- For $t \in T$, ω_t is piecewise linear with finitely many linear segments on any closed interval and each of those linear segments has a slope of η^C if $t = t^+$ or ζ^C if $t = t^-$.
- ω_{t^+} is continuous from the right, ω_{t^-} is continuous from the left.
- ω_{t^+} and ω_{t^-} are both lower-semicontinuous.

Theorem

- *z is piecewise-linear with finitely many linear segments on any closed interval and each of those linear segments has a slope of η^C or ζ^C.*
- *(Meyer 1975) z is lower-semicontinuous.*
- *$\eta^C < \infty$ if and only if z is continuous from the right.*
- *$\zeta^C > -\infty$ if and only if z is continuous from the left.*
- *Both η^C and ζ^C are finite if and only if z is continuous everywhere.*

Maximal Subadditive Extension

- Let $f : [0,h] \to \mathbb{R}, h > 0$ be subadditive and $f(0) = 0$.
- The maximal subadditive extension of f from $[0,h]$ to $\mathbb{R}_+$ is

$$f_S(d) = \begin{cases} f(d) & \text{if } \ d \in [0,h] \\ \inf\limits_{C \in \mathcal{C}(d)} \sum\limits_{\rho \in C} f(\rho) & \text{if } \ d > h \end{cases},$$

  - $\mathcal{C}(d)$ is the set of all finite collections $\{\rho_1, ..., \rho_R\}$ such that $\rho_i \in [0,h], i = 1, ..., R$ and $\sum_{i=1}^{R} \rho_i = d$.
  - Each collection $\{\rho_1, ..., \rho_R\}$ is called an h-partition of d.
  - We can also extend a subadditive function $f : [h,0] \to \mathbb{R}, h < 0$ to $\mathbb{R}_-$ similarly.
- (Bruckner 1960) f_S is subadditive and if g is any other subadditive extension of f from $[0,h]$ to $\mathbb{R}_+$, then $g \leq f_S$ (maximality).

Extending the Value Function

- Suppose we use z itself as the seed function.
- Observe that we can change the "inf" to "min":

Lemma

Let the function $f : [0,h] \rightarrow \mathbb{R}$ *be defined by* $f(d) = z(d)\ \forall d \in [0,h]$. *Then,*

$$f_S(d) = \begin{cases} z(d) & \text{if} \quad d \in [0,h] \\ \min_{\mathcal{C} \in \mathcal{C}(d)} \sum_{\rho \in \mathcal{C}} z(\rho) & \text{if} \ \ d > h \end{cases} .$$

- For any $h > 0$, $z(d) \leq f_S(d)\ \forall d \in \mathbb{R}_+$.
- Observe that for $d \in \mathbb{R}_+$, $f_S(d) \rightarrow z(d)$ while $h \rightarrow \infty$.
- Is there an $h < \infty$ such that $f_S(d) = z(d)\ \forall d \in \mathbb{R}_+$?

Yes! For large enough h, maximal extension produces the value function itself.

Theorem

Let $d_r = \max\{a_i \mid i \in N\}$ and $d_l = \min\{a_i \mid i \in N\}$ and let the functions f_r and f_l be the maximal subadditive extensions of z from the intervals $[0, d_r]$ and $[d_l, 0]$ to $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively. Let

$$F(d) = \begin{cases} f_r(d) & d \in \mathbb{R}_+ \\ f_l(d) & d \in \mathbb{R}_- \end{cases}$$

then, $z = F$.

Outline of the Proof.

- $z \leq F$: By construction.
- $z \geq F$: Using MILP duality, F is dual feasible.

In other words, the value function is completely encoded by the breakpoints in $[d_l, d_r]$ and 2 slopes.

General Procedure

- We will construct the value function in two steps
  - Construct the value function on $[d_l, d_r]$.
  - Extend the value function to the entire real line from $[d_l, d_r]$.
- For the rest of the talk
  - We assume $\eta^C < \infty$ and $\zeta^C < \infty$.
  - We construct the value function over $\mathbb{R}_+$ only.
  - These assmuptions are only needed to simplify the presentation.

Constructing the Value Function on $[0, d_r]$

- If both η^C and ζ^C are finite, the value function is continuous and the slopes of the linear segments alternate between η^C and ζ^C.
- For $d_1, d_2 \in [0, d_r]$, if $z(d_1)$ and $z(d_2)$ are connected by a line with slope η^C or ζ^C, then z is linear over $[d_1, d_2]$ with the respective slope (subadditivity).
- With these observations, we can formulate a finite algorithm to evaluate z in $[d_l, d_r]$.

Example (cont'd)

$d_r = 6$:

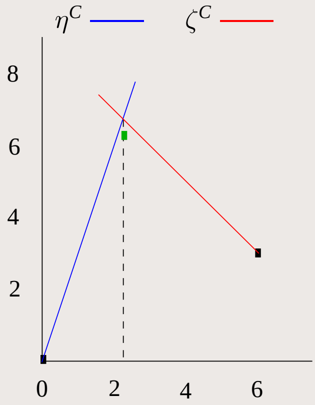


Figure: Evaluating z in $[0, 6]$

Example (cont'd)

$d_r = 6$:

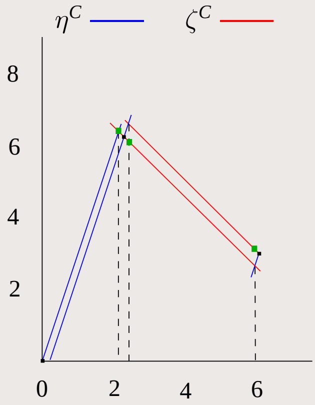


Figure: Evaluating z in $[0, 6]$

Example (cont'd)

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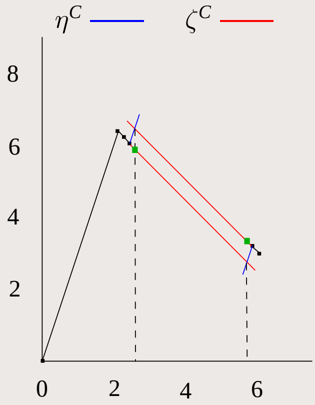

Figure: Evaluating z in $[0, 6]$

Example (cont'd)

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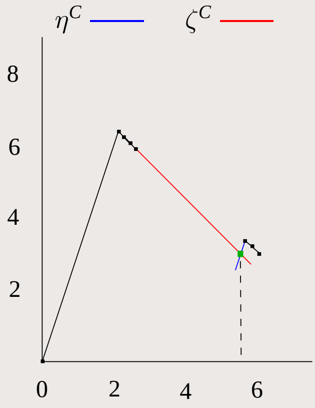


Figure: Evaluating z in $[0, 6]$

Example (cont'd)

$d_r = 6$:

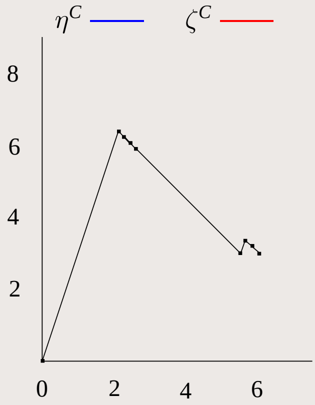


Figure: Evaluating z in $[0, 6]$

Extending the Value Function

Consider evaluating

$$z(d) = \min_{\mathcal{C} \in \mathcal{C}(d)} \sum_{\rho \in \mathcal{C}} z(\rho) \quad \text{for } d \notin [0, d_r].$$

- Can we limit $|\mathcal{C}|$, $\mathcal{C} \in \mathcal{C}(d)$? Yes!
- Can we limit $|\mathcal{C}(d)|$? Yes!

Theorem

Let $d > d_r$ and let $k_d \geq 2$ be the integer such that $d \in (\frac{k_d}{2} d_r, \frac{k_d+1}{2} d_r]$. Then

$$z(d) = \min\{\sum_{i=1}^{k_d} z(\rho_i) \mid \sum_{i=1}^{k_d} \rho_i = d, \rho_i \in [0, d_r], i = 1, \ldots, k_d\}.$$

- Therefore, $|\mathcal{C}| \leq k_d$ for any $\mathcal{C} \in \mathcal{C}(d)$.
- How about $|\mathcal{C}(d)|$?

Lower Break Points

Let Ψ be the lower break points of z in $[0, d_r]$.

Theorem

For any $d \in \mathbb{R}_+ \backslash [0, d_r]$ there is an optimal d_r-partition $\mathcal{C} \in \mathcal{C}(d)$ such that $|\mathcal{C} \backslash \Psi| \leq 1$.

- In particular, we only need to consider the collection

$$\Lambda(d) \equiv \{\mathcal{H} \cup \{\mu\} \mid \mathcal{H} \in \mathcal{C}(d-\mu) \cap \Psi^{k_d - 1}, \sum_{\rho \in \mathcal{H}} \rho + \mu = d, \\ \mu \in [0, d_r]\}$$

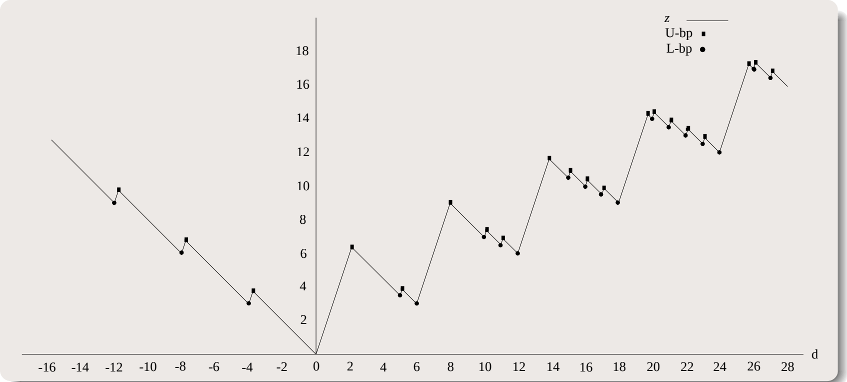
  In other words,

$$z(d) = \min_{\mathcal{C} \in \Lambda(d)} \sum_{\rho \in \mathcal{C}} z(\rho) \quad \forall d \in \mathbb{R}_+ \backslash [0, d_r]$$

- Observe that the set $\Lambda(d)$ is finite.

Example (cont'd)

For the interval $[0, 6]$, we have $\Psi = \{0, 5, 6\}$. For $b = \frac{31}{2}$, $\mathcal{C} = \{5, 5, \frac{11}{2}\}$ is an optimal d_r-partition with $|\mathcal{C} \setminus \Psi|$ = 1.



Getting z over $\mathbb{R}_+$

- Recursive Construction:
- Let $\Psi((0,p])$ to the set of the lower break points of z in the interval $(0,p]$ $p \in \mathbb{R}_+$.
  1. Let $p := d_r$.
  2. For any $d \in \left(p, p + \frac{p}{2}\right]$, let

$$z(d) = \min\{z(\rho_1) + z(\rho_2) \mid \rho_1 + \rho_2 = d, \rho_1 \in \Psi((0,p]), \rho_2 \in (0,p]\}$$

Let $p := p + \frac{p}{2}$ and repeat this step.

In other words, we do the following at each iteration:

$$z(d) = \min_j g^j(d) \ \forall d \in \left(p, p + \frac{p}{2}\right]$$

where, for each $d^j \in \Psi((0,p])$, the functions $g^j : \left[0, p + \frac{p}{2}\right] \rightarrow \mathbb{R} \cup \{\infty\}$ are defined as

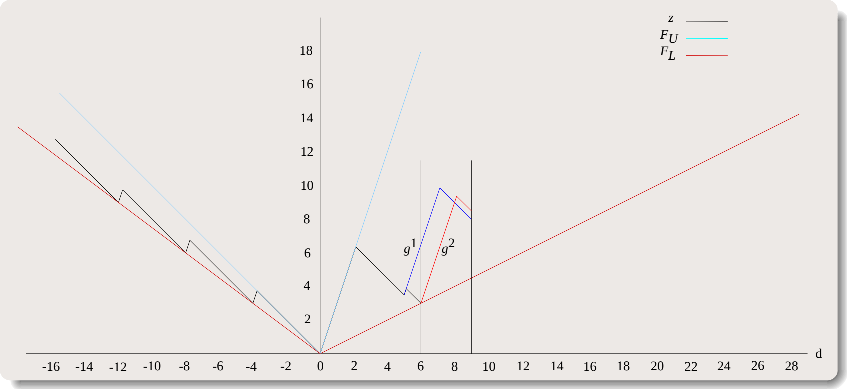
$$g^j(d) = \begin{cases} z(d) & \text{if} \quad d \leq d^j, \\ z(d^j) + z(d - d^j) & \text{if} \quad d^j < d \leq p + d^j, \\ \infty & \text{otherwise.} \end{cases}$$

Because of subadditivity, we can then write

$$z(d) = \min_j g^j(d) \ \forall d \in \left(0, p + \frac{p}{2}\right].$$

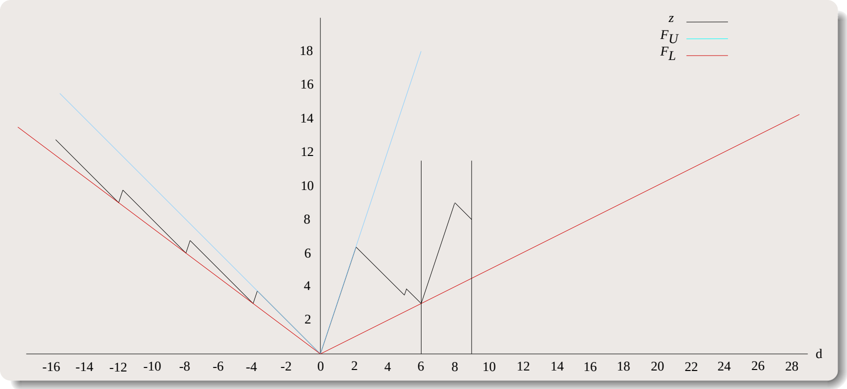
Example (cont'd)

Extending the value function of (SP) from $[0, 6]$ to $[0, 9]$



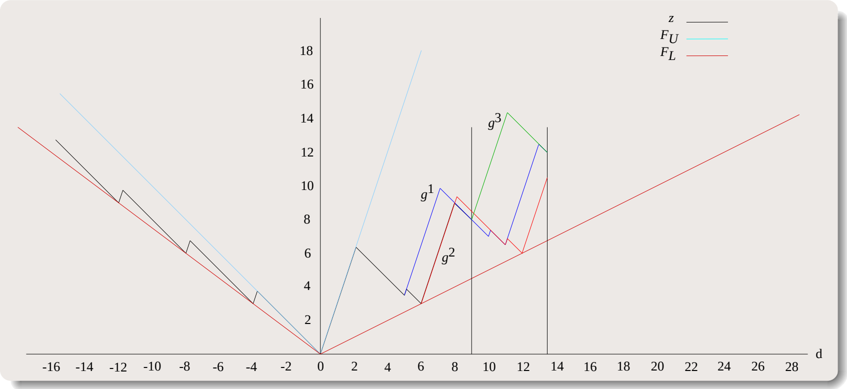
Example (cont'd)

Extending the value function of (SP) from $[0, 6]$ to $[0, 9]$



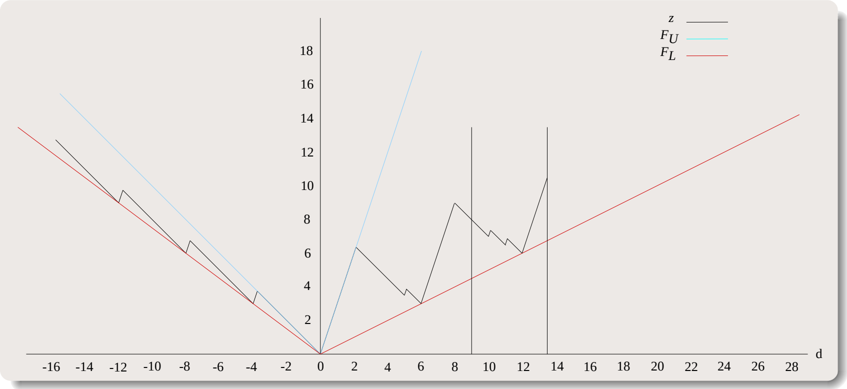
Example (cont'd)

Extending the value function of (SP) from $[0, 9]$ to $\left[0, \frac{27}{2}\right]$



Example (cont'd)

Extending the value function of (SP) from $[0, 9]$ to $\left[0, \frac{27}{2}\right]$



A Combinatorial Procedure

Observe that it is enough to get the lower break points and this can be done more easily.

Theorem

If d is a lower break-point of z on $(p, p+\frac{p}{2}]$ then there exist $\rho_1, \rho_2 \in \Psi((0,p])$ such that $z(d) = z(\rho_1) + z(\rho_2)$ and $d = \rho_1 + \rho_2$.

- Set $\Upsilon(p) \equiv \{z(\rho_1) + z(\rho_2) \mid p < \rho_1 + \rho_2 \leq p + \frac{p}{2}, \rho_1, \rho_2 \in \Psi((0,p])\}$.
- Then, z is obtained by connecting the points on the "lower envelope" of $\Upsilon(p)$.
- Can we make the procedure finite?

Termination

Yes! Periodicity

- Let $\mathcal{D} = \{d \mid z(d) = F_L(d)\}$. Note that $\mathcal{D} \neq \emptyset$.
- Furthermore, let $\lambda = \min\{d \mid d \geq d_r, d \in \mathcal{D}\}$.
- Define the functions $f_j : \mathbb{R}_+ \rightarrow \mathbb{R}, j \in \mathbb{Z}_+ \backslash \{0\}$ as follows

$$f_j(d) = \begin{cases} z(d) & , d \leq j\lambda \\ kz(\lambda) + z(d - k\lambda) & , d \in ((k+j-1)\lambda, (k+j)\lambda], k \in \mathbb{Z}_+ \backslash \{0\}. \end{cases}$$

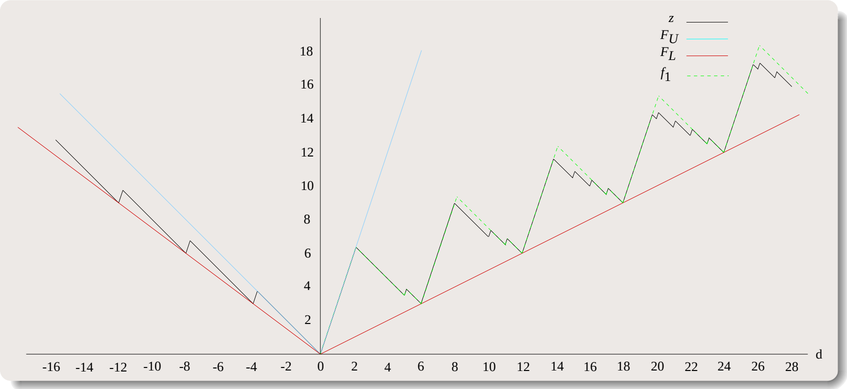
Theorem

1. $f_j(d) \geq f_{j+1}(d) \geq z(d)$ *for all* $d \in \mathbb{R}_+, j \in \mathbb{Z}_+ \backslash \{0\}$.
2. *There exists* $q \in \mathbb{Z}_+ \backslash \{0\}$ *such that* $z(d) = f_q(d)\ \forall d \in \mathbb{R}_+$.
3. *In addition,* $z(d) = f_q(d)\ \forall d \in \mathbb{R}_+$ *if and only if* $f_q(d) = f_{q+1}(d)\ \forall d \in \mathbb{R}_+$.

- Therefore, we can extend over the intervals of size λ and stop when we reach the 3. condition above.

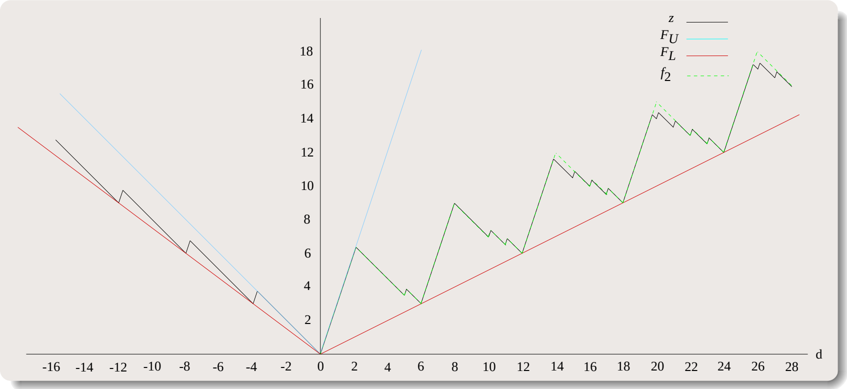
Example (cont'd)

$$\lambda = 6, \quad f_1(d) = \begin{cases} z(d) & , d \leq 6 \\ kz(6) + z(d - 6k) & , d \in (6k, 6(k+1)], k \in \mathbb{Z}_+ \backslash \{0\}. \end{cases}$$



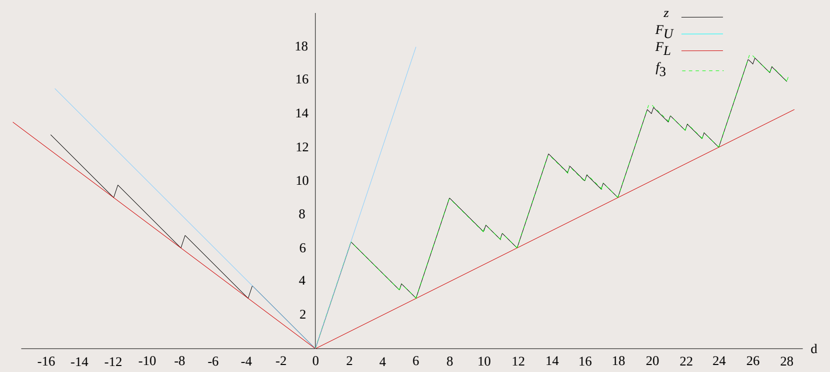
Example (cont'd)

$$f_2(d) = \begin{cases} z(d) & , d \leq 12 \\ kz(6) + z(d - 6k) & , d \in (6(k+1), 6(k+2)], k \in \mathbb{Z}_+ \backslash \{0\}. \end{cases}$$



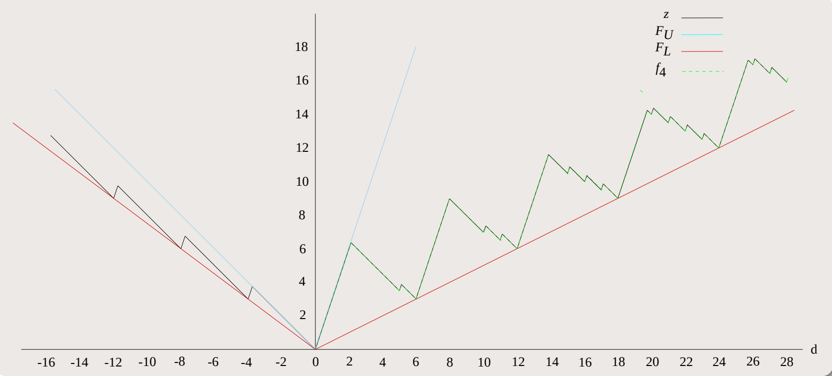
Example (cont'd)

$$f_3(d) = \begin{cases} z(d) & , d \leq 18 \\ kz(6) + z(d - 6k) & , d \in (6(k+2), 6(k+3)], k \in \mathbb{Z}_+ \backslash \{0\}. \end{cases}$$



Example (cont'd)

$$f_4(d) = \begin{cases} z(d) & , d \le 24 \\ kz(6) + z(d-6k) & , d \in (6(k+3), 6(k+4)], k \in \mathbb{Z}_+ \setminus \{0\}. \end{cases}$$



- Note that $f_4(d) = f_5(d) \forall d \in \mathbb{R}_+$. Therefore, $z(d) = f_4(d) \forall d \in \mathbb{R}_+$.

A Finite Procedure

We can further restrict the search space by again using maximal extension and the fact that $z(k\lambda) = kz(\lambda)$ and $\lambda \geq d_r$.

Theorem

For a given $k \geq 2, k \in \mathbb{Z}_+$,

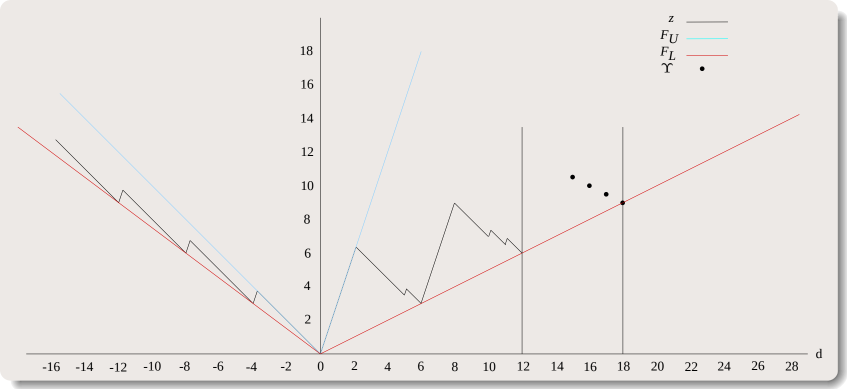
$$z(d) = \min\{z(\rho_1) + z(\rho_2) \mid \rho_1 + \rho_2 = d, \rho_1 \in (0, 2\lambda], \rho_2 \in ((k-1)\lambda, k\lambda]\}$$

$\forall d \in (k\lambda, (k+1)\lambda]$.

- Revised Recursive Construction:
  1. Let $p := 2\lambda$.
  2. Set $\Upsilon(p) \equiv \{z(\rho_1) + z(\rho_2) \mid p < \rho_1 + \rho_2 \leq p + \lambda, \rho_1 \in \Psi((0, 2\lambda]), \rho_2 \in \Psi((p-\lambda, p])\}$ and obtain z over $[p, p+\lambda]$ by considering the "lower subadditive envelope" of $\Upsilon(p)$.
  3. If $z(d) = z(d-\lambda) + z(\lambda) \forall d \in \Psi((p, p+\lambda))$, then stop. Otherwise, let $p := p + \lambda$ and repeat the last step.

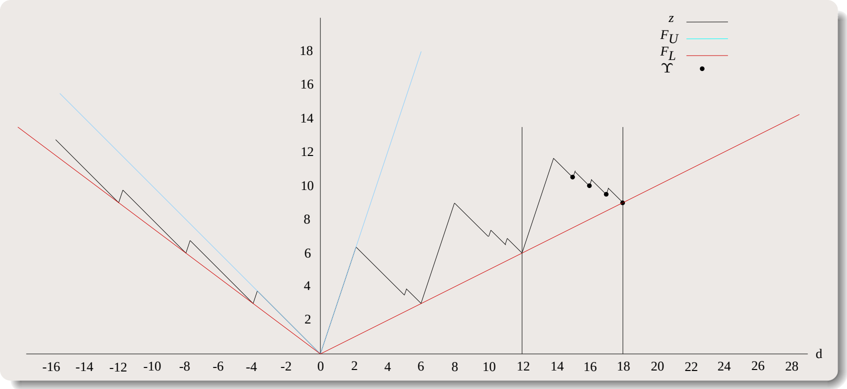
Example (cont'd)

Extending the value function of (SP) from $[0, 12]$ to $[0, 18]$



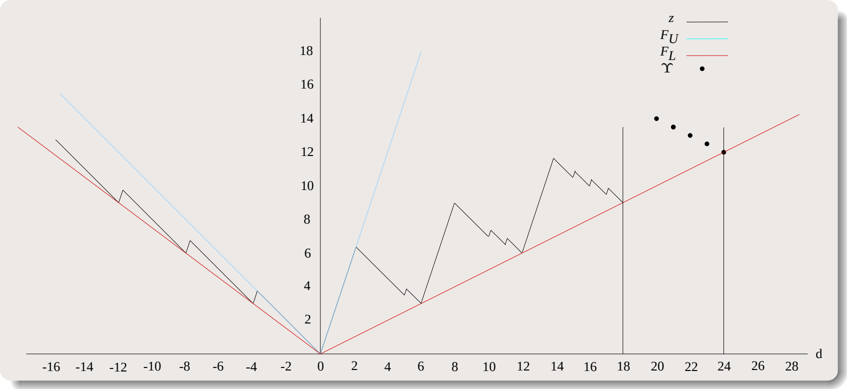
Example (cont'd)

Extending the value function of (SP) from $[0, 12]$ to $[0, 18]$



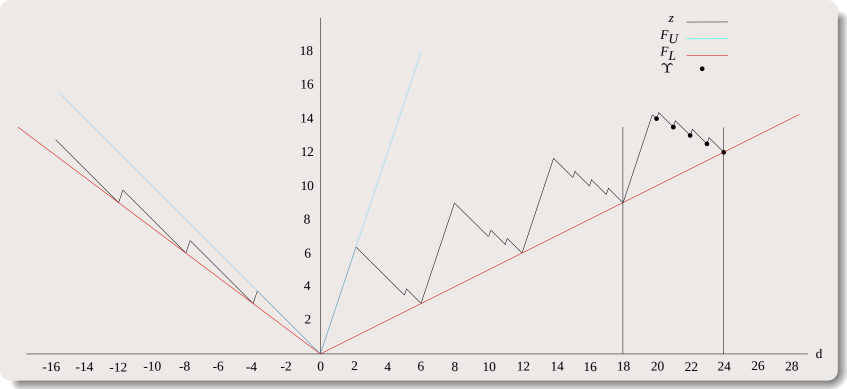
Example (cont'd)

Extending the value function of (SP) from $[0, 18]$ to $[0, 24]$



Example (cont'd)

Extending the value function of (SP) from $[0, 18]$ to $[0, 24]$



General Case

- Consider a general mixed integer linear program (MILP)

$$z_P = \min_{x \in \mathcal{S}} cx, \tag{P}$$

$c \in \mathbb{R}^n$, $S = \{x \in \mathbb{Z}^r_+ \times \mathbb{R}^{n-r}_+ \mid Ax = b\}$ with $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{R}^m$.

- The value function of the primal problem (P) is

$$z(d) = \min_{x \in \mathcal{S}(d)} cx,$$

where for a given $d \in \mathbb{R}^m$, $\mathcal{S}(d) = \{x \in \mathbb{Z}^r_+ \times \mathbb{R}^{n-r}_+ \mid Ax = d\}$.

Jeroslow Formula for General MILP

Let the set $\mathcal{E}$ consist of the index sets of dual feasible bases of the linear program

$$\min\{\frac{1}{M}c_C x_C \ : \ \frac{1}{M}A_C x_C = b, x \geq 0\}$$

where $M \in \mathbb{Z}_+$ such that for any $E \in \mathcal{E}$, $MA_E^{-1}a^j \in \mathbb{Z}^m$ for all $j \in I$.

Theorem (Jeroslow Formula)

There is a $g \in \mathcal{G}^m$ such that

$$z(d) = \min_{E \in \mathcal{E}} g(\lfloor d \rfloor_E) + v_E(d - \lfloor d \rfloor_E) \ \ \forall d \in \mathbb{R}^m \ \ \textit{with} \ \ \mathcal{S}(d) \neq \emptyset,$$

where for $E \in \mathcal{E}$, $\lfloor d \rfloor_E = A_E \lfloor A_E^{-1} d \rfloor$ and v_E is the corresponding basic feasible solution.

- For $E \in \mathscr{E}$, setting

$$\omega_E(d) = g(\lfloor d \rfloor_E) + v_E(d - \lfloor d \rfloor_E) \ \forall d \in \mathbb{R}^m \ \text{ with } \ \mathcal{S}(d) \neq \emptyset,$$

we can write

$$z(d) = \min_{E \in \mathscr{E}} \omega_E(d) \ \forall d \in \mathbb{R}^m \ \text{ with } \ \mathcal{S}(d) \neq \emptyset.$$

- Many of our previous results can be extended to general case in the obvious way.
- Similarly, we can use maximal subadditive extensions to construct the value function..
- However, an obvious combinatorial explosion occurs.
- Therefore, we consider using single row relaxations to get a subadditive approximation.

Basic Idea

Consider the value functions of each single row relaxation:

$$z_i(q) = \min\{cx \mid a_i x = q, x \in \mathbb{Z}^r_+ \times \mathbb{R}^{n-r}_+\} \; q \in \mathbb{R}, i \in M \equiv \{1, \ldots, m\}$$

where a_i is the i^{th} row of A.

Theorem

Let $F(d) = \max_{i \in M}\{z_i(d_i)\}, \; d = (d_1, \ldots, d_m), \; d \in \mathbb{R}^m$. Then F is subadditive and $z(d) \geq F(d) \; \forall d \in \mathbb{R}^m$.

Maximal Subadditive Extension

Assume that $A \in \mathbb{Q}^m_+$. Let $S \subseteq M$ and $q^r \in \mathbb{Q}^{|S|}_+$ be the vector of the maximum of the coefficients of rows $a_i, i \in S$. Define

$$G_S(q) = \begin{cases} \max_{i \in S}\{z_i(q_i)\} & q_i \in [0, q_i^r] \ i \in M \\ \max\left\{ \max_{i\in K}\{z_i(q_i)\}, \inf_{\mathcal{C}\in\mathcal{C}(q_{S\setminus K})} \sum_{\rho\in\mathcal{C}} G_S(\rho) \right\} & \begin{array}{l} q_i \in [0, q_i^r] \ \ i \in K \\ q_i > q_i^r \ \ i \in S\setminus K \\ K \subset S \end{array} \\ \inf_{\mathcal{C}\in\mathcal{C}(q)} \sum_{\rho\in\mathcal{C}} G_S(\rho) & q_i \in \mathbb{R}_+\setminus[0, q_i^r] \ i \in M \end{cases}$$

for all $q \in \mathbb{R}^{|S|}$ where for $T \subseteq S$, $\mathcal{C}(q_T)$ is the set of all finite collections $\{\rho_1, ..., \rho_R\}, \rho_j \in \mathbb{R}^{|T|}$ such that $\rho_j \in \times_{i\in T}[0, q_i^r], j = 1, ..., R$ and $\sum_{j=1}^R \rho_j = q_T$.

Maximal Subadditive Extension

G_S is simply the maximal subadditive extension of the function $\max_{i \in S}\{z_i(q_i)\}$ from the box $\times_{i \in S}[0, q_i^r]$ to $\mathbb{R}_+^{|S|}$.

Theorem

Let $F_S(d) = \max\left\{G_S(d_S), \max_{i \in M \setminus S}\{z_i(d_i)\}\right\}$. $F_S \geq \max_{i \in M}\{z_i(d_i)\}$, *is subadditive and* $z(d) \geq F_S(d)$ *for all* $d \in \mathbb{R}_+^m$.

Aggregation

For $S \subseteq M$, $\omega \in \mathbb{R}^{|S|}$, set

$$G_S(q,\omega) = \min\{cx \mid \omega a_S x = \omega q, x \in \mathbb{Z}^r_+ \times \mathbb{R}^{n-r}_+\} \ \forall q \in \mathbb{R}^{|S|}$$

Theorem

Let

$$F_S(\omega, d) = \max\left\{G_S(d_S,\omega), \max_{i \in M \setminus S}\{z_i(d_i)\}\right\}, \ d \in \mathbb{R}^m.$$

F_S *is subadditive and* $z(d) \geq F_S(\omega, d)$ *for any* $\omega \in \mathbb{R}^{|S|}$, $d \in \mathbb{R}^m$.

As with cutting planes, different aggregation procedures are possible.

Using Cuts

- Assume that $\mathcal{S}(d) = \{x \in \mathbb{Z}^n_+ \mid Ax \leq d\}$.
- Consider the set of Gomory cuts $\Pi x \geq \Pi^0$, $\Pi \in Q^{k \times n}, \Pi^0 \in \mathbb{Q}^k$ defined by the sets of multipliers $\Omega = \{\omega^1, \ldots \omega^{k-1}\}$, $\omega^i \in \mathbb{Q}^{m+i-1}_+$ as follows

$$\Pi_{ij} = \left\lceil \sum_{l=1}^{m} \omega_l^i A_{lj} + \sum_{l=1}^{i-1} \omega_{m+l}^i \Pi_{lj} \right\rceil \quad \forall i = 1, \ldots, k, j = 1, \ldots, n$$

$$\Pi_i^0 = \left\lceil \sum_{l=1}^{m} \omega_l^i d_l + \sum_{l=1}^{i-1} \omega_{m+l}^i \Pi_l^0 \right\rceil \quad \forall i = 1, \ldots, k$$

Theorem

For $\Omega = \{\omega^1, \ldots \omega^{k-1}\}, \omega^i \in \mathbb{Q}^{m+i-1}_+$, $k \in \mathbb{Z}_+$, let $z_{m+i}(\omega^i, d)$ denote the value function of row $m+i, i = 1, \ldots, k-1$ and

$$F(\Omega, d) = \max\left\{ \max_{i \in M} z(d_i), \max_{i=1,\ldots,k-1,\omega^i \in \Omega} z_{m+i}(\omega^i, d) \right\}.$$

Then, F is subadditive and $z(d) \geq F(\Omega, d)$ for any $d \in \mathbb{R}^m$.

Current and Future Work

- Extending the theory and algorithms to the general case.
- Developing upper bounding approximations.
- Integrating these procedures in with applications
  - Bilevel programming
  - Combinatorial auctions
- Answering the question

"Can we do anything practical with any of this?"