The SYMPHONY Callable Library for Mixed-Integer Linear Programming

Ted Ralphs and Menal Guzelsoy
Industrial and Systems Engineering
Lehigh University

INFORMS Computing Society Conference, Annapolis, MD, Thursday, January 6, 2005

Outline of Talk

- Introduction to SYMPHONY
- A little bit of theory
  - Duality
  - Sensitivity analysis
  - Warm starting
  - Parametric analysis
- A little bit of computation
  - Implementation in SYMPHONY
  - Examples
  - Computational results

A Really Brief Overview of SYMPHONY

- SYMPHONY is an open-source software package for solving and analyzing mixed-integer linear programs (MILPs).
- SYMPHONY can be used in three distinct modes.
  - Black box solver: Solve generic MILPs (command line or shell).
  - Callable library: Call SYMPHONY from a C/C++ code.
  - Framework: Develop a customized solver or callable library.
- Available as part of the Computational Infrastructure for Operations Research (COIN-OR) (`www.coin-or.org`).
- Packaged releases available for download on `www.branchandcut.org`.
- The new interface and features of SYMPHONY give it the look and feel an LP solver.
- This talk will focus on these new features—for detailed information on using SYMPHONY, please attend yesterday's SYMPHONY tutorial :).

A Really Brief Introduction to Duality

- For an optimization problem

$$z = \min\{f(x) \mid x \in X\},$$

called the *primal problem*, an optimization problem

$$w = \max\{g(u) \mid u \in U\}$$

such that $w \leq z$ is called a *dual problem*.

- It is a *strong dual* if $w = z$.
- Uses for the dual problem
  - Bounding
  - Deriving optimality conditions
  - Sensitivity analysis
  - Warm starting

Some Previous Work

- R. Gomory (and W. Baumol) ('60–'73)
- G. Roodman ('72)
- E. Johnson (and Burdet) ('72–'81)
- R. Jeroslow (and C. Blair) ('77-'85)
- A. Geoffrion and R. Nauss ('77)
- D. Klein and S. Holm ('79–'84)
- L. Wolsey (and L. Schrage) ('81–'84)
- ...
- D. Klabjan ('02)

Duals for ILP

- Let $\mathcal{P} = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ nonempty for $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$.
- We consider the (bounded) pure integer linear program $\min_{x \in \mathcal{P} \cap \mathbb{Z}^n} c^\top x$ for $c \in \mathbb{R}^n$.
- The most common dual for this ILP is the well-known *Lagrangian dual*.
  - The Lagrangian dual is not generally strong.
  - Blair and Jeroslow discussed how to make the Lagrangian dual strong by for ILP by introducing a quadratic penalty term.
- How do we derive a strong dual? Consider the following more formal notion of dual (Wolsey).

$$w_{IP}^g = \max_{g:\mathbb{R}^m \to \mathbb{R}} \{g(b) \mid g(Ax) \leq c^\top x, x \geq 0\} \quad (1)$$

$$= \max_{g:\mathbb{R}^m \to \mathbb{R}} \{g(b) \mid g(d) \leq z_{IP}(d), d \in \mathbb{R}^m\}, \quad (2)$$

where $z_{IP}(d) = \min_{x \in \mathcal{P}^I(d)} c^\top x$ is the *value function* and $\mathcal{P}^I(d) = \{x \in \mathbb{Z}^n \mid Ax = d, x \geq 0\}$

Dual Solutions from Primal Algorithms

- Sensitivity analysis and warm starting procedures for LP are based on optimality conditions arising from LP duality.
- The *optimal basis* contains all the information needed to construct optimal primal and dual solutions.
- This information can be obtained as a by-product of the primal simplex algorithm.
- We extend this to ILP by considering the implicit optimality conditions associated with branch and bound.

Dual Solutions for ILP from Branch and Bound

- An extension of the optimality conditions for LP to ILP is straightforward.
- Let $\mathcal{P}_1, \ldots, \mathcal{P}_s$ be a partition of $\mathcal{P}$ into (nonempty) subpolyhedra.
- Let LP_i be the linear program $\min_{x^i \in \mathcal{P}_i} c^\top x^i$ associated with the subpolyhedron $\mathcal{P}_i$.
- Let B^i be an optimal basis for LP_i.
- Then the following is a valid lower bound

$$L = \min\{c_{B^i}(B^i)^{-1}b + \gamma_i \mid 1 \leq i \leq s\},$$

  where γ_i is the constant factor associated with the nonbasic variables fixed at nonzero bounds.
- A similar function yields an upper bound.
- A partition that yields equal lower and upper bounds is called an *optimal partition*.

Sensitivity Analysis for ILP

- The function

$$L(d) = \min\{c_{B^i}(B^i)^{-1}d + \gamma_i \mid 1 \leq i \leq s\},$$

provides an optimal solution to (2).

- The corresponding upper bounding function is

$$U(c) = \min\{c_{B^i}(B^i)^{-1}b + \beta_i \mid 1 \leq i \leq s, \hat{x}^i \in \mathcal{P}^I\}$$

- These functions can be used for local sensitivity analysis, just as one would do in linear programming.
  - For changes in the right-hand side, the lower bound remains valid.
  - For changes in the objective function, the upper bound remains valid.
  - One can also add cuts and variables.
- One can compute an "allowable range" for changes to the instance data. as the intersection of the ranges for each member of the partition.

Sensitivity Analysis in SYMPHONY

- Using the functions on the previous slide, SYMPHONY can calculate bounds after changing the objective or right-hand side vectors.

```
int main(int argc, char **argv)
{
   OsiSymSolverInterface si;
   si.parseCommandLine(argc, argv);
   si.loadProblem();
   si.setSymParam(OsiSymSensitivityAnalysis, true);
   si.initialSolve();
   int ind[2];
   double val[2];
   ind[0] = 4;    val[0] = 7000;
   ind[1] = 7;    val[1] = 6000;
   lb = si.getLbForNewRhs(2, ind, val);
   ub = si.getUbForNewRhs(2, ind, val);
}
```

A Few Caveats

- The method presented only applies to pure branch and bound.
- Cut generation complicates matters.
- Fixing by reduced cost also complicates matters.
- Have to deal with infeasibility of subproblems.
- These issues can all be addressed, but the methodology is more involved.
- Question: What happens outside the allowable range?
- Answers:
  - Continue solving from a "warm start."
  - Perform a parametric analysis.

Warm Starting

- Question: What is "warm starting"?
- Question: Why are we interested in it?
- There are many examples of algorithms that solve a sequence of related ILPs.
  - Decomposition algorithms
  - Stochastic ILP
  - Parametric/Multicriteria ILP
  - Determining irreducible inconsistent subsystem
  -
- For such problems, warm starting can potentially yield big improvements.
- Warm starting is also important for performing sensitivity analysis outside of the allowable range.

Warm Starting Information

- Question: What is "warm starting information"?
- Many optimization algorithms can be viewed as iterative procedures for satisfying a set of optimality conditions, often based on duality.
- These conditions provide a measure of "distance from optimality."
- Warm starting information can be seen as additional input data that allows an algorithm to quickly get "close to optimality."
- In linear and integer linear programming, the *duality gap* is the usual measure.
- A starting basis can reduce the initial duality gap in LP.
- The corresponding concept in ILP is a *starting partition*.
- It is not at all obvious what makes a good starting partition.
- The most obvious choice for a starting partition is to use the optimal partition from a previous computation.

Warm Starts for MILP

- To allow resolving from a warm start, we have defined a SYMPHONY warm start class, which is derived from `CoinWarmStart`.
- The class stores a snapshot of the search tree, with node descriptions including:
  - lists of active cuts and variables,
  - branching information,
  - warm start information, and
  - current status (candidate, fathomed, etc.).
- The tree is stored in a compact form by storing the node descriptions as differences from the parent.
- Other auxiliary information is also stored, such as the current incumbent.
- A warm start can be saved at any time and then reloaded later.
- The warm starts can also be written to and read from disk.

Warm Starting Procedure

- After modifying parameters
  - If only parameters have been modified, then the candidate list is recreated and the algorithm proceeds as if left off.
  - This allows parameters to be tuned as the algorithm progresses if desired.
- After modifying problem data
  - Currently, we only allow modification of rim vectors.
  - After modification, all leaf nodes must be added to the candidate list.
  - After constructing the candidate list, we can continue the algorithm as before.
- There are many opportunities for improving the basic scheme, especially when solving a known family of instances (Geoffrion and Nauss)

Using Warm Starting (Parameter Modification)

- The following example shows a simple use of warm starting to create a dynamic algorithm.

```
int main(int argc, char **argv)
{
   OsiSymSolverInterface si;
   si.parseCommandLine(argc, argv);
   si.loadProblem();
   si.setSymParam(OsiSymFindFirstFeasible, true);
   si.setSymParam(OsiSymSearchStrategy, DEPTH_FIRST_SEARCH);
   si.initialSolve();
   si.setSymParam(OsiSymFindFirstFeasible, false);
   si.setSymParam(OsiSymSearchStrategy, BEST_FIRST_SEARCH);
   si.resolve();
}
```

Using Warm Starting (Problem Modification)

- The following example shows how to warm start after problem modification.

```
int main(int argc, char **argv)
{
   OsiSymSolverInterface si;
   CoinWarmStart ws;
   si.parseCommandLine(argc, argv);
   si.loadProblem();
   si.setSymParam(OsiSymNodeLimit, 100);
   si.initialSolve();
   ws = si.getWarmStart();
   si.resolve();
   si.setObjCoeff(0, 1);
   si.setObjCoeff(200, 150);
   si.setWarmStart(ws);
   si.resolve();
}
```

Using Warm Starting: Generic Mixed-Integer Programming

- Applying the code from the previous slide to the MIPLIB 3 problem p0201, we obtain the results below.
- Note that the warm start doesn't reduce the number of nodes generated, but does reduce the solve time dramatically.

	CPU Time	Tree Nodes
Generate warm start	28	100
Solve orig problem (from warm start)	3	118
Solve mod problem (from scratch)	24	122
Solve mod problem (from warm start)	6	198

Using Warm Starting: Generic Mixed-Integer Programming

- Here, we show the effect of using warm starting to solve generic MILPs whose objective functions have been perturbed.
- The coefficients were perturbed by a random percentage between α and $-\alpha$ for $\alpha = 1, 10, 20$.

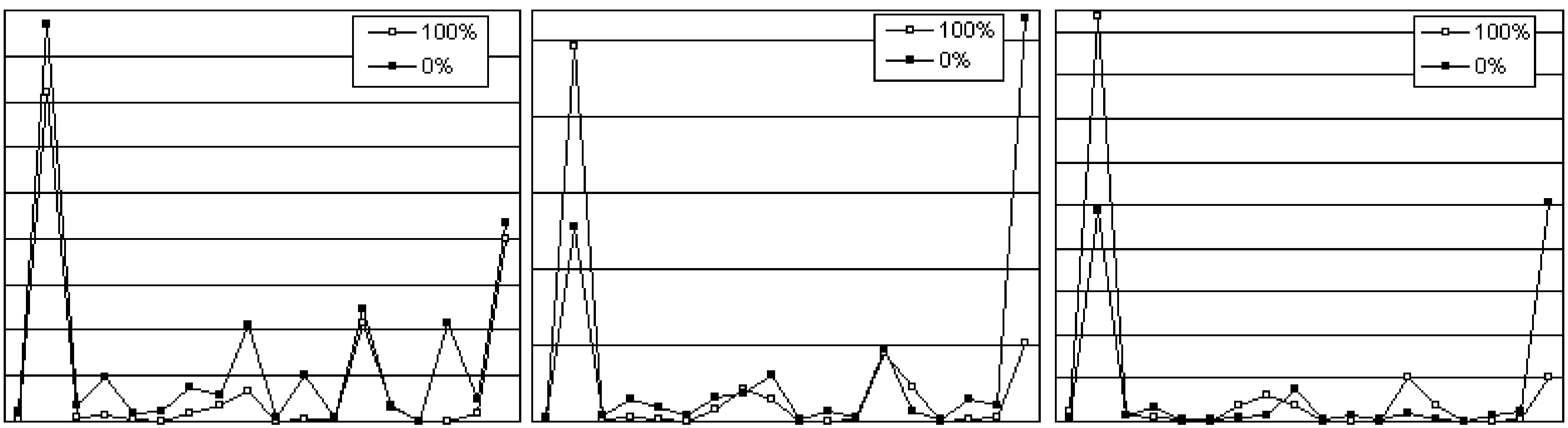


Table 1: Results of using warm starting to solve multi-criteria optimization problems.

Using Warm Starting: Stochastic Integer Programming

Problem	Tree Size Without WS	Tree Size With WS	% Gap Without WS	% Gap With WS	CPU Without WS	CPU With WS
storm8	1	1	-	-	14.75	8.71
storm27	5	5	-	-	69.48	48.99
storm125	3	3	-	-	322.58	176.88
LandS27	71	69	-	-	6.50	4.99
LandS125	37	29	-	-	15.72	12.72
LandS216	39	35	-	-	30.59	24.80
dcap233_200	39	61	-	-	256.19	120.86
dcap233_300	111	89	0.387	-	1672.48	498.14
dcap233_500	21	36	24.701	14.831	1003	1004
dcap243_200	37	53	0.622	0.485	1244.17	1202.75
dcap243_300	64	220	0.0691	0.0461	1140.12	1150.35
dcap243_500	29	113	0.357	0.186	1219.17	1200.57
sizes3	225	165	-	-	789.71	219.92
sizes5	345	241	-	-	964.60	691.98
sizes10	241	429	0.104	0.0436	1671.25	1666.75

Parametric Analysis

- For global sensitivity analysis, we need to solve parametric programs.
- Along with Saltzman and Wiecek, we have developed an algorithm for determining all Pareto outcomes for a bicriteria MILP.
- The algorithm consists of solving a sequence of related ILPs and is asymptotically optimal.
- Such an algorithm can be used to perform global sensitivity analysis by constructing a "slice" of the value function.
- Warm starting can be used to improve efficiency.

Bicriteria MILPs

- The general form of a bicriteria (pure) ILP is

$$\begin{aligned} &\text{vmax}\,[cx, dx], \\ &\text{s.t.} \qquad Ax \le b, \\ &\qquad\qquad x \in \mathbb{Z}^n. \end{aligned}$$

- Solutions don't have single objective function values, but pairs of values called *outcomes*.
- A feasible $\hat{x}$ is called *efficient* if there is no feasible $\bar{x}$ such that $c\bar{x} \geq c\hat{x}$ and $d\bar{x} \geq d\hat{x}$, with at least one inequality strict.
- The outcome corresponding to an efficient solution is called *Pareto*.
- The goal of a bicriteria ILP is to enumerate Pareto outcomes.

Example: Bicriteria ILP

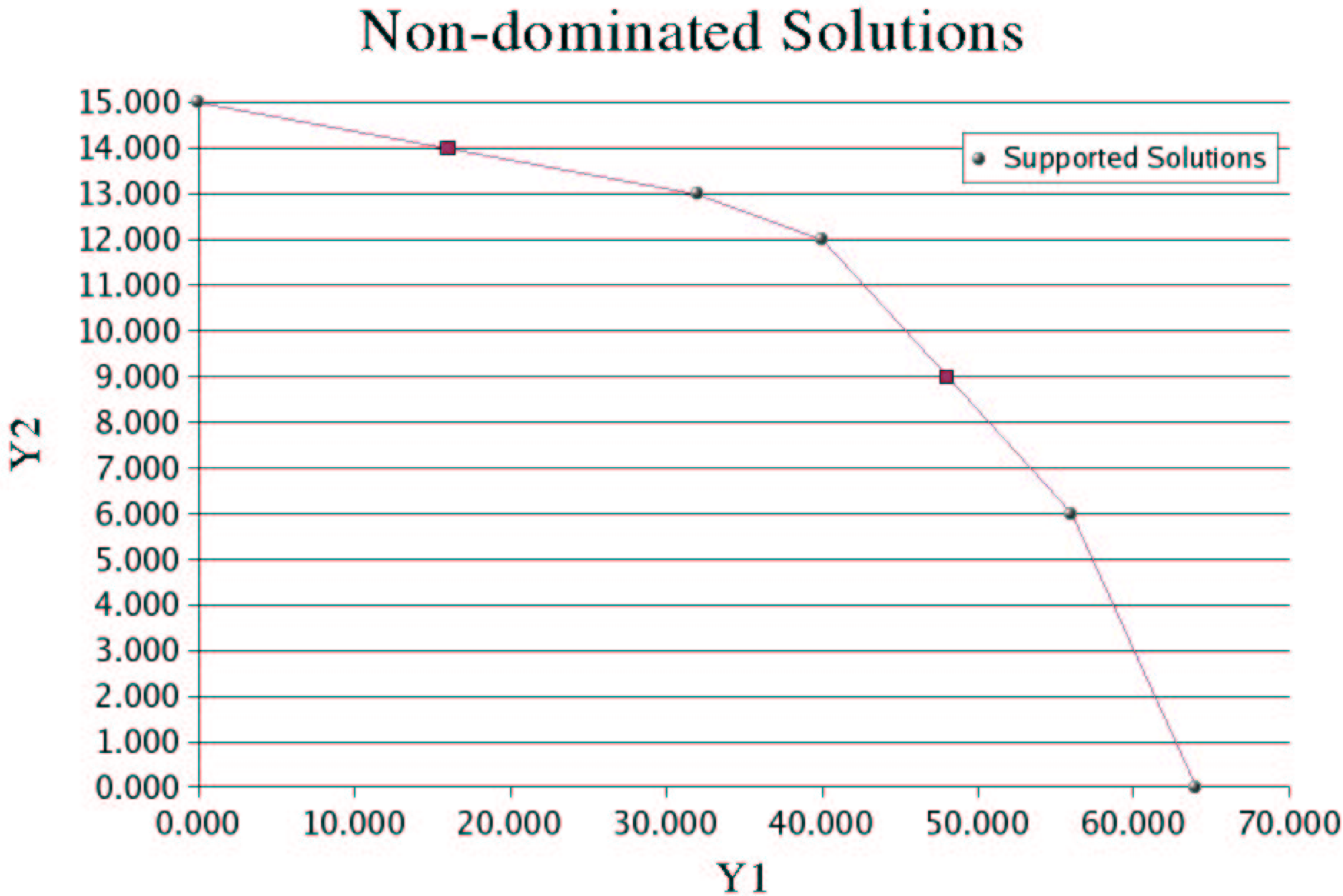
- Consider the following bicriteria ILP:

$$\begin{aligned} \text{vmax} \quad & [8x_1, x_2] \\ \text{s.t.} \quad & 7x_1 + x_2 \leq 56 \\ & 28x_1 + 9x_2 \leq 252 \\ & 3x_1 + 7x_2 \leq 105 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- The following code solves this model.

```
int main(int argc, char **argv)
{
   OsiSymSolverInterface si;
   si.parseCommandLine(argc, argv);
   si.loadProblem();
   si.setObj2Coeff(1, 1);
   si.multiCriteriaBranchAndBound();
}
```

Example: Pareto Outcomes for Example

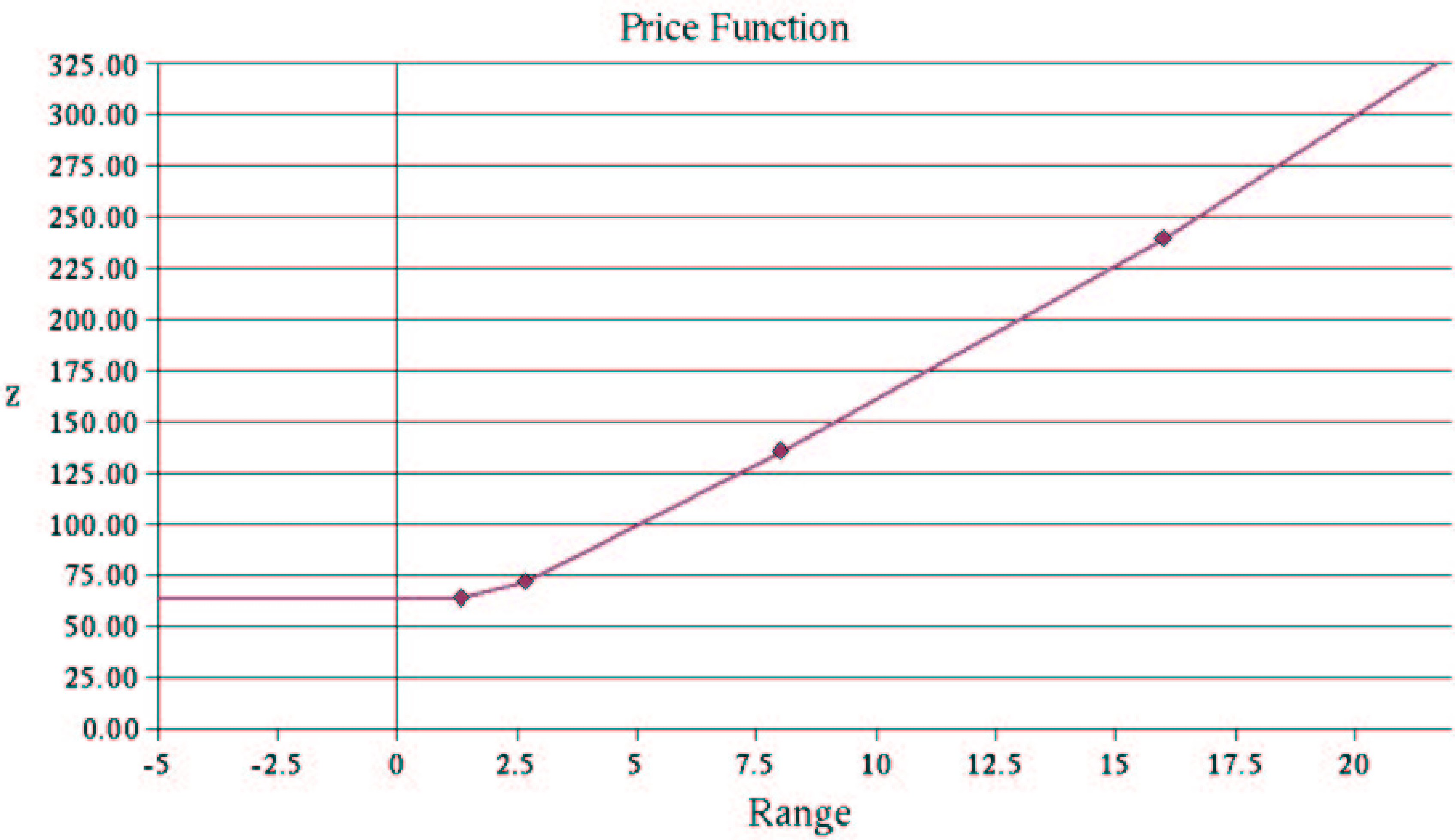


Example: Bicriteria Solver

- By examining the supported solutions and break points, we can easily determine $p(\theta)$, the optimal solution to the ILP with objective $8x_1 + \theta$.

θ range	$p(\theta)$	x_1^*	x_2^*
$(-\infty, 1.333)$	64	8	0
$(1.333, 2.667)$	$56 + 6\theta$	7	6
$(2.667, 8.000)$	$40 + 12\theta$	5	12
$(8.000, 16.000)$	$32 + 13\theta$	4	13
$(16.000, \infty)$	15θ	0	15

Example: Graph of Price Function



Using Warm Starting: Bicriteria Optimization

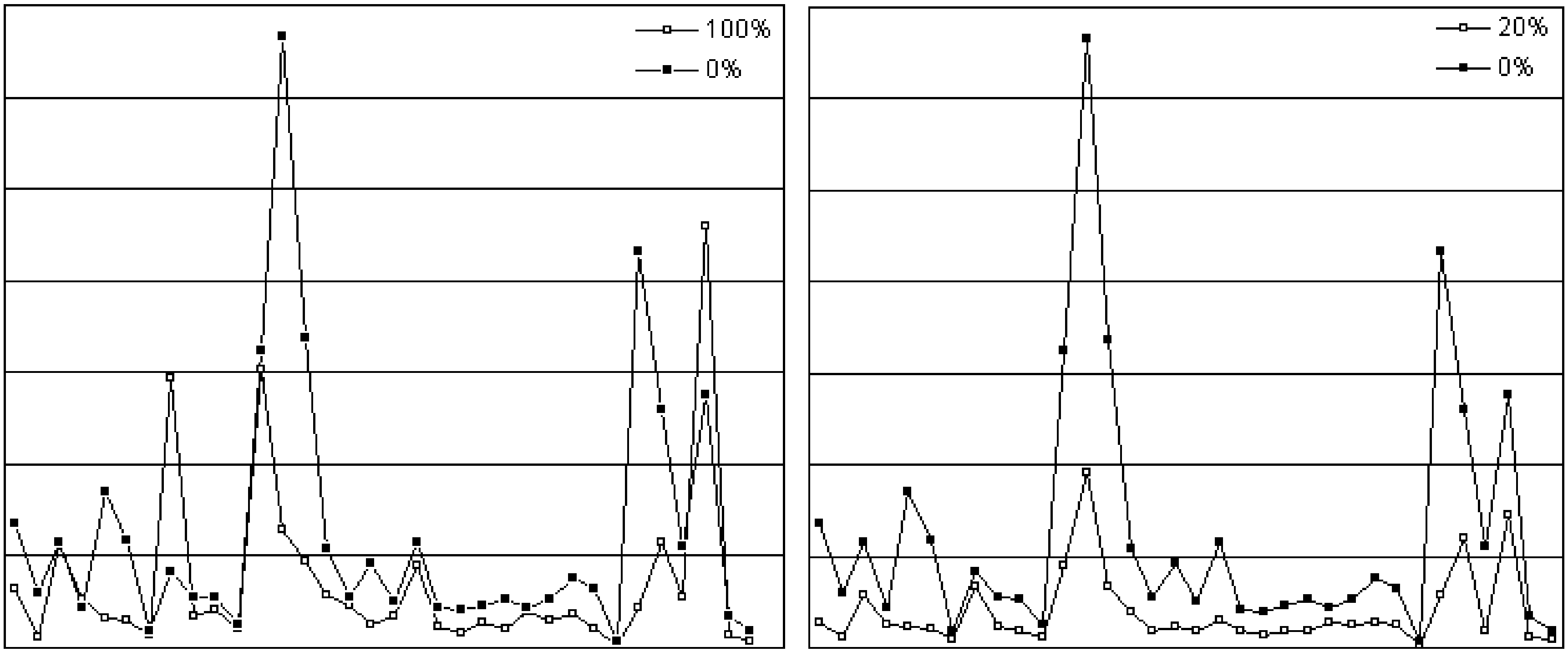


Table 2: Results of using warm starting to solve bicriteria optimization problems.

Conclusion

- We have briefly introduced the issues surrounding warm starting and sensitivity analysis for integer programming.
- An examination of early literature has yielded some ideas that can be useful in today's computational environment.
- We presented a new version of the SYMPHONY solver supporting warm starting and sensitivity analysis for MILPs.
- We have also demonstrated SYMPHONY's multicriteria optimization capabilities.
- This work has only scratched the surface of what can be done.
- In future work, we plan on refining SYMPHONY's warm start and sensitivity analysis capabilities.
- We will also provide more extensive computational results.