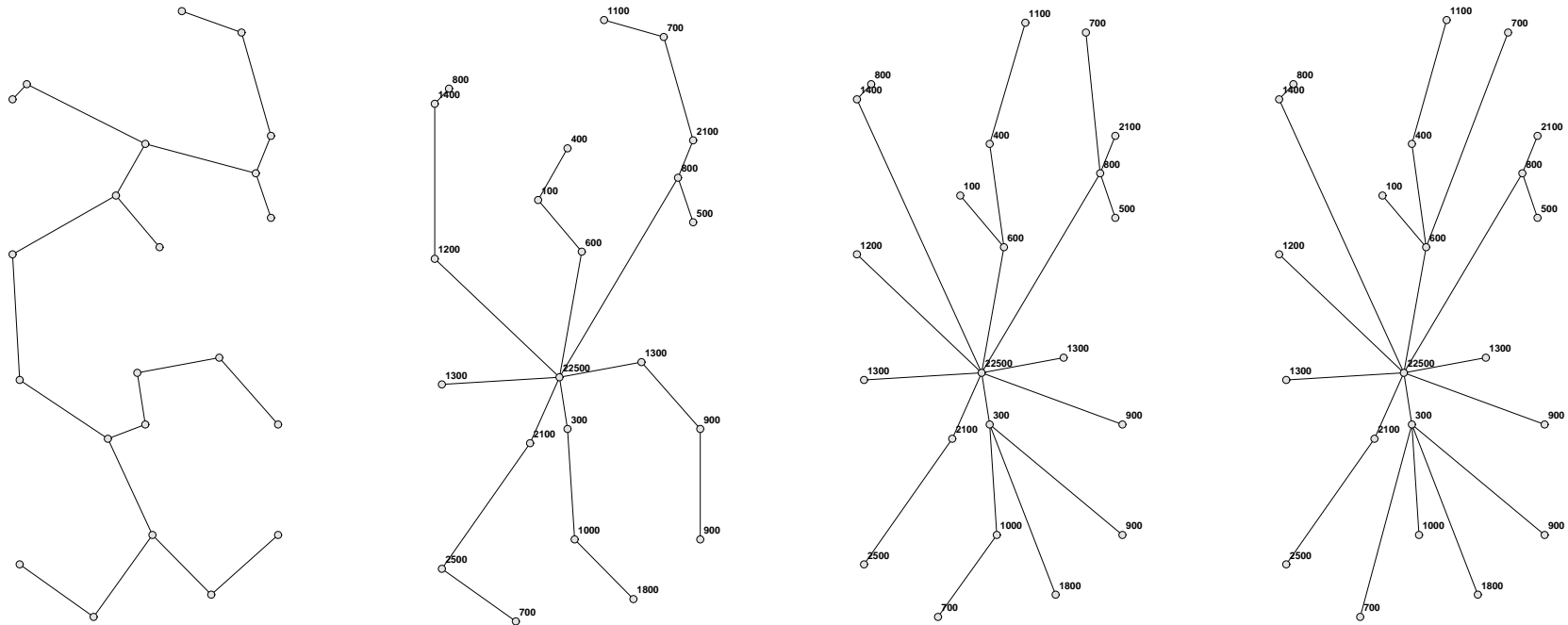


Biobjective Integer Programming

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Outline of Talk

- Preliminaries
- The WCN Algorithm
- Variants
 - Interactive algorithm
 - Approximation algorithm
- Enhancements
 - Avoiding weakly dominated solutions
 - Improving efficiency
- Examples and Applications
 - Parametric Programming
 - Network Routing
- Computational Results

Biobjective Mixed-integer Programs

A *biobjective* or *bicriterion mixed-integer program* (BMIP) is an optimization problem of the form

$$\begin{array}{ll} \text{vmax} & f(x) \\ \text{subject to} & x \in X, \end{array}$$

where

- $f : \mathbb{R}^n \rightarrow \mathbb{R}^2$ is the *(bicriteria) objective function*, and
- $X \subset \mathbb{Z}^p \times \mathbb{R}^{n-p}$ is the *feasible region*, usually defined to be

$$\{x \in \mathbb{Z}^p \times \mathbb{R}^{n-p} \mid g_i(x) \leq 0, i = 1, \dots, m\}$$

for functions $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1, \dots, m$.

The *vmax* operator indicates that the goal is to generate the set of *efficient solutions* (defined next).

Some Definitions

- We define the set of *outcomes* to be $Y = f(X) \subset \mathbb{R}^2$.
- In outcome space, BMIP can be restated as

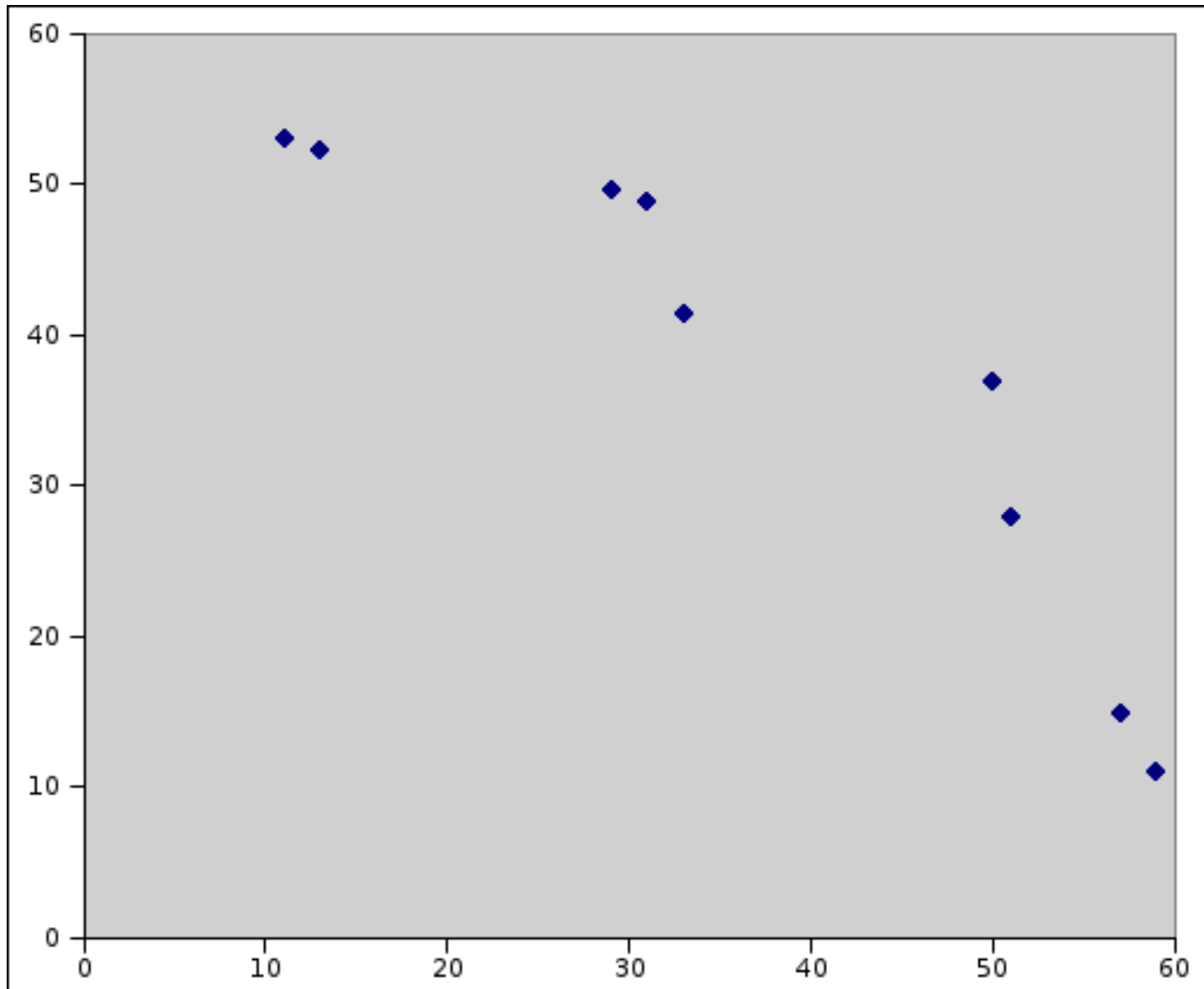
$$\begin{array}{ll} \text{vmax} & y \\ \text{subject to} & y \in f(X), \end{array}$$

- For convenience, we will work primarily in outcome space.
- $x^1 \in X$ *dominates* $x^2 \in X$ if $f_i(x_1) \leq f_i(x_2)$ for $i = 1, 2$ and at least one inequality is strict.
- If both inequalities are strict the dominance is *strong* (otherwise *weak*).
- Any $x \in X$ not dominated by another member of X is said to be *efficient*.
- If $x \in X$ is efficient, then $y = f(x)$ is a *Pareto outcome*.
- Our goal is to generate the set of *all Pareto outcomes*.

More Definitions

- We will denote the set of efficient solutions by X_E .
- The set of Pareto outcomes is then $Y_E = f(X_E)$.
- We assume that $|Y_E|$ is finite.
- If $x \in X_E$ strongly dominates all members of $X \setminus X_E$, then x is said to be *strongly efficient*.
- Likewise, if $x \in X_E$ is strongly efficient, then $y = f(x)$ is *strongly Pareto*.
- If all members of Y_E are strongly Pareto, then Y_E is said to be *uniformly dominant*.
- The assumption of uniform dominance simplifies computation substantially, but is not satisfied in most practical settings.

Illustrating Pareto Outcomes



Algorithms for Generating Pareto Outcomes

- A number of algorithms for generating Pareto outcomes have been proposed.
- These can be **categorized** in several ways:
 - By **output**: complete enumeration, partial enumeration, or heuristic enumeration of Y_E .
 - By **user interaction**: Interactive or non-interactive.
 - By **methodology**: branch and bound, dynamic programming, implicit enumeration, weighted sums, weighted norms, probing.
- We present an algorithm
 - that can either partially or completely enumerate the Pareto set,
 - has both interactive and non-interactive variants,
 - is based on a modified branch and bound algorithm.

Probing Algorithms

- We will focus on *probing algorithms* that *scalarize* the objective, i.e., replace it with a single criterion.
- Such algorithms reduce solution of a BMIP to a series of MIPs.
- The main factor in the running time is then the number of *probes*.
- The most obvious scalarization is the *weighted sum objective*.
- We replace the original objective with

$$\max_{y \in f(X)} \beta y_1 + (1 - \beta) y_2$$

to obtain a parameterized family of MIPs.

Supported Outcomes

- Optimal solutions to weighted sum MIPs are extreme points of $\text{conv}(Y_E)$.
- Such outcomes are called *supported outcomes*.
- The set of all supported outcomes can easily be generated by solving a sequence of MIPs.
- Every supported outcome is Pareto, but the converse is not true.
- This makes it difficult as a tool to generate all Pareto outcomes.
- Chalmet (1986) suggested restricting the subproblems so that each Pareto outcome is supported on some subregion.
- Using this technique, it is possible to generate all Pareto outcomes.

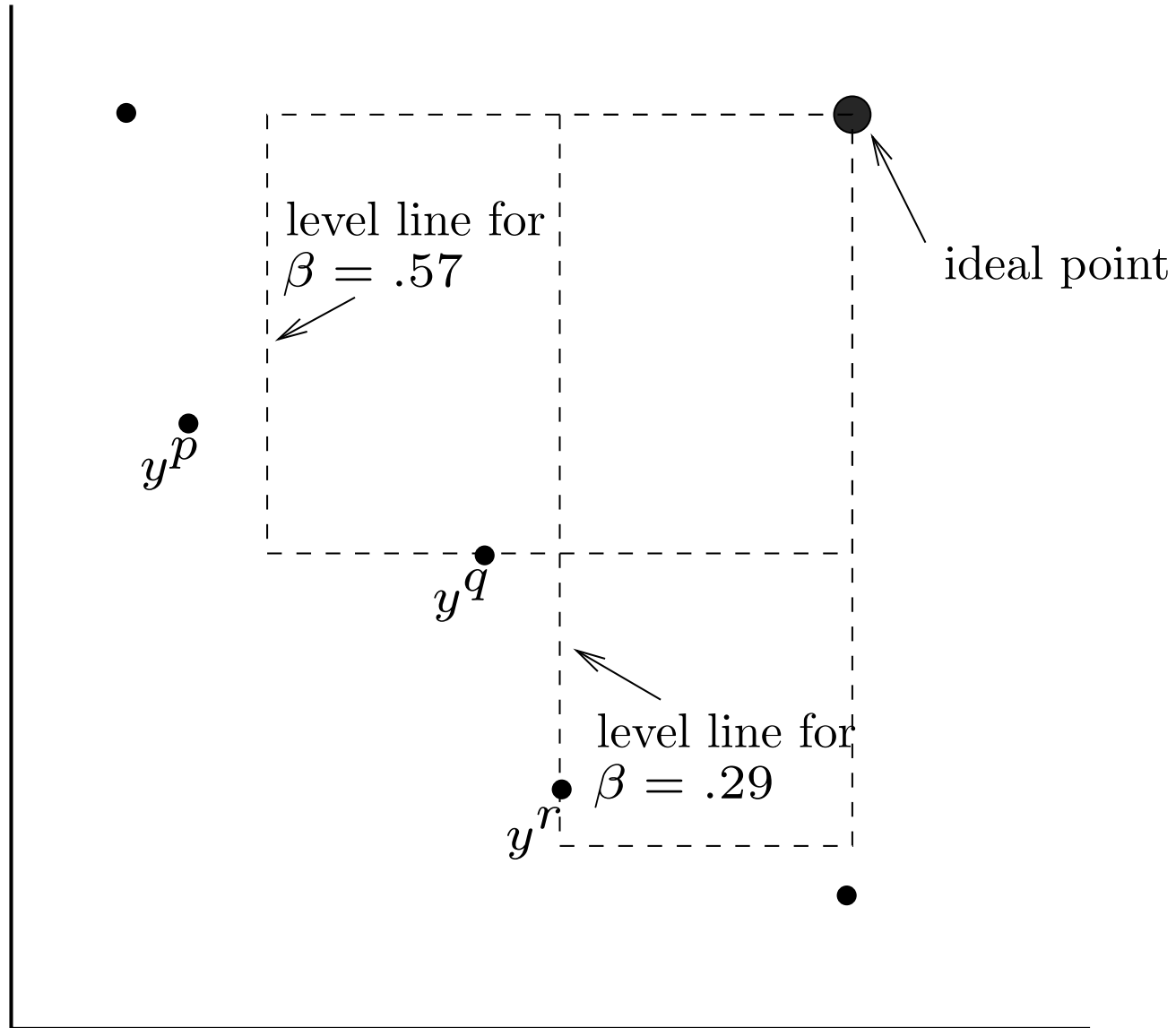
The Weighted Chebyshev Norm

- Another option is to replace the weighted sum objective with a *weighted Chebyshev norm* (WCN) objective.
- The *Chebyshev norm* (l_∞ norm) in \mathbb{R}^2 is defined by $\|y\|_\infty = \max\{|y_1|, |y_2|\}$.
- The *weighted Chebyshev norm* with weight $0 \leq \beta \leq 1$ is defined by $\|y\|_\infty = \max\{\beta|y_1|, (1 - \beta)|y_2|\}$.
- The *ideal point* y^* is (y_1^*, y_2^*) where $y_i^* = \max_{x \in X} (f(x))_i$.
- Methods based on the WCN select outcomes with minimum WCN distance from the ideal point by solving

$$\min_{y \in f(X)} \{ \|y^* - y\|_\infty^\beta \}. \quad (1)$$

- **Bowman** (1976) showed that every Pareto outcome is a solution to (1) for some $0 \leq \beta \leq 1$.
- The converse only holds if Y_E is **uniformly dominant**.

Illustrating the WCN



Ordering the Pareto Outcomes

- **Eswaran** (1989) suggested ordering the Pareto outcomes so that
 - $Y_E = \{y_p \mid 1 \leq p \leq N\}$, and
 - if $p < q$, then $y_1^p < y_1^q$ (and hence $y_2^p > y_2^q$).
- For any Pareto outcome y_p , if we define

$$\beta_p = (y_2^* - y_2^p) / (y_1^* - y_1^p + y_2^* - y_2^p),$$

then y^p is the unique optimal outcome for (1) with $\beta = \beta_p$.

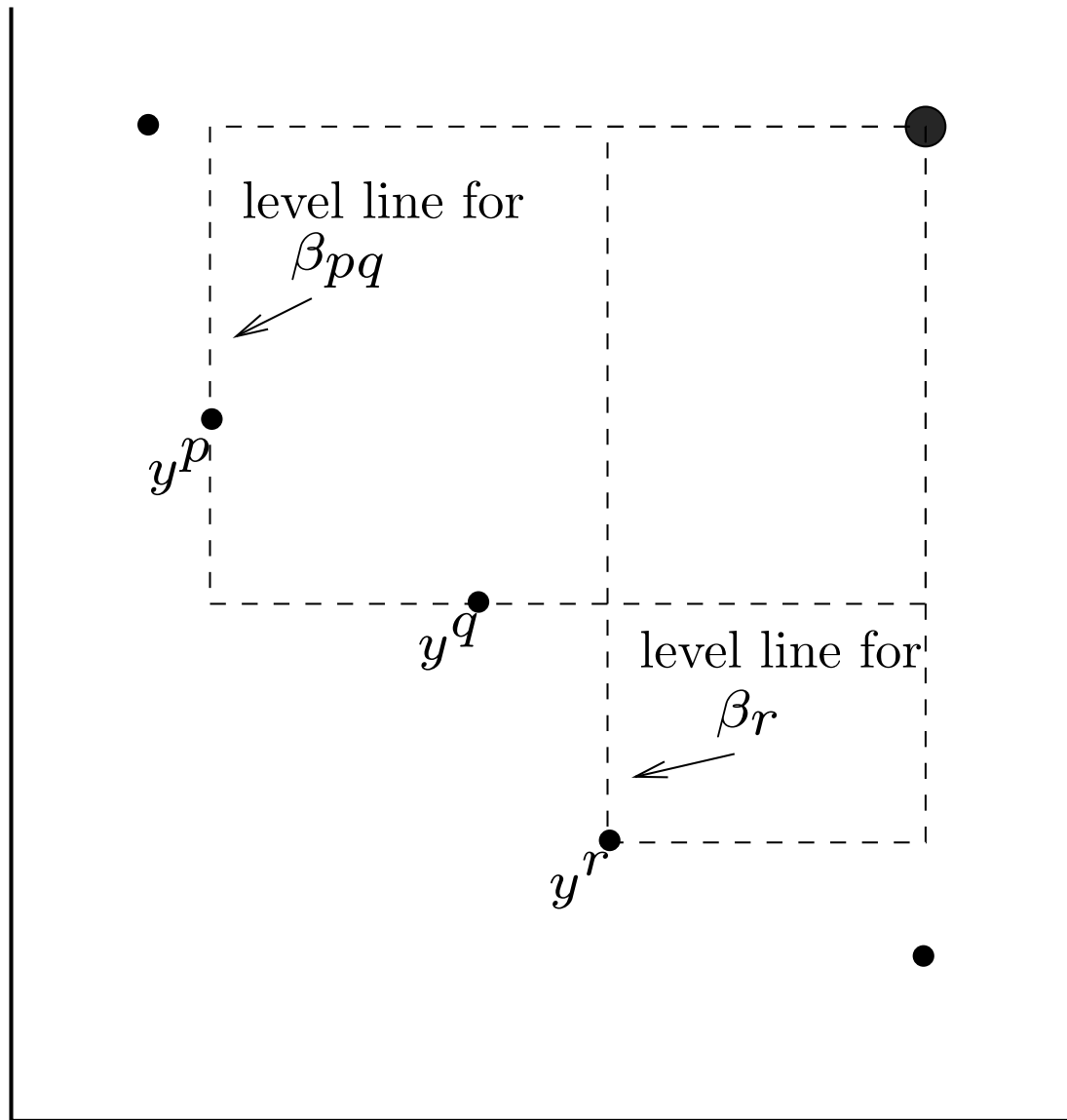
- For any pair of Pareto outcomes y^p and y^q with $p < q$, if we define

$$\beta_{pq} = (y_2^* - y_2^q) / (y_1^* - y_1^p + y_2^* - y_2^q), \quad (2)$$

then y^p and y^q are both optimal outcomes for (1) with $\beta = \beta_{pq}$.

- This provides us with a notion of *adjacency* and *breakpoints*.

Breakpoints Between Pareto Outcomes with the WCN



Algorithms Based on the WCN

- **Solanki** (1991) proposed an algorithm to generate an approximation to the Pareto set using the WCN.
 - The algorithm probes between pairs of known outcomes for new outcomes by restricting the domain ala Chalmet.
 - The search is controlled by an “error measure,” which can be set to zero to get complete enumeration.
 - The number of probes is asymptotically optimal, but the algorithm does not produce breakpoints (directly).
- **Eswaran** (1989) proposed an algorithm based on binary search over the values of β .
 - In the worst case, the number of probes is

$$|Y_E|(1 - \lg(\xi(|Y_E| - 1))),$$

where ξ is a chosen error parameter.

- The algorithm produces only approximate breakpoint information.

The WCN Algorithm

Let $P(\beta)$ be the parameterized subproblem defined by (1) for a given weight β . The WCN algorithm is then:

Initialization Solve $P(1)$ and $P(0)$ to identify optimal outcomes y^1 and y^N , respectively, and the ideal point $y^* = (y_1^1, y_2^N)$. Set $I = \{(y^1, y^N)\}$.

Iteration While $I \neq \emptyset$ do:

1. Remove any (y^p, y^q) from I .
2. Compute β_{pq} as in (2) and solve $P(\beta_{pq})$. If the outcome is y^p or y^q , then y^p and y^q are adjacent in the list (y^1, y^2, \dots, y^N) .
3. Otherwise, a new outcome y^r is generated. Add (y^p, y^r) and (y^r, y^q) to I .

This reduces solution of the original BMIP to solution of a sequence of $2N - 1$ MIPs, but still requires the assumption of uniform dominance.

Solving $P(\beta)$

- Problem (1) is equivalent to

$$\begin{array}{ll} \text{minimize} & z \\ \text{subject to} & z \geq \beta(y_1^* - y_1), \\ & z \geq (1 - \beta)(y_2^* - y_2), \text{ and} \\ & y \in f(X). \end{array} \quad (3)$$

- This is a MIP, which can be solved by standard methods.
- This reformulation can still produce weakly dominated outcomes.

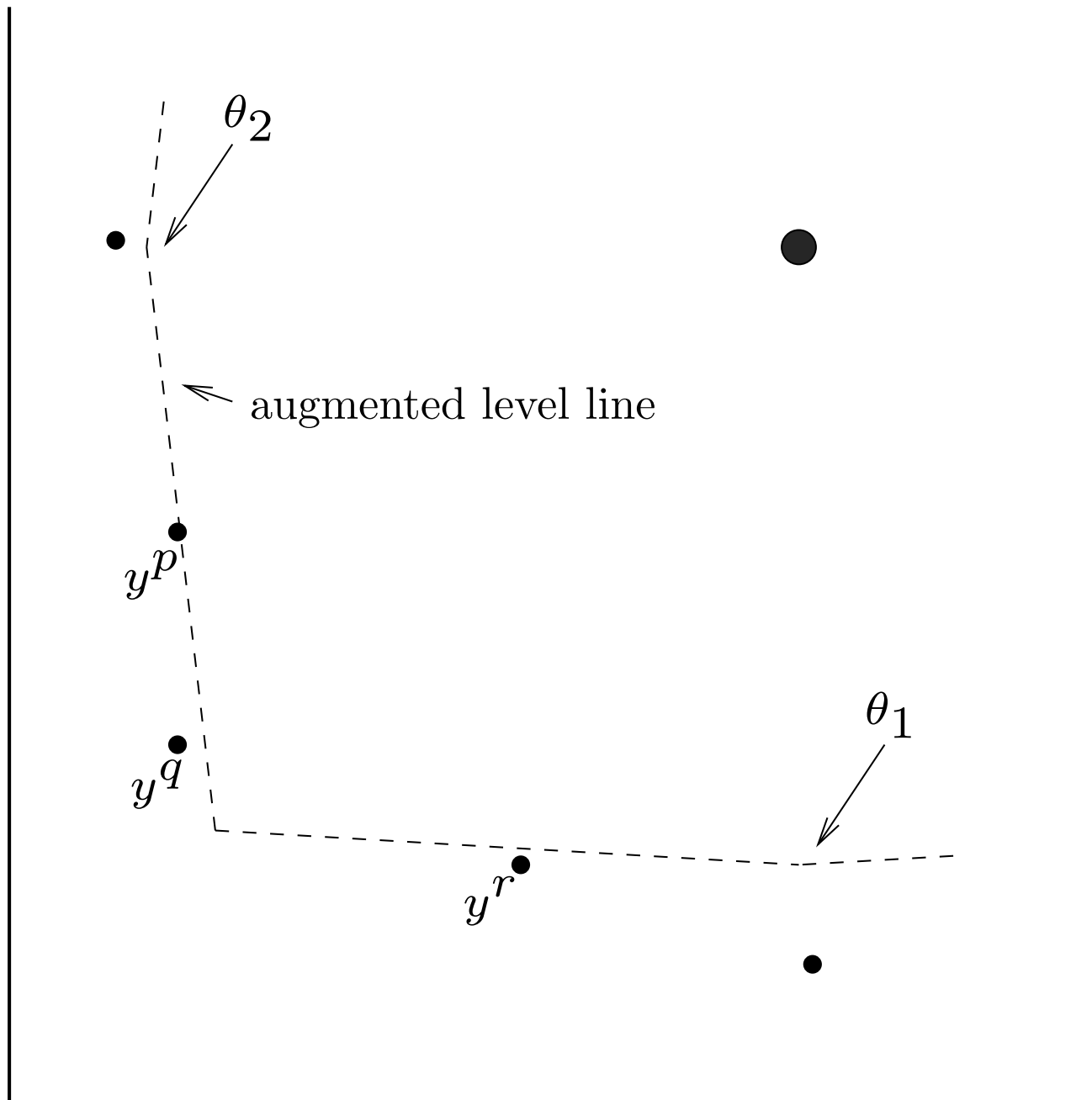
Relaxing the Uniform Dominance Requirement

- Dealing with weakly dominated outcomes is the most challenging aspect of these methods.
- We need a method of preventing $P(\beta)$ from producing weakly dominated outcomes.
- Weakly dominated outcomes are the same WCN distance from the ideal point as the outcomes they are dominated by.
- However, they are farther from the ideal point as measured by the l_p norm for $p < \infty$.
- One solution is to replace the WCN with the *augmented Chebyshev norm* (ACN), defined by

$$\|(y_1, y_2)\|_{\infty}^{\beta, \rho} = \max\{\beta|y_1|, (1 - \beta)|y_2|\} + \rho(|y_1| + |y_2|),$$

where ρ is a small positive number.

Illustrating the ACN



Solving $P(\beta)$ with the ACN

- The problem of determining the outcome closest to the ideal point under this metric is

$$\begin{array}{ll}
 \min & z + \rho(|y_1^* - y_1| + |y_2^* - y_2|) \\
 \text{subject to} & z \geq \beta(y_1^* - y_1) \\
 & z \geq (1 - \beta)(y_2^* - y_2) \\
 & y \in f(X).
 \end{array} \tag{4}$$

- Because $y_k^* - y_k \geq 0$ for all $y \in f(X)$, the objective function can be rewritten as

$$\min z - \rho(y_1 + y_2).$$

- For fixed $\rho > 0$ small enough:
 - all optimal outcomes for problem (4) are Pareto (in particular, they are not weakly dominated), and
 - for a given Pareto outcome y for problem (4), there exists $0 \leq \hat{\beta} \leq 1$ such that y is the unique outcome to problem (4) with $\beta = \hat{\beta}$.
- In practice, choosing a proper value for ρ can be problematic.

Combinatorial Methods for Eliminating Weakly Dominated Solutions

- In the case of *biobjective linear integer programs* (BLIPs), we can employ combinatorial methods.
- Such a strategy involves implicitly enumerating alternative optimal solutions to $P(\beta)$.
- Weakly dominated outcomes are eliminated with cutting planes during the branch and bound procedure.
- Instead of pruning subproblems that yield feasible outcomes, we continue to search for alternative optima that dominate the current incumbent.
- To do so, we determine which of the two constraints

$$\begin{aligned}z &\geq \beta(y_1^* - y_1) \\z &\geq (1 - \beta)(y_2^* - y_2)\end{aligned}$$

from problem (1) is binding at \hat{y} .

Combinatorial Methods for Eliminating Weakly Dominated Solutions (cont'd)

- Let ϵ_1 and ϵ_2 be such that if y_r is a new outcome between y^p and y^q , then $y_i^r \geq \min\{y_i^p, y_i^q\} + \epsilon_i$, for $i = 1, 2$.
- If only the first constraint is binding, then the cut

$$y_1 \geq \hat{y}_1 + \epsilon_1$$

is valid for any outcome that dominates \hat{y} .

- If only the second constraint is binding, then the cut

$$y_2 \geq \hat{y}_2 + \epsilon_2$$

is valid for any outcome that dominates \hat{y} .

- If both constraints are binding, either cut can be imposed.

Hybrid Methods

- In practice, the ACN method is fast, but choosing the proper value of ρ is problematic.
- Combinatorial methods are less susceptible to numerical difficulties, but are slower.
- Combining the two methods improves running times and reduces dependence on the magnitude of ρ .

Other Enhancements to the Algorithm

- In Step 2, any new outcome y^r will have $y_1^r > y_1^p$ and $y_2^r > y_2^q$.
- If no such outcome exists, then the subproblem solver must still re-prove the optimality of y^p or y^q .
- Then it must be the case that

$$\|y^* - y^r\|_{\infty}^{\beta_{pq}} + \min\{\beta_{pq}\epsilon_1, (1 - \beta_{pq})\epsilon_2\} \leq \|y^* - y^p\|_{\infty}^{\beta_{pq}} = \|y^* - y^q\|_{\infty}^{\beta_{pq}}$$

- Hence, we can impose an a priori upper bound of

$$\|y^* - y^p\|_{\infty}^{\beta_{pq}} - \min\{\beta_{pq}\epsilon_1, (1 - \beta_{pq})\epsilon_2\}$$

when solving the subproblem $P(\beta_{pq})$.

- With this upper bound, each subproblem will either be infeasible or produce a new outcome.

Using Warm Starting

- We have been developing methodology for *warm starting* branch and bound computations.
- Because the WCN algorithm involves solving a sequence of slightly modified MILPs, warm starting can be used.
- **Three approaches**
 - Warm start from the result of the previous iteration.
 - Solve a “base” problem first and warm each subsequent problem from there.
 - Warm start from the “closest” previously solved subproblem.
- In addition, we can optionally save the global cut pool from iteration to iteration.

Approximating the Pareto Set

- If the number of Pareto outcomes is large, it may not be desirable to generate the entire set.
- If only part of the set is generated, it is important that the subset be *well-distributed* among the entire set.
- Any probing algorithm can generate an approximation to the Pareto set by terminating early.
 - In such case, the key is to avoid **failed probes** whenever possible.
 - The order in which the intervals are explored affects both the *distribution of solutions* and the *number of failed probes*.
 - Empirically, **FIFO selection schemes** tend to distribute the points well and also minimize the number of failed probes.
- Another approach is to generate the set of *supported solutions*.
- This can be an extremely bad approximation in some cases.

Interactive Algorithms

- Interactive algorithms offer another method of avoiding enumeration of the entire set.
- In an interactive algorithm, the user guides the solution process by providing **real-time feedback**.
- This feedback provides information of the user's unknown **utility function**.
- A simple feedback mechanism for the WCN algorithm is to allow the user to select the next interval to be explored.
- In this way, the user is able to zero in on the portion of the tradeoff curve that is most attractive.
- There are a number of mechanisms for providing estimated **tradeoff information** to the user as the algorithm progresses.

Implementation: A Brief Overview of SYMPHONY

- **SYMPHONY** is an open-source software package for solving and analyzing mixed-integer linear programs (MILPs).
- **SYMPHONY** can be used in three distinct modes.
 - Black box solver: Solve generic MILPs (command line or shell).
 - Callable library: Call SYMPHONY from a C/C++ code.
 - Framework: Develop a customized black box solver or callable library.
- Makes extensive use of the **Computational Infrastructure for Operations Research** (COIN-OR) libraries (www.coin-or.org).
- Complete documentation, code samples, data sets, and application plug-ins are available (www.BranchAndCut.org).
- Advanced features
 - Warm starting
 - Bicriteria solve
 - Sensitivity analysis
 - Parallel execution mode

Example: Bicriteria ILP

- Consider the following bicriteria ILP:

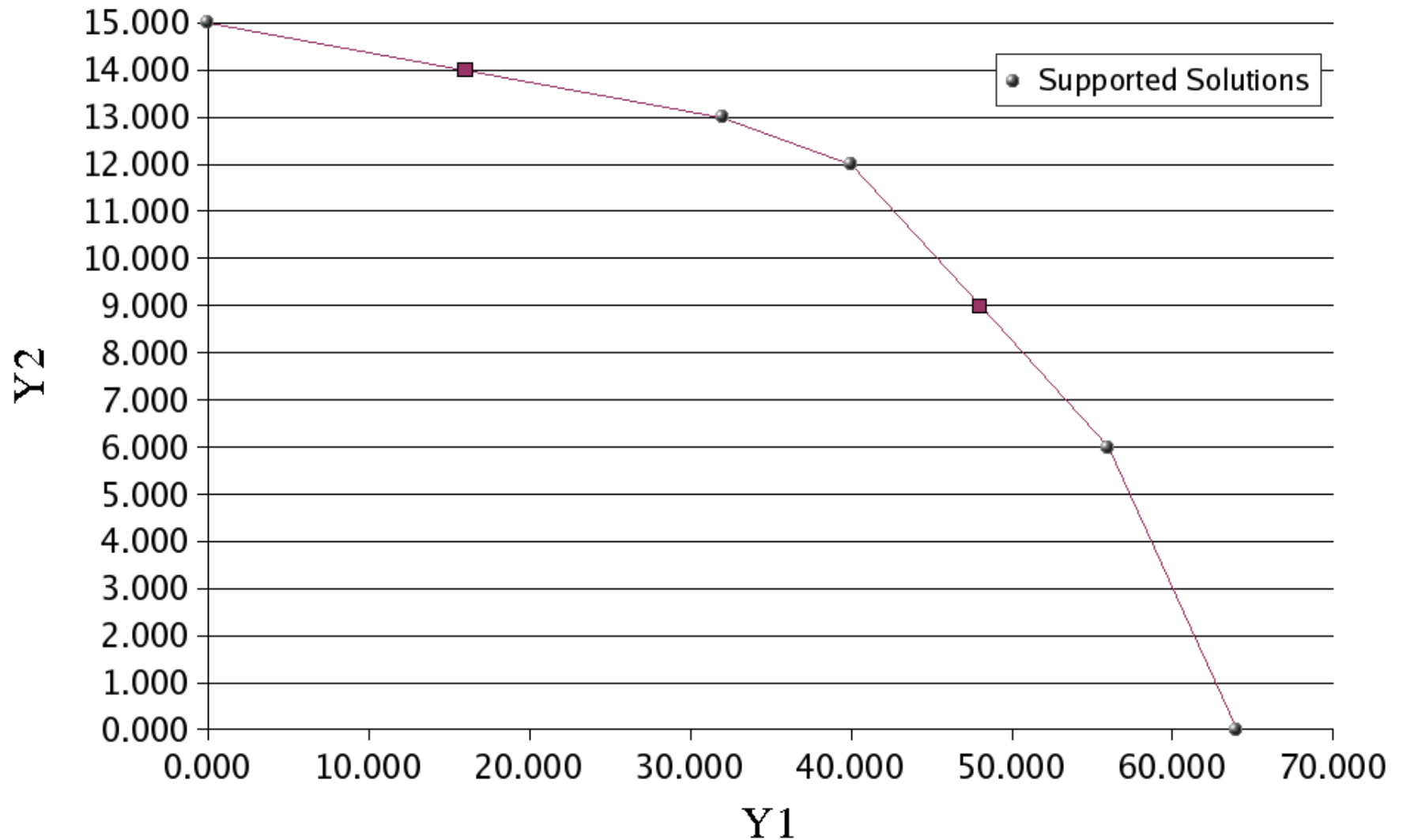
$$\begin{aligned} \text{vmax} \quad & [8x_1, x_2] \\ \text{s.t.} \quad & 7x_1 + x_2 \leq 56 \\ & 28x_1 + 9x_2 \leq 252 \\ & 3x_1 + 7x_2 \leq 105 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- The following code solves this model using **SYMPHONY**.

```
int main(int argc, char **argv)
{
    OsiSymSolverInterface si;
    si.parseCommandLine(argc, argv);
    si.setObj2Coeff(1, 1);
    si.loadProblem();
    si.multiCriteriaBranchAndBound();
}
```

Example: Pareto Outcomes for Example

Non-dominated Solutions

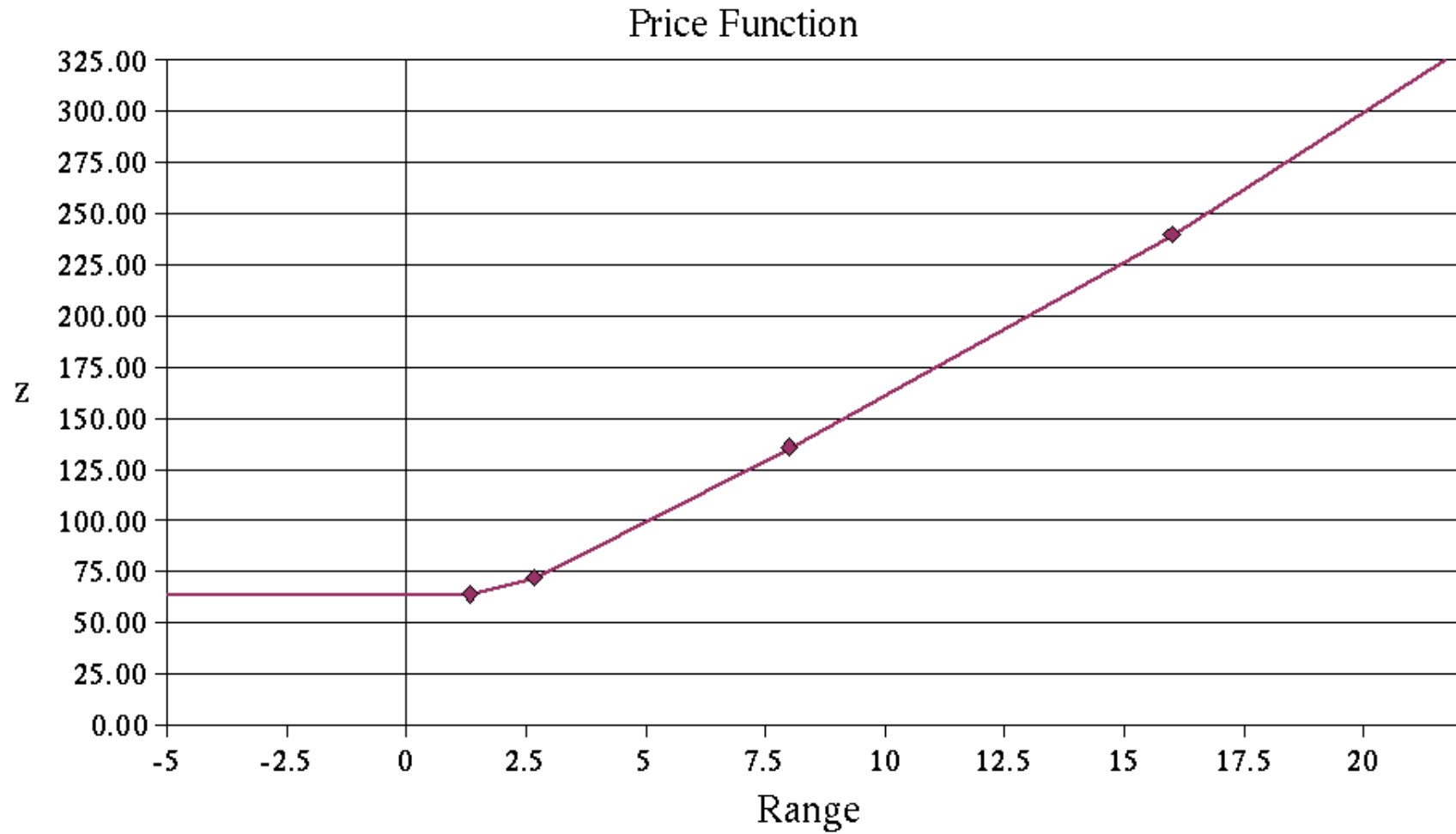


Example: Sensitivity Analysis

- By examining the supported solutions and break points, we can easily determine $p(\theta)$, the optimal solution to the ILP with objective $8x_1 + \theta x_2$.

θ range	$p(\theta)$	x_1^*	x_2^*
$(-\infty, 1.333)$	64	8	0
$(1.333, 2.667)$	$56 + 6\theta$	7	6
$(2.667, 8.000)$	$40 + 12\theta$	5	12
$(8.000, 16.000)$	$32 + 13\theta$	4	13
$(16.000, \infty)$	15θ	0	15

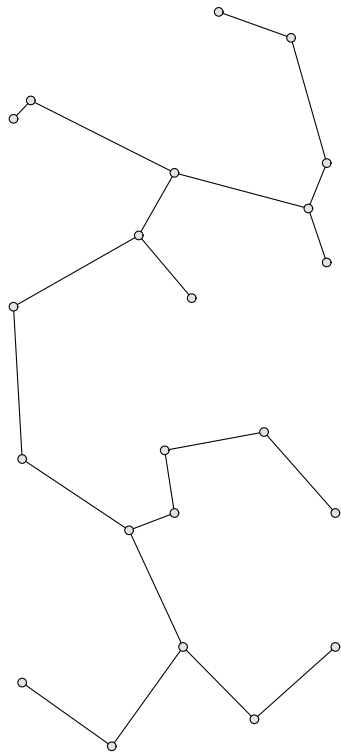
Example: Price Function



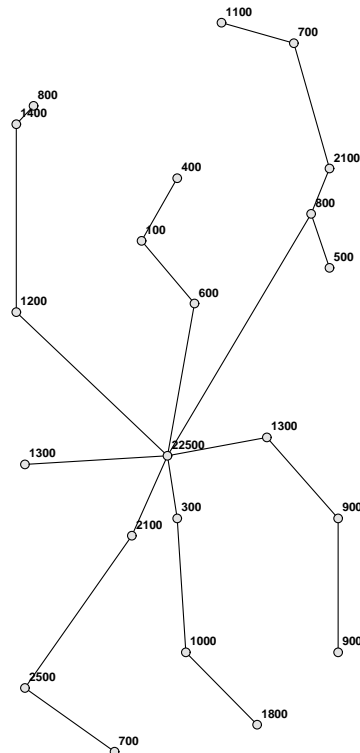
Application: Capacitated Network Routing Problems

- Using **SYMPHONY**, we developed a custom solver for a class of **capacitated network routing problems** (CNRPs).
- A single commodity is supplied to a set of customers from a single supply point.
- We must design the network and route the demand, obeying capacity and other side constraints.
- We wish to consider both
 - the **cost of construction** (the sum of lengths of all links), and
 - the **latency of the resulting network** (the sum of length multiplied by demand carried for all links).
- These are competing objectives, so we can analyze the tradeoff by using the **SYMPHONY** multicriteria solver.

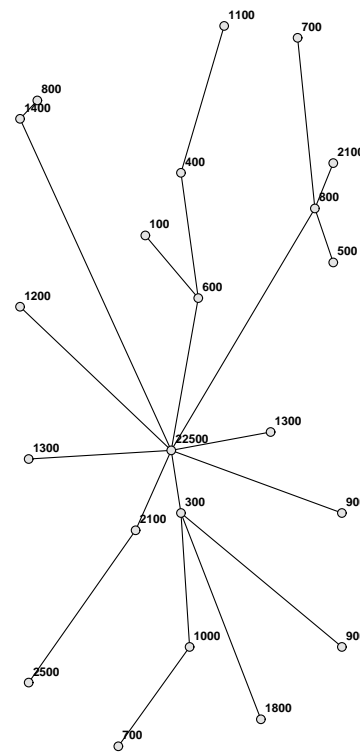
Application: Efficient Solutions for a Small CNRP



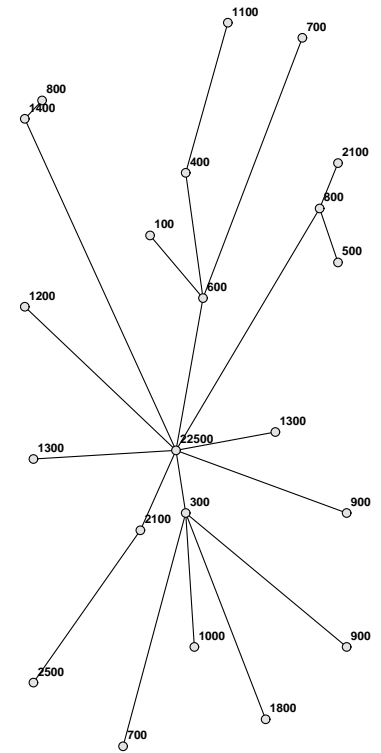
(a)



(b)

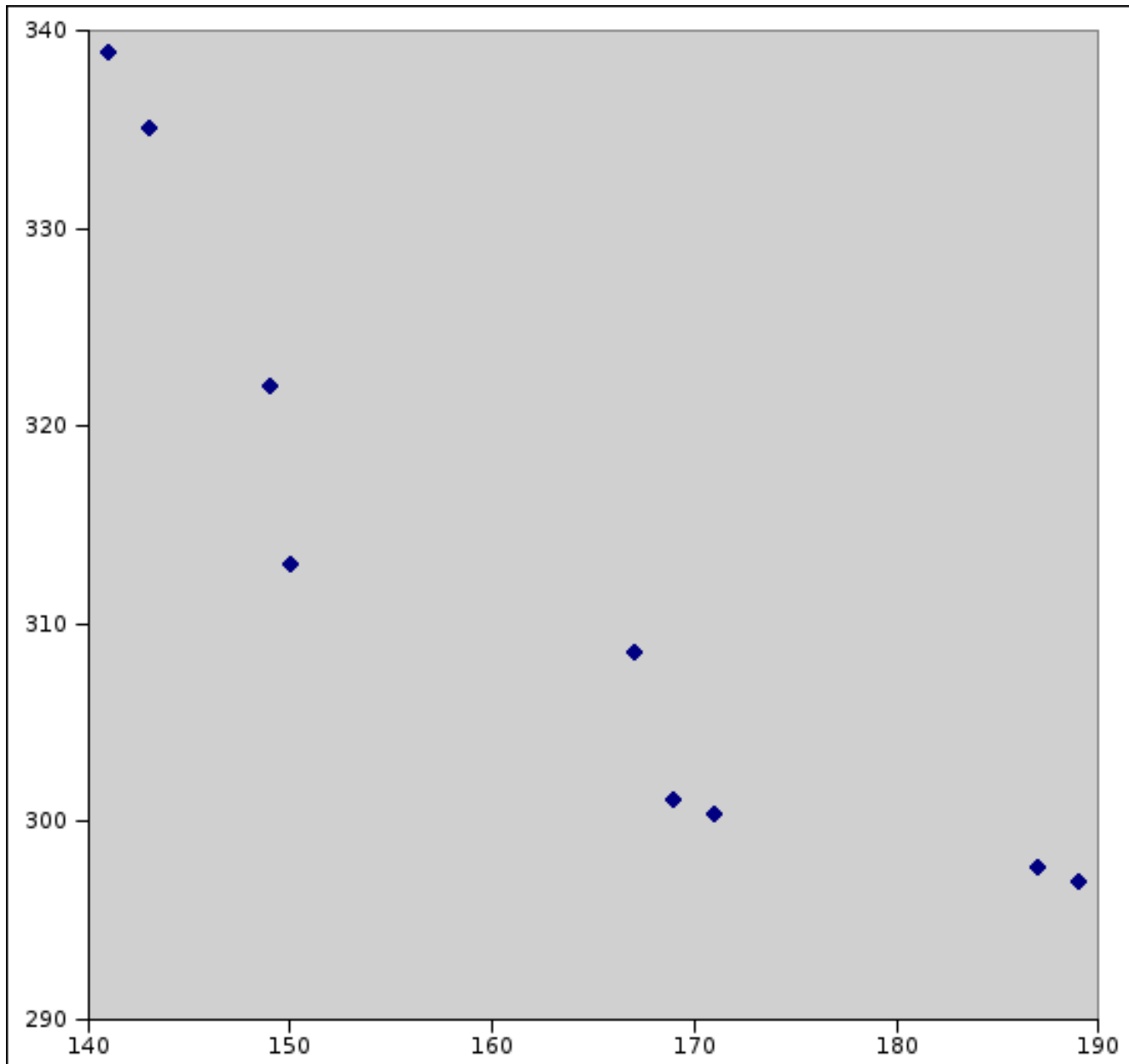


(c)

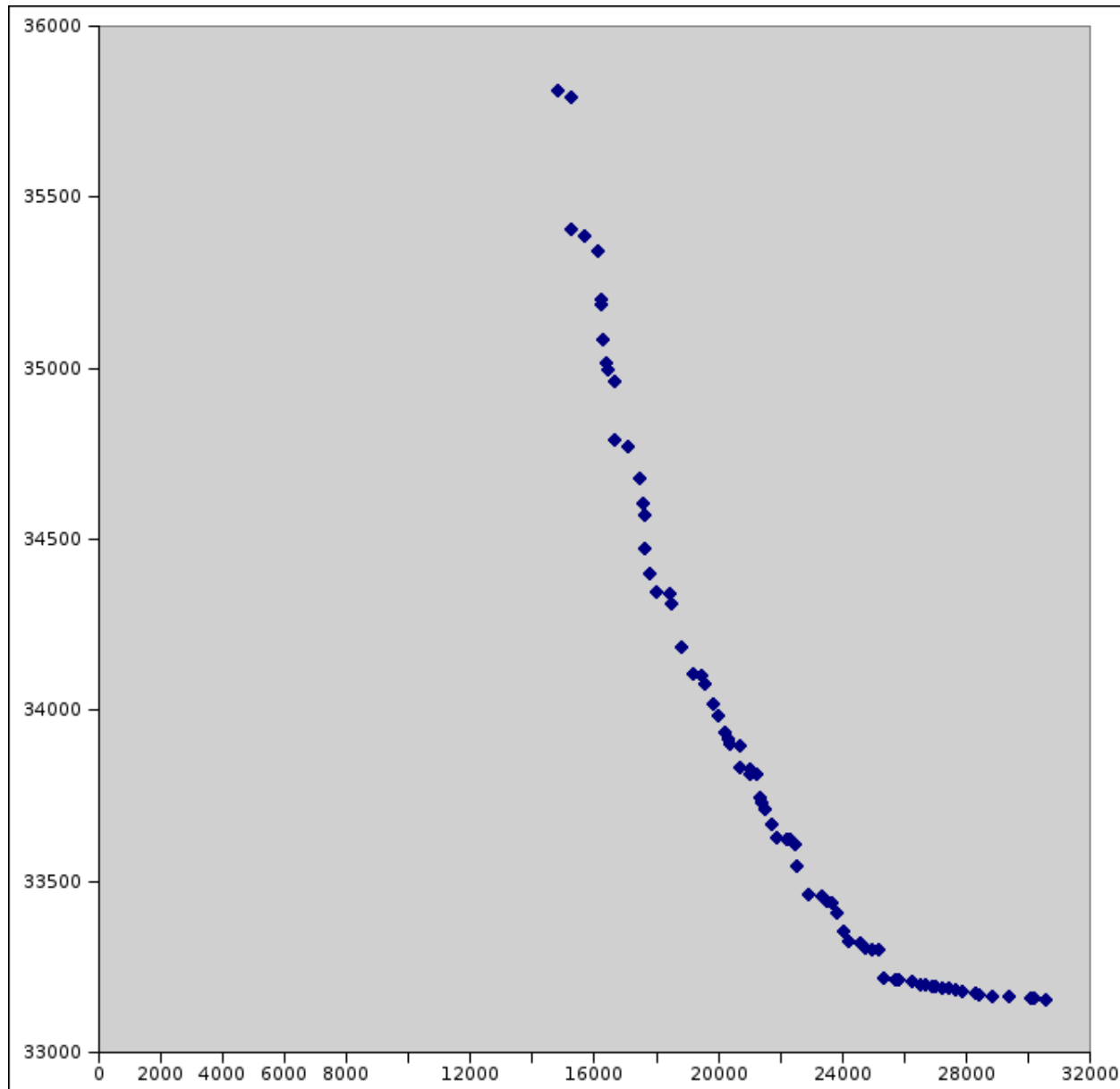


(d)

Application: Pareto Outcomes for a Small CNRP



Application: Pareto Outcomes for a Larger CNRP



Computational Results: Comparing WCN with Bisection Search

Knapsack

Size	Iterations				Outcomes Found				Max Missed		
	WCN	Δ from WCN			WCN	Δ from WCN					
	0	10^{-1}	10^{-2}	10^{-3}	0	10^{-1}	10^{-2}	10^{-3}	10^{-1}	10^{-2}	10^{-3}
10	278	12	300	679	149	-17	0	0	6	0	0
20	364	-1	390	896	192	-22	-2	0	6	1	0
30	324	-43	246	712	167	-25	0	0	4	0	0
40	490	-108	235	898	250	-55	-11	0	5	2	0
50	686	-138	235	1123	348	-69	-9	-1	11	1	1
Totals	2142	-278	1406	4308	1106	-188	-22	-1	11	2	1

CNRP

Name	Iterations				Outcomes Found				Max Missed		
	WCN	Δ from WCN			WCN	Δ from WCN					
	0	10^{-1}	10^{-2}	10^{-3}	0	10^{-1}	10^{-2}	10^{-3}	10^{-1}	10^{-2}	10^{-3}
att48	147	-35	-9	104	74	-18	-15	-4	3	3	1
Totals	2381	-264	724	3794	1207	-135	-13	0	5	1	0

Computational Results: Comparing WCN with ACN

Knapsack

Size	Iterations				Outcomes Found				Max Missed		
	WCN	Δ from WCN			WCN	Δ from WCN					
	0	10^{-2}	10^{-3}	10^{-4}	0	10^{-2}	10^{-3}	10^{-4}	10^{-2}	10^{-3}	10^{-4}
10	278	-4	0	0	149	-2	0	0	1	0	0
20	364	-6	0	0	192	-3	0	0	1	0	0
30	324	-6	0	0	167	-3	0	0	1	0	0
40	490	-24	0	0	250	-12	0	0	1	0	0
50	686	-28	-4	0	348	-24	-2	0	3	2	0
Totals	2142	-70	0	0	1106	-34	-2	0	3	2	0

CNRP

Name	Iterations				Outcomes Found				Max Missed		
	WCN	Δ from WCN			WCN	Δ from WCN					
	0	10^{-2}	10^{-3}	10^{-4}	0	10^{-2}	10^{-3}	10^{-4}	10^{-2}	10^{-3}	10^{-4}
att48	147	-140	-106	-62	74	-70	-53	-31	44	17	8
Totals	2381	-2056	-1012	-34	1207	-1028	-506	-17	18	5	1

Computational Results: Comparing WCN with Hybrid ACN

Knapsack

Size	Iterations				Outcomes Found				Max Missed		
	WCN	Δ from WCN			WCN	Δ from WCN					
	0	10^{-2}	10^{-3}	10^{-4}	0	10^{-2}	10^{-3}	10^{-4}	10^{-2}	10^{-3}	10^{-4}
10	278	-4	0	0	149	-2	0	0	1	0	0
20	364	-6	0	0	192	-3	0	0	1	0	0
30	324	-6	0	0	167	-3	0	0	1	0	0
40	490	-24	0	0	250	-12	0	0	1	0	0
50	686	-28	-4	0	348	-14	-2	0	3	2	0
Totals	2142	-68	-4	0	1106	-34	-2	0	3	2	0

CNRP

Name	Iterations				Outcomes Found				Max Missed		
	WCN	Δ from WCN			WCN	Δ from WCN					
	0	10^{-3}	10^{-4}	10^{-5}	0	10^{-3}	10^{-4}	10^{-5}	10^{-3}	10^{-4}	10^{-5}
att48	147	-106	-62	-6	74	-53	-31	-3	17	8	2
Totals	2381	-1012	-44	-2	1207	-612	-22	-1	5	1	1

Computational Results: Comparing WCN with ACN and Hybrid ACN (CPU Time)

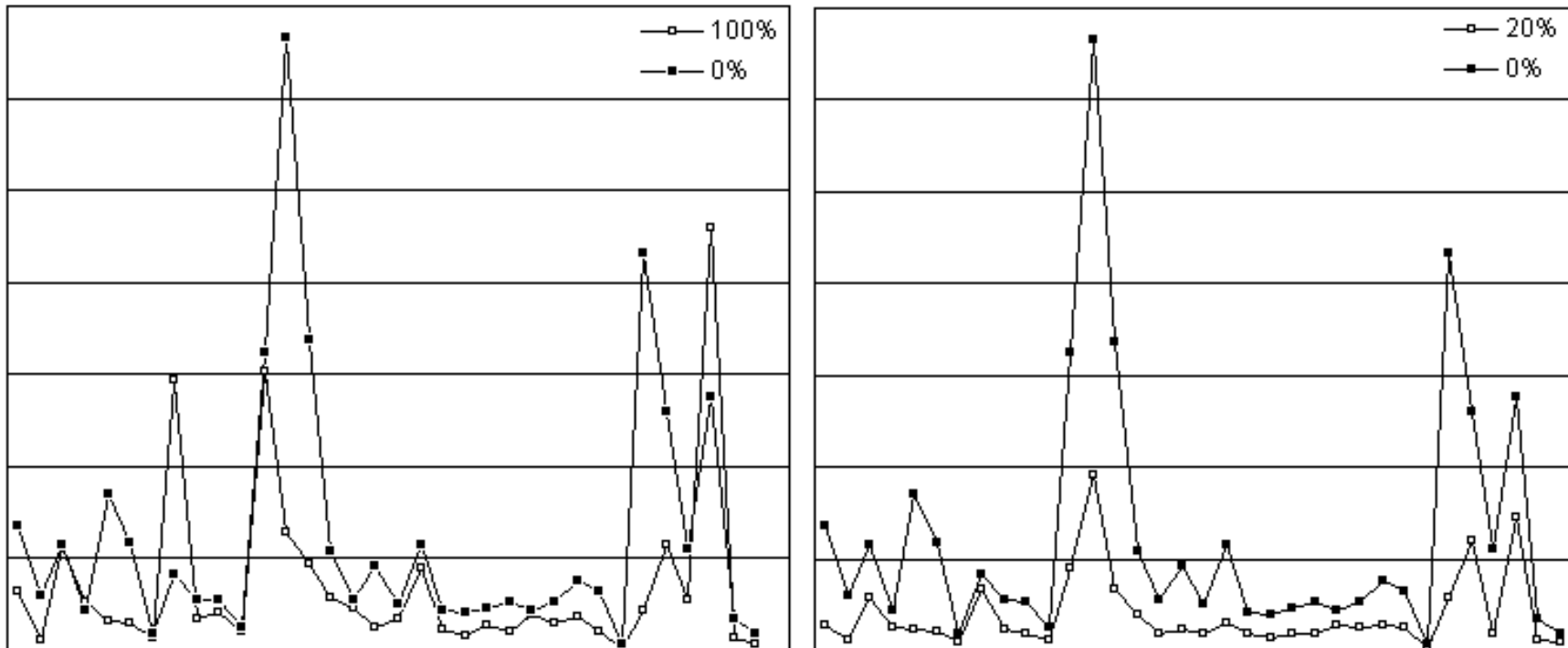
Knapsack

Size	CPU Time (ACN)				CPU Time (Hybrid)			
	WCN	Δ from WCN			WCN	Δ from WCN		
	0	10^{-2}	10^{-3}	10^{-4}	0	10^{-2}	10^{-3}	10^{-4}
10	13.18	0.06	-0.23	-0.10	13.18	0.34	0.12	0.16
20	17.46	-1.33	-0.41	-0.21	17.46	-1.17	0.03	-0.16
30	24.93	-1.28	-0.43	-0.43	24.93	-1.02	-0.11	0.10
40	65.88	-5.69	-1.70	-0.66	24.93	-1.02	-0.11	0.10
50	139.42	-27.18	-3.78	-1.35	65.88	-4.89	-1.09	-0.30
60	260.87	-35.42	-6.55	-2.75	139.42	-13.04	-3.37	-1.17
Totals	260.87	-35.42	-6.55	-2.75	260.87	-19.78	-4.42	-1.37

CNRP

Name	CPU Time (ACN)				CPU Time (Hybrid)			
	WCN	Δ from WCN			WCN	Δ from WCN		
	0	10^{-2}	10^{-3}	10^{-4}	0	10^{-2}	10^{-3}	10^{-4}
att48	83.67	-80.14	-59.83	-28.48	83.67	-59.34	-30.19	-1.12
Totals	8122.36	-7728.51	-5244.54	-1451.37	8122.36	-5481.53	-1531.35	-589.90

Computational Results: Using Warm Starting to Solve CNRP Instances



These are results using **SYMPHONY** to solve CNRP instances with two different warm starting strategies.

Conclusion

- Generating the complete set of Pareto outcomes is a challenging computational problem.
- We presented a new algorithm for solving bicriteria mixed-integer programs.
- The algorithm is
 - asymptotically optimal,
 - generates exact breakpoints,
 - has good numerical properties, and
 - can exploits modern solution techniques.
- We have shown how this algorithm is implemented in the SYMPHONY MILP solver framework.
- Future work
 - Improvements to warm starting procedures
 - Parallelization
 - More than two objective

Shameless Plug

- The software discussed in this talk is available for free download from the **Computational Infrastructure for Operations Research** Web site

`www.coin-or.org`

- **The COIN-OR Project**

- An **initiative** promoting the development and use of interoperable, open-source software for operations research.
- A **consortium** of researchers in both industry and academia dedicated to improving the state of computational research in OR.
- A non-profit educational foundation known as the COIN-OR Foundation.

- **The COIN-OR Repository**

- A **library** of interoperable software tools for building optimization codes, as well as some stand-alone packages.
- A **venue for peer review** of OR software tools.
- A **development platform** for open source projects, including a CVS repository.

More Information

- SYMPHONY

- Prepackaged releases can be obtained from www.BranchAndCut.org.
- Up-to-date source is available from www.coin-or.org.
- Available Solvers

- Generic MILP
- Traveling Salesman Problem
- Vehicle Routing Problem
- Mixed Postman Problem
- Bicriteria Knapsack Solver
- Set Partitioning Problem
- Matching Problem
- Network Routing

- For references and further details, see *An Improved Algorithm for Biobjective Integer Programming*, to appear in *Annals of OR*, available from

www.lehigh.edu/~tkr2

- Overviews of multiobjective integer programming
 - Climaco (1997)
 - Ehrgott and Gandibleux (2002)
 - Ehrgott and Wiecek (2005)