A Value Function Approach to Two-Stage Stochastic Programs With Mixed Integer Recourse

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Outline

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1. Introduction
2. Value Function
3. Algorithms
4. Conclusions

Two-Stage Stochastic Program with Recourse

$$\begin{aligned} \min f(x) = \min\ & c^\top x + \mathbb{E}_{w \in \Omega}[Q(x,w)] \\ \text{s.t. } & x \in X \end{aligned} \tag{SP}$$

$$\begin{aligned} Q(x,w) = \min\ & q(w)^\top y \\ \text{s.t. } & W(w)y = h(w) - T(w)x \\ & y \in Y \end{aligned} \tag{RP}$$

where X and Y are the feasible regions of the first and second stages and may be discrete sets. In this talk, we assume

- w follows a discrete distribution with a finite support, and
- W, q, and T are fixed.

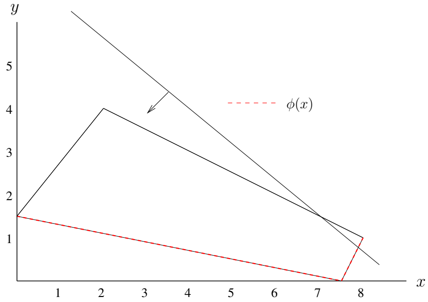
Overview

- We present an algorithmic framework for solving two-stage stochastic integer programs.
- Solution of the problem requires analysis of how the solution to the second-stage problem varies as a function of the first stage solution.
- The first part of this talk will focus on properties of the *value function* of a mixed integer linear program.
- In the second part, we describe a Benders-like algorithm based on approximation of the value function.
- We aim to develop an algorithm that can be implemented in practice.

$$
\begin{aligned}
z_{\mathrm{LP}} &= \min_{(x,y)\in\mathbb{R}^n} \left\{ c'x + c''y \mid A'x + A''y \geq b \right\} \\
&= \min_{x\in\mathbb{R}^{n'}} \left\{ c'x + \phi(b - A'x) \right\},
\end{aligned}
$$

where

$$
\begin{aligned}
\phi(d) = \quad & \min c''y \\
& \text{s.t. } A''y \geq d \\
& y \in \mathbb{R}^{n''}
\end{aligned}
$$

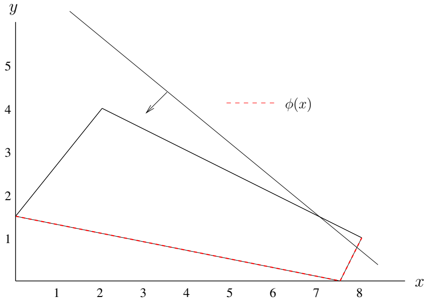


Basic Strategy:

- The function ϕ is the *value function* of a linear program.
- The value function is piecewise linear and convex.
- We iteratively generate a lower approximation by sampling the domain.

Benders' Principle (Linear Programming)

$$
\begin{aligned}
z_{LP} \quad = \quad \min \quad & x + y \\
\text{s.t.} \quad & 25x - 20y \geq -30 \\
& -x - 2y \geq -10 \\
& -2x + y \geq -15 \\
& 2x + 10y \geq 15 \\
& x, y \in \mathbb{R}
\end{aligned}
$$



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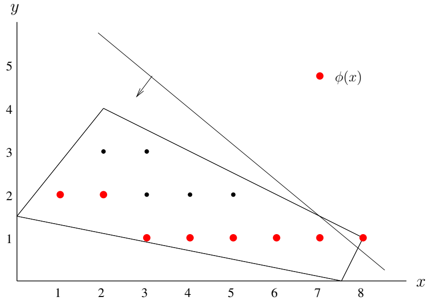
$$z_{\text{IP}} = \min_{(x,y)\in\mathbb{Z}^n} \{c'x + c''y \mid A'x + A''y \geq b\}$$
$$= \min_{x\in\mathbb{R}^{n'}} \{c'x + \phi(b - A'x)\},$$

where

$$\phi(d) = \min c''y$$
$$\text{s.t. } A''y \geq d$$
$$y \in \mathbb{Z}^{n''}$$



Basic Strategy:

- Here, ϕ is the value function of an *integer program*.
- In the general case, the function ϕ is piecewise linear but not convex.
- Here, we also iteratively generate a lower approximation by evaluating ϕ.

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\begin{aligned}
z_{IP} \quad = \quad \min \quad & x + y \\
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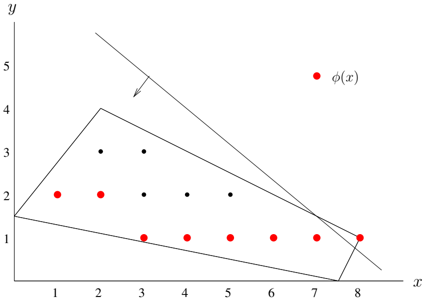
$$



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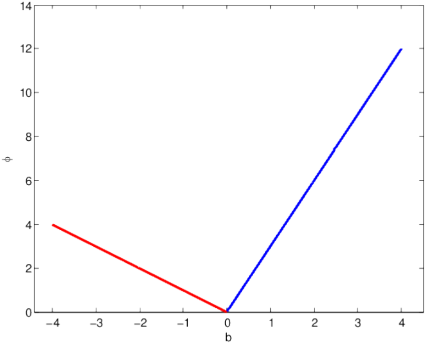
- In the general case, the function ϕ is piecewise linear but not convex.
- Here, we also iteratively generate a lower approximation by evaluating ϕ.

Outline

LP Value Function

Example

$$\begin{aligned} \phi_{LP}(b) = \min\ & 6x_1 + 7x_2 + 5x_3 \\ \text{s.t. }\ & 2x_1 - 7x_2 + x_3 = b \\ & x_1, x_2, x_3 \in \mathbb{R}_+ \end{aligned} \qquad \text{(Ex.LP)}$$



LP Value Function Structure

$$\phi_{LP}(b) = \min \; c^\top x \quad \text{s.t. } Ax = b, \quad x \in \mathbb{R}^n_+ \qquad \text{(LP)}$$

- Assume the dual of (LP) is feasible.
- The epigraph of ϕ_{LP} is a convex cone, call it $\mathcal{L}$:

$$\mathcal{L} := cone\{(A_1, c_1), (A_2, c_2), \ldots, (A_n, c_n), (0, 1)\}$$

- Let $u_1, \ldots, u_k$ be extreme points of the feasible region of the dual of (LP) and $d_1, \ldots, d_p$ be its extreme directions. Then

$$\mathcal{L} := \{(b, z) : z \geq u_i^\top b, i = 1, \ldots, k, d_j^\top b \leq 0, j = 1, \ldots, p\}.$$

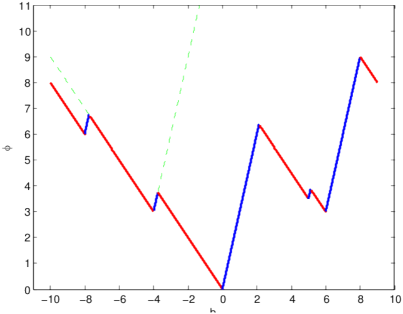
- Note that the value function has an underlying discrete structure.

Example: MILP Value Function

MILP value function is non-convex and discontinuous piecewise polyhedral.

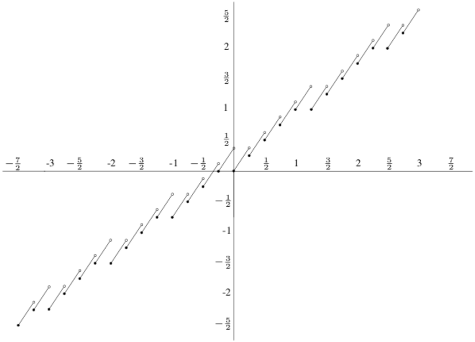
Example

$$\begin{aligned} \phi(b) = \min\ & 3x_1 + \frac{7}{2}x_2 + 3x_3 + 6x_4 + 7x_5 + 5x_6 \\ \text{s.t. } & 6x_1 + 5x_2 - 4x_3 + 2x_4 - 7x_5 + x_6 = b \\ & x_1, x_2, x_3 \in \mathbb{Z}_+,\ x_4, x_5, x_6 \in \mathbb{R}_+ \end{aligned} \qquad \text{(Ex1.MILP)}$$



Example

$$\begin{aligned} \phi(b) = \min\ & x_1 - \frac{3}{4}x_2 + \frac{3}{4}x_3 \\ \text{s.t. } & \frac{5}{4}x_1 - x_2 + \frac{1}{2}x_3 = b \\ & x_1, x_2 \in \mathbb{Z}_+,\ x_3 \in \mathbb{R}_+ \end{aligned} \tag{Ex2.MILP}$$



Discrete Structure of the Value Function

- Our goal is to develop a finite procedure for constructing the value function.
- To accomplish this, we want to exploit its discrete structure.
- This structure arises as a combination of the discrete structures of two underlying value functions.
  - The continuous restriction.
  - The integer restriction.

Continuous and Integer Restriction of an MILP

Consider

$$\begin{aligned} \phi(b) = \min\ & c_I^\top x_I + c_C^\top x_C \\ \text{s.t. }\ & A_I x_I + A_C x_C = b, \\ & x \in \mathbb{Z}_+^r \times \mathbb{R}_+^{n-r} \end{aligned} \tag{MILP}$$

Define the *continuous restriction* of (**??**) as

$$\begin{aligned} \phi_C(b) = \min\ & c_C^\top x_C \\ \text{s.t. }\ & A_C x_C = b, \\ & x \in \mathbb{R}_+^{n-r} \end{aligned} \tag{CR}$$

and its *integer restriction* as

$$\begin{aligned} \phi_I(b) = \min\ & c_I^\top x_I \\ \text{s.t. }\ & A_I x_I = b \\ & x_I \in \mathbb{Z}_+^r \end{aligned} \tag{IR}$$

Discrete Representation of the Value Function

For $b \in \mathbb{R}^m$, we have that

$$\begin{aligned} \phi(b) = \min\ & c_I x_I + \phi_C(b - A_I x_I) \\ \text{s.t. } & x_I \in \mathbb{Z}^r_+ \end{aligned} \quad (1)$$

- From this we see that the value function is comprised of the minimum of a set of shifted copies of ϕ_C.
- The set of shifts, along with ϕ_C describe the value function exactly.
- For $\hat{x}_I \in \mathbb{Z}^r_+$, let

$$\phi_C(b, \hat{x}_I) = \phi_C(b - A_I \hat{x}_I) + c_I x_I \ \ \forall b \in \mathbb{R}^m.$$

- Then we have that $\phi(b) = \min_{x_I \in \mathbb{Z}^r_+} \phi_C(b, \hat{x}_I)$.

Properties of MILP Value Function

We define

- $\mathcal{S}_D = \{\nu : A_C^\top \nu \leq c_C\}$.
- $\mathcal{E} = \{E \in \mathbb{R}^n : E$ is the index set of a dual feasible basis of (CR)$\}$.
- For $E \in \mathcal{E}$, $\nu_E^\top = c_E^\top A_E^{-1}$ (extreme points of $\mathcal{S}_D$).

Proposition 2.1

Consider $\mathcal{N} \subseteq B$ over which ϕ is differentiable. Then, there exist an integral part of the solution $x_I^* \in \mathbb{Z}^r$ and $E \in \mathcal{E}$ such that $\phi(b) = c_I^\top x_I^* + \nu_E^\top (b - A_I x_I^*)$ for all $b \in \mathcal{N}$.

Proposition 2.2

The gradient of ϕ on a neighborhood of a differentiable point is a unique optimal dual feasible solution to (CR).

Properties of MILP Value Function

We now attempt to characterize the points at which the shifts of the LP value function occur.

Definition 2.1

A point $\hat{b}$ is called a point of strict local convexity of a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ if for some $\epsilon > 0$ and $g \in \partial f(\hat{b})$

$$f(b) > f(\hat{b}) + g^\top (b - \hat{b}) \text{ for all } b \in \mathcal{N}_\epsilon(\hat{b}),\ b \neq \hat{b}$$

- For $epi(\phi_C)$, the single extreme point (if there is one), is the only point of strict local convexity.
- For ϕ, these occur wherever one of the LP value function cones is "anchored."
- Let $\mathcal{P}$ be this set of points of strict local convexity of ϕ.

Consider the following scaled MILP, where $M \in \mathbb{Z}_+$ such that $MA_E^{-1}A_I^j$ is a vector of integers for all $E \in \mathcal{E}, j = 1 \ldots r$.

$$\begin{aligned} \min\ & c_I^\top x_I + \frac{1}{M} c_C^\top x_C \\ \text{s.t. } & A_I x_I + \frac{1}{M} A_C x_C = b \\ & (x_I, x_C) \in \mathbb{Z}_+^r \times \mathbb{R}_+^{n-r} \end{aligned} \tag{2}$$

Jeroslow Formula [Blair, 1995]

The value function (**??**) can be written as

$$\phi(b) = \min_{E \in \mathcal{E}} G(\lfloor b \rfloor_E) + \nu_E^\top (b - \lfloor b \rfloor_E),$$

where $\lfloor b \rfloor_E = \frac{1}{M} A_E \left\lfloor M A_E^{-1} b \right\rfloor$ and G is the value function of a related pure integer program.

Properties of MILP Value Function

Let $\mathcal{T}_E = \{b \in B : \lfloor b \rfloor_E = b\}$, $\mathcal{T} = \bigcap_{E \in \mathcal{E}} \mathcal{T}_E$. From [Blair, 1995], we know for $b \in \mathcal{T}$,

$$\begin{aligned} \phi(b) = \min\ & c_I^\top x_I + \frac{1}{M} c_C^\top x_C \\ \text{s.t. } & A_I x_I + \frac{1}{M} A_C x_C = b \\ & (x_I, x_C) \in \mathbb{Z}_+^r \times \mathbb{Z}_+^{n-r} \end{aligned} \tag{3}$$

Theorem 2.1

$\mathcal{P} \subseteq \mathcal{T}$

Corollary 1

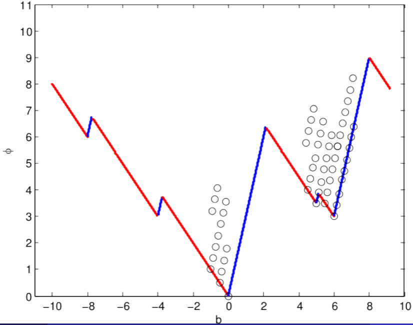
For $b \in \mathcal{P}$, $\phi(b)$ is the optimal value of a pure integer program.

Scaled IP from Jeroslow

For the example:

$$\phi(b) = \min\ 3x_1 + \frac{7}{2}x_2 + 3x_3 + \frac{3}{7}x_4 + \frac{1}{2}x_5$$
$$\text{s.t. } 6x_1 + 5x_2 - 4x_3 + \frac{1}{7}x_4 - \frac{1}{2}x_5 = b \qquad \text{(Ex.Scaled)}$$
$$x_1, x_2, x_3, x_4, x_5 \in \mathbb{Z}_+$$

The solutions to the above problem for $\{[-1, 0] \cup [4.4, 7.1]\}$



Properties of MILP Value Function

Theorem 2.2

For $b \in \mathcal{P}$, *there exists* $x_I \in \mathbb{Z}^r_+$ *such that* $A_I x_I = b$.

Theorem 2.3

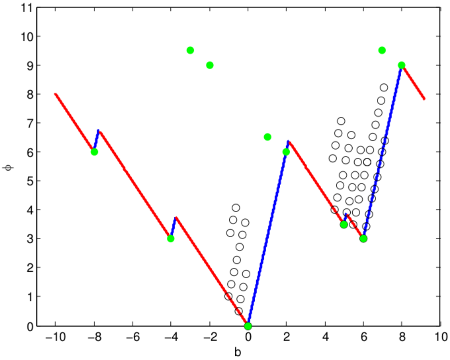
For $b \in \mathcal{P}$, *we have*

$$\begin{aligned} G(b) = \phi(b) = \phi_I(b) = \min\ & c_I^\top x_I \\ \text{s.t. } & A_I x_I = b \\ & x_I \in \mathbb{Z}^r_+ \end{aligned} \tag{IR}$$

Corollary 2

$\mathcal{P} \subseteq \{A_I x_I : x_I \in \mathbb{Z}^r_+\}$.

Integer Restriction of an MILP



Bottom Line: The value function of a MILP has discrete structure arising from the integer restriction and can be constructed without solving the original MILP.

Outline

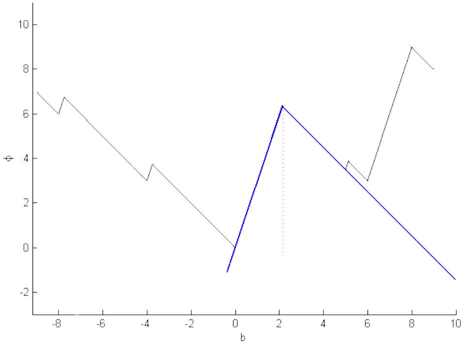
Related Algorithms

The algorithmic framework we utilize builds on a number of previous works.

- Modification to the L-shaped framework [Laporte and Louveaux, 1993, Carøe and Tind, 1998, Sen and Higle, 2005]
  - Linear cuts in first stage for binary first stage
  - Optimality cuts from B&B and cutting plane, applied to pure integer second stage
  - Disjunctive programming approaches and cuts in the second stage
- Value function approaches: Pure integer case [Ahmed et al., 2004, Kong et al., 2006]
- Scenario decomposition [Carøe and Schultz, 1998]
- Enumeration/Gröbner basis reduction [Schultz et al., 1998]

Two-Stage Problem and Value Function

- Benders' original method does not apply directly when Y contains integer variables.
- To generalize it, we need lower bounding functions to approximate the MILP value function.



Dual Functions

A dual function $\varphi : \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$ is

$$\varphi(b) \leq \phi(b) \quad \forall b \in \Lambda$$

For a particular instance $\hat{b}$, the dual problem is

$$\phi_D = \max\{\varphi(\hat{b}) : \varphi(b) \leq \phi(b) \quad \forall b \in \mathbb{R}^m, \ \varphi : \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}\}$$

Value Function Reformulation of the Two-Stage Problem

Let

- $\mathcal{B} = \{\beta : \beta = Tx, x \in X\}$
- $S_1(\beta) = \{x \in X : Tx = \beta\}$
- $S_2(\beta) = \{Wy = \beta, \ y \in Y\}$
- $\psi(\beta) = \min\{c^\top x : x \in S_1(\beta)\}$
- $\phi(\beta) = \{q^\top y : y \in S_2(\beta)\}$
- $f(\beta) = \{\psi(\beta) + \min \mathbb{E}_s[\phi(h_s - \beta)] : \beta \in \mathcal{B}\}$

Then our problem is to determine $\min_{\beta \in \mathcal{B}} f(\beta)$.

Assumptions:

- q, T, and W are fixed.
- The dual of the LP relaxation of the recourse problem is feasible, i.e.,

$$\{\nu \in \mathbb{R}^{m_2} : W_I^\top \nu \leq q_I, W_C^\top \nu \leq q_C\} \neq \emptyset$$

- X is non-empty and bounded.

The Algorithm

Step 0. Initialize

a) Set $\beta^1 = Tx^1$ where $x^1 \in \text{argmin}\{c^\top x : x \in X\}$
b) Initialize the dual function lists $\mathcal{F}_1 = \emptyset, \mathcal{F}_s = \emptyset$.
c) Set $k = 1$.

The Algorithm

Step 0. Initialize

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Step 1. Lower bound the problem and check for termination

a) Find optimal dual functions F_1^k and F_s^k for each $s \in 1 \ldots S$ to $\psi(\beta^k)$ and $\phi(h_s - \beta^k)$ respectively.

b) If

$$\max_{f_1 \in \mathcal{F}_1, f_s \in \mathcal{F}_s} \{f_1(\beta^k) + \mathbb{E}_s[f_s(h_s - \beta^k)]\} = F_1^k(\beta^k) + \mathbb{E}_s[F_s^k(h_s - \beta^k)]$$

then stop, $x^* \in \text{argmin}\{c^\top x : x \in X, Tx = \beta^k\}$ is an optimal solution.

Step 2. Update the lower bound

a) Update the dual functions lists: $\mathcal{F}_1 = \mathcal{F}_1 \cup F_1^k$ and let $\mathcal{F}_s = \mathcal{F}_s \cup_{s \in \Omega} F_s^k$.

b) Solve the problem

$$z^k = \min_{\beta \in \mathcal{B}} \max_{f_1 \in \mathcal{F}_1, f_s \in \mathcal{F}_s} \{f_1(\beta) + \mathbb{E}_s[f_s(h_s - \beta)]\}$$

and set its optimal solution to β^{k+1}.

c) Go to Step 1.

MILP Duals from Branch-and-Bound

Let T be set of the terminating nodes of the tree. Then in a terminating node $t \in T$ we solve:

$$\begin{aligned} &\min\ c^\top x \\ &\text{s.t. } Ax = b, \\ &\quad l^t \le x \le u^t, x \ge 0 \end{aligned} \tag{4}$$

The dual at node t:

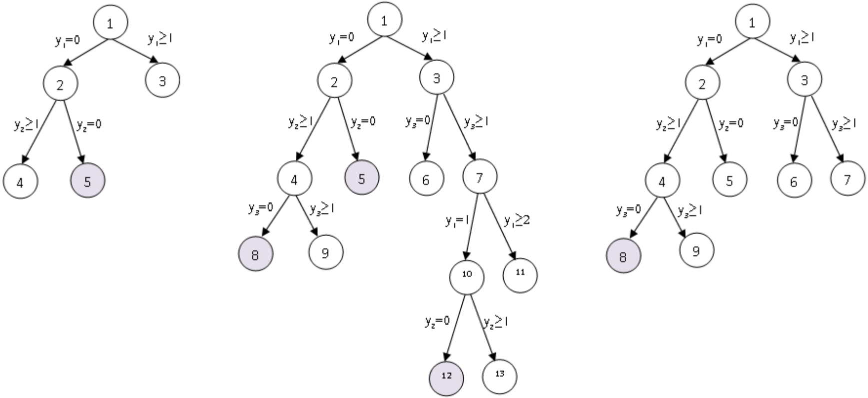
$$\begin{aligned} &\max\ \{\pi^t b + \underline{\pi}^t l^t + \bar{\pi}^t u^t\} \\ &\text{s.t. } \pi^t A + \underline{\pi}^t + \bar{\pi}^t \le c^\top \\ &\quad \underline{\pi} \ge 0, \bar{\pi} \le 0 \end{aligned} \tag{5}$$

We obtain the following strong dual function:

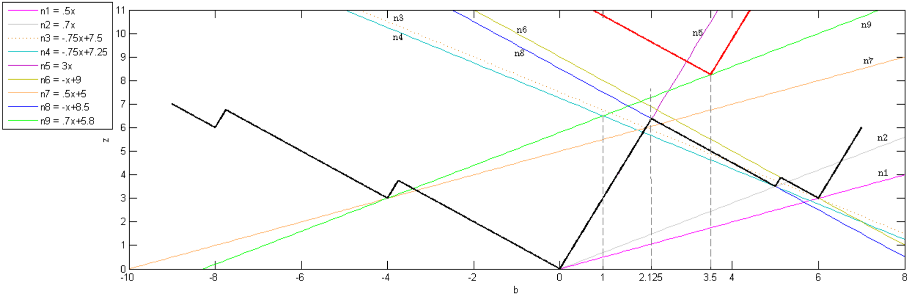
$$\min_{t \in T}\{\pi^t b + \underline{\pi}^t l^t + \bar{\pi}^t u^t\} \tag{6}$$

MILP Duals from Branch-and-Bound

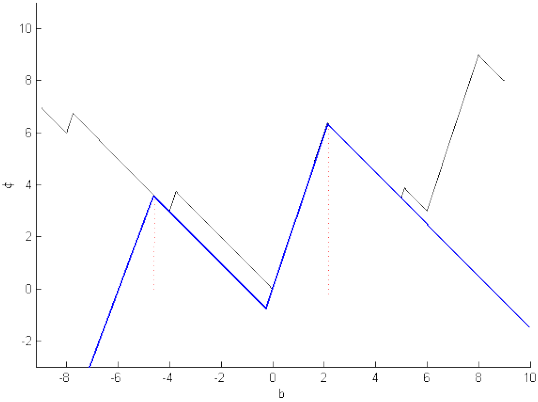
Figure : Dual Functions from B&B for right hand sides 1, 2.125, 3.5



MILP Duals from Branch-and-Bound



MILP Duals from Branch-and-Bound



Master Problem Formulation

Notation:

- $s, r \in \{1, \ldots, S\}$ where S is the number of scenarios
- $p \in \{1, \ldots, k\}$ where k is the iteration number
- $n \in \{1, \ldots, N(s)\}$ where $N(s)$ is the number of terminating nodes in the B&B tree solved for scenario s.
- $\theta_s = \mathcal{F}_s(\beta)$
- $t_{spr} = F_r^p(h(s) - \beta)$
- a_{prn}, ν_{prn} respectively, the dual vector and intercept obtained from node n of the B&B tree solved for scenario r in iteration p.
- p_s probability of scenario s
- $M > 0$ an appropriate large number

Master Problem Formulation

Solving the second stage problem with B&B, in Step 2, the following problem is solved to get β^{k+1} :

$$
\begin{aligned}
f^k = \min\ & c^\top x + \sum_{s=1}^{S} p_s \theta_s \\
\text{s.t. } & Tx = \beta \\
& \theta_s \geq t_{spr} && \forall s, p, r \\
& t_{spr} \leq a_{prn} + \nu_{prn}^\top (h(s) - \beta) && \forall s, r, p, n \quad \text{(master)} \\
& t_{spr} \geq a_{prn} + \nu_{prn}^\top (h(s) - \beta) - M u_{sprn} && \forall s, p, r, n \\
& \sum_{n=1}^{N} u_{sprn} = N(s) - 1 && \forall s, p, r \\
& x \in X, u_{sprn} \in \mathbb{B} && \forall s, p, r, n
\end{aligned}
$$

Example

Consider

$$\begin{aligned} \min f(x) = \min \ & -3x_1 - 4x_2 + \sum_{s=1}^{2} 0.5Q(x,s) \\ \text{s.t. } & x_1 + x_2 \leq 5 \\ & x \in \mathbb{Z}_+ \end{aligned} \tag{7}$$

where

$$\begin{aligned} Q(x,s) = \min \ & 3x_1 + \frac{7}{2}x_2 + 3x_3 + 6x_4 + 7x_5 \\ \text{s.t. } & 6x_1 + 5x_2 - 4x_3 + 2x_4 - 7x_5 = h(s) - 2x_1 - \frac{1}{2}x_2 \\ & x_1, x_2, x_3 \in \mathbb{Z}_+,\ x_4, x_5 \in \mathbb{R}_+ \end{aligned} \tag{8}$$

with $h(s) \in \{-4, 10\}$.

Iteration 1

Step 0

- $\mathcal{F} = \emptyset$
- $k = 1.$
- Solve

$$\min f(x) = \min \; -3x_1 - 4x_2$$
$$\text{s.t. } x_1 + x_2 \leq 5$$
$$x_1, x_2 \in \mathbb{Z}_+$$

$f^0 = 20, x_1^* = 0, x_2^* = 5, \; \beta^1 = \frac{5}{2}$

Example

Step 1

- Solve the second stage problem for each scenario. That is, with $h(1) - \beta^1 = -6.5$ and $h(2) - \beta^1 = 7.5$.
- The respective dual functions are

$$F^1_{s=1}(\beta) = \min\{-\beta - 1, 0.5\beta + 10\} \text{ and}$$
$$F^1_{s=2}(\beta) = \min\{3\beta - 15, -0.75\beta + 14.5\}.$$

Then, $\mathcal{F}(\beta) = \max\{F^1_{s=1}, F^1_{s=2}\}$.

Step 2

- Solve the master problem

$$\begin{aligned} f^1 = \min \ & -3x_1 - 4x_2 + 0.5(\mathcal{F}_s(-4-\beta) + \mathcal{F}_s(10-\beta)) \\ \text{s.t. } & x_1 + x_2 \leq 5 \\ & 2x_1 + \frac{1}{2}x_2 = \beta \\ & x_1, x_2 \in \mathbb{Z}_+ \end{aligned}$$

Example

The MILP reformulation of the master problem is

$$
\begin{aligned}
\min\ & -3x_1 - 4x_2 + 0.5\theta_1 + 0.5\theta_2 \\
\text{s.t. } & \theta_s \geq t_{s1r} \quad s, r \in \{1,2\} \\
& t_{11r} \leq a_{1rn} + \nu_{1rn}(-4-\beta) && r, n \in \{1,2\} \\
& t_{11r} \geq a_{1rn} + \nu_{1rn}(-4-\beta) - Mu_{11rn} && r, n \in \{1,2\} \\
& t_{21r} \leq a_{1rn} + \nu_{1rn}(10-\beta) && r, n \in \{1,2\} \\
& t_{21r} \geq a_{1rn} + \nu_{1rn}(10-\beta) - Mu_{21rn} && r, n \in \{1,2\} \\
& u_{11r1} + u_{11r2} = 1 && r \in \{1,2\} \\
& u_{21r1} + u_{21r2} = 1 && r \in \{1,2\} \\
& 2x_1 + \frac{1}{2}x_2 = \beta \\
& x_1, x_2 \in \mathbb{Z}_+, u_{s1rn} \in \mathbb{B} && s, r, n \in \{1,2\}
\end{aligned}
$$

Example

For example, for $t_{111} = \min\{-(-4-\beta)-1, 0.5(-4-\beta)+10\}$ we add:

$$
\begin{aligned}
t_{111} &\leq -(-4-\beta)-1 \\
t_{111} &\geq -(-4-\beta)-1-Mu_{1111} \\
t_{111} &\leq 0.5(-4-\beta)+10 \\
t_{111} &\geq 0.5(-4-\beta)+10-Mu_{1112} \\
u_{1111} &+ u_{1112} = 1
\end{aligned}
$$

The solution to the master problem is $f^1 = -16.75$ with $\beta^1 = 7$.

Example

Iteration 2

Step 1

- Solve the second stage problem with right hand sides: -11 and 3.
- The respective dual functions are:
  $F^2_{s=1}(\beta) = \min\{-\beta - 2, 0.5\beta + 15\}$ and
  $F^2_{s=2}(\beta) = \min\{3\beta, -\beta + 8.5, 0.7\beta + 5.8\}$.
- Since $\mathcal{F}(-11) + \mathcal{F}(3) < F^2_{s=1}(-11) + F^2_{s=2}(3)$, we continue:
- Update $\mathcal{F}(\beta) = \max\{F^1_{s=1}, F^1_{s=2}, F^2_{s=1}, F^2_{s=2}\}$.

Step 2

- Solve the updated master problem. We obtain $f^2 = -14.5$ with $\beta^2 = 4$.

Example

Iteration 3

Step 1

- Solve the second stage problem with right hand sides: -8 and 6.
- The respective dual functions are:
  $F^3_{s=1}(\beta) = -0.75\beta$ and $F^3_{s=2}(\beta) = 0.5\beta$.
- $\mathcal{F}(-8) + \mathcal{F}(6) = F^3_{s=1}(-8) + F^3_{s=2}(6) = 9$, the approximation is exact and the optimal solution to the problem is $f^3 = -14.5$ and $\beta^3 = 4$.

Implementation Challenges

- To make the algorithm practical, several issues need to be addressed.
- The master problem includes a piecewise linear function which grows in dimensions.
- In each iteration, for a scenario s, $S \times N(s)$ binary variables are added, where $N(s)$ is the number of new pieces of the function.
- Therefore, some "cut pool management" techniques need to be used to keep the size of the master problem manageable.
- This requires using an appropriate database.
- The examined right hand sides and their corresponding dual functions also need to be stored in an efficient manner.

Implementation for a Single Constrained Recourse

- For storing dual functions, a "nested hash table" is used.
- The first level of hashing consists of pairs (key = r.h.s, value = collection of linear pieces of dual function). the number of terminating nodes of the corresponding B&B tree determines the number of dual pieces.
- The value itself consists of pairs (key = optimal dual vector of the LP solved in a terminating node, value = intercept).
- Therefore, look ups are cheap.
- A linear piece of B&B tree node is only added if it is stronger than the previously found ones.

Right Hand Side Modification

- Can we do better than blindly solving the master problem to get candidate right hand sides β^k ?
- In theory, we get more pieces of the value function by checking for $h(s) - \beta^k \in \mathcal{P}$.
- Sensitivity analysis on terminating nodes of the B&B tree tells us about strong pieces for right hand sides around $h(s) - \beta^k$.
- This allows us to build pieces of the value function locally around the examined right hand side.

Outline

Conclusions

- We aim to develop a practical algorithm for the two-stage problem with general mixed integer recourse.
- The algorithm uses the Benders' framework.
- The master problem suggests right hand sides to the recourse problem.
- We use piece-wise linear dual functions obtained from B&B tree to approximate the value function of the recourse problem.
- For implementation, we are looking for ways to keep the size of the approximation small.
- Cut pool management is needed to restrict the function description to the local area of interest and discard irrelevant parts.
- We are looking for ways to tweak the right hand sides to get stronger lower bounds.

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