Parametric Valid Inequalities and the Solution of Multistage Optimization Problems

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Attributions

Many students and collaborators contributed to development of this material.

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Thanks!

Outline

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Overview

- In many settings, we need ether to
  - ⇒ solve a sequence of related MILPs;
  - ⇒ analyze a parametric family of MILPs; or
  - ⇒ solve a problem with multiple stages in which later-stage problems are parameterzed on the solutions to earlier stage problems.

Examples

- Decomposition-based algorithm (Lagrangean relaxation, Dantzig-Wolfe)
- Parametric optimization
- Multistage/multilevel Optimization

- Branch-and-bound algorithms themselves consist of solving a sequence of related subproblems!
- Algorithms for MILP depend heavily on the generation of valid inequalities, but such inequalities are typically only valid for a single instance.
- This talk is about some ideas about how to make the inequalities themselves parametric.

Setting: (Multi-Stage) Mixed Integer Linear Optimization

- In this talk, we'll initially consider (single-level) mixed integer linear optimization problems (MILPs).

$$\min_{x \in \mathcal{P}(b) \cap X} c^\top x, \qquad \text{(MILP)}$$

where, $c \in \mathcal{R}^n$; $\mathcal{P}(\beta) = \{x \in \mathbb{R}^{n_1} \mid Ax \geq \beta\}$ with $A \in \mathbb{Q}^{m \times n}$, $b \in \mathcal{R}^m$; and $X = \mathbb{Z}^r_+ \times \mathbb{R}^{n-r}_+$.

- We'll later consider a multi-stage generalization.

MSMILP

$$\min_{x \in \mathcal{P}(b) \cap X} \left\{ c^\top x + \Xi(x) \right\}, \qquad \text{(MSMILP)}$$

where Ξ is the *risk function* that represents the impact of future uncertainty due to stochasticity or other unknowns.

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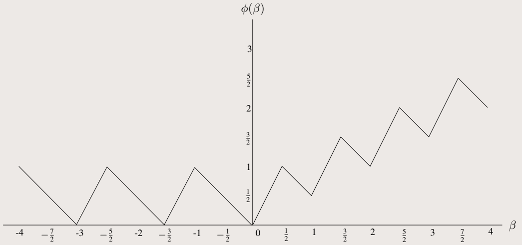
The Value Function

- The *value function* associated with (MILP) is

MILP Value Function

$$\phi(\beta) = \min_{x \in \mathcal{P}(\beta) \cap X} c^\top x \tag{VF}$$

for $\beta \in \mathcal{R}^m$. We let $\phi(\beta) = \infty$ if $\beta \in \Omega = \{\beta \in \mathbb{R}^m \mid \mathcal{S}(\beta) = \emptyset\}$.



The General Dual and Dual Functions

Dual Functions

A *dual function* $F : \mathbb{R}^m \rightarrow \mathbb{R}$ is one that satisfies $F(\beta) \leq \phi(\beta)$ for all $\beta \in \mathbb{R}^m$.

- The problem of finding a dual function for which $F(b) \approx \phi(b)$ is the *general dual problem* associated with (MILP).

$$\max \{F(b) : F(\beta) \leq \phi(\beta),\ \beta \in \mathcal{R}^m, F \in \Upsilon^m\} \qquad \text{(D)}$$

  where $\Upsilon^m \subseteq \{f \mid f : \mathcal{R}^m \rightarrow \mathcal{R}\}$

- We call F^* *strong* for this instance if F^* is a *feasible* dual function and $F^*(b) = \phi(b)$.
- This dual instance always has a solution F^* that is strong if $\phi \in \Upsilon^m$

Outline

Relaxations

Relaxation

A relaxation of (MILP) is an optimization problem

$$\min_{x \in \mathcal{R}} f(x)$$

such that $\mathcal{R} \supseteq (\mathcal{P}(b) \cap X)$ and $f(x) \leq c^\top x \ \forall x \in \mathcal{P}(b) \cap X$.

- The linear optimization problem (LP) obtained by relaxing the requirement $x \in X$ is the *LP relaxation* of (MILP)

$$\min_{x \in \mathcal{P}(b)} \ c^\top x, \qquad \text{(LPR)}$$

- Solving the LP relaxation is the first step in many/most algorithms for solving (MILP).

Disjunctive Relaxations

Valid Disjunction

A *valid disjunction* for (MILP) is a disjoint collection

$$X_1(b), X_2(b), \ldots, X_k(b) \tag{VD}$$

of subsets of $\mathbb{R}^n$ such that

$$\mathcal{P}(b) \cap X \subseteq \bigcup_{1 \le i \le k} X_i(b)$$

- Any valid disjunction can be added to the LP relaxation to obtain a stronger relaxation of (MILP).

$$\min_{x \in \mathcal{P}(b) \cap (\bigcup_{1 \le i \le k} X_i(b))} c^\top x, \tag{DR}$$

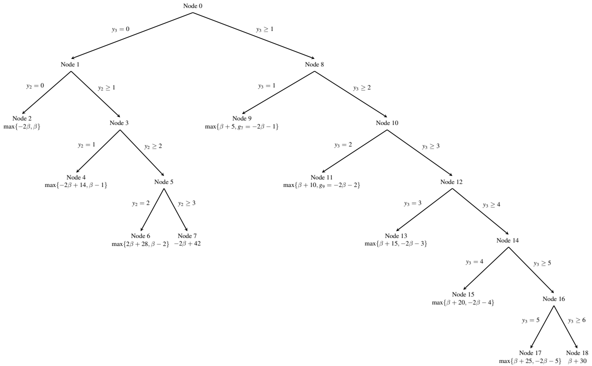
- Thus, the branch-and-bound tree encodes a relaxation of (MILP).
- We can also derive such disjunctions from problems structure (integrality).

Disjunctions via Branch and Bound

- Branch and bound can be viewed as an algorithm for iteratively constructing and solving disjunctive relaxations.
- In the context of branch and bound, each sets $X_i(b)$ corresponds to a *subproblem*

$$\min_{x \in \mathcal{P}(b) \cap X \cap X_i(b)} c^\top x \tag{1}$$

associated with a leaf node of the branch-and-bound tree.

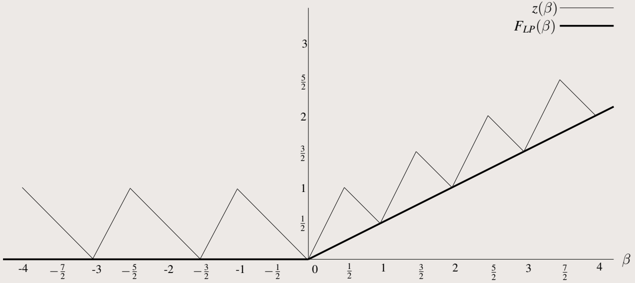


Dual Functions from Relaxations

- The value function of any relaxation is a valid dual function.
- The value function of the LP relaxation is

$$\phi_{LP}(\beta) = \min_{x \in \mathcal{P}(\beta)} c^\top x. \quad \text{(LPVF)}$$

- To define the value function of (DR), we need the notion of a *parametric valid disjunction*, which we discuss next.



Dual Functions from Disjunctive Relaxation

Parametric Valid Disjunction

A *parametric valid disjunction* for (MILP) is a parametric family of disjoint collections

$$X_1(\beta), X_2(\beta), \ldots, X_k(\beta) \tag{PVD}$$

of subsets of $\mathbb{R}^n$ such that

$$\mathcal{P}(\beta) \cap X \subseteq \bigcup_{1 \leq i \leq k} X_i(\beta)$$

With any parametric valid disjunction of the form (PVD), we can associate the following value function.

$$\begin{aligned}\phi_D(\beta) &= \min_{x \in \mathcal{P}(\beta) \cap (\bigcup_{1 \leq i \leq k} X_i(\beta))} c^\top x, \\ &= \min_{1 \leq i \leq k} \phi_D^i(\beta),\end{aligned} \tag{DR}$$

where $\phi_D^i(\beta) = \min_{x \in \mathcal{P}(\beta) \cap X_i(\beta)} c^\top x$ (value function of relaxation of subproblem i).

Outline

The Gauge Function

The *gauge function* associated with (MILP) is a function that returns the largest valid right-hand side for an inequality with left-hand side vector α.

MILP Gauge Function

$$\Gamma(\zeta) = \min_{x \in \mathcal{P}(b) \cap X} \zeta^\top x \; \forall \alpha \in \mathbb{R}^n \tag{GF}$$

Note that we have

$$\alpha x \geq \Gamma(\alpha) \;\; \forall x \in \mathcal{P}(b)$$

Valid Inequality

An *inequality* defined by $(\alpha, \eta) \in \mathbb{Q}^n \times \mathbb{Q}$ is *valid* for $\mathcal{P}(b)$ if $\eta \leq \Gamma(\alpha)$.

Inequalities from Relaxations

- Valid inequalities are often obtained more efficiently by considering a relaxation.
- Let $\mathcal{R} \supseteq \mathcal{P}(b)$ be the feasible set of a relaxation whose gauge function is

$$\Gamma_{\mathcal{R}}(\zeta) = \min_{x \in \mathcal{R}} \zeta^{\top} x \tag{RGF}$$

- Then $(\alpha, \Gamma_{\mathcal{R}}(\alpha))$ is valid for $\mathcal{P}(b)$.

Disjunctive Inequalities

- Let $\mathcal{R} = \mathcal{P}(\beta) \cap \left(\bigcup_{1 \leq i \leq k} X_i(\beta)\right)$ be the feasible set of a disjunctive relaxation.
- Then $(\alpha, \Gamma_{\mathcal{R}}(\alpha))$ is valid for $\mathcal{P}(b)$, where $\Gamma_{\mathcal{R}}$ is the gauge function (RGF).
- Inequalities derived in this way are known as *disjunctive inequalities*.
- Many known classes of valid inequalities can be derived by applying this framework to the relaxation (DR) for various classes of disjunction.

Outline

Parametric Valid Inequalities

Parametric Valid Inequality

A *parametric valid inequality* is a pair (α, F), where $\alpha \in \mathbb{R}^n$ and F is a dual function for the MILP

$$\min_{x \in \mathcal{P}(b) \cap X} \alpha^\top x, \qquad \text{(MILP-}\alpha\text{)}$$

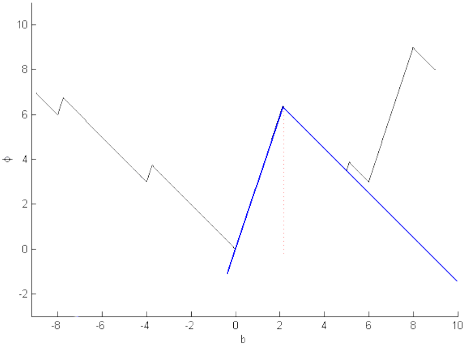
- The right-hand side of a parametric valid inequality is a *function*.
- The parametric inequality corresponds to a parametric family of inequalities with the same left-hand side.
- We have

$$\alpha^\top x \geq F(\beta) \ \forall x \in \mathcal{P}(\beta)$$

- Note that (α, ϕ_α) is a valid parametric inequality, where ϕ_α is the value function of (MILP-α).
- These inequalities are related to subadditive inequalities.

Dual Functions from Branch and Bound

- Recall that a *dual function* $F : \mathbb{R}^m \to \mathbb{R}$ is one that satisfies $F(\beta) \leq \phi(\beta)$ for all $\beta \in \mathbb{R}^m$.
- Observe that any branch-and-bound tree yields a lower approximation of the value function.



Dual Functions from Branch-and-Bound

Let T be set of the terminating nodes of the tree. Then in a terminating node $t \in T$ we solve:

$$\begin{aligned} \phi^t(\beta) = \min\ & c^\top x \\ \text{s.t. } & Ax = \beta, \\ & l^t \leq x \leq u^t, x \geq 0 \end{aligned} \tag{2}$$

The dual at node t:

$$\begin{aligned} \phi^t(\beta) = \max\ & \{\pi^t \beta + \underline{\pi}^t l^t + \bar{\pi}^t u^t\} \\ \text{s.t. } & \pi^t A + \underline{\pi}^t + \bar{\pi}^t \leq c^\top \\ & \underline{\pi} \geq 0, \bar{\pi} \leq 0 \end{aligned} \tag{3}$$

We obtain the following strong dual function:

$$\min_{t \in T} \{\hat{\pi}^t \beta + \hat{\underline{\pi}}^t l^t + \hat{\bar{\pi}}^t u^t\}, \tag{4}$$

where $(\hat{\pi}^t, \hat{\underline{\pi}}^t, \hat{\bar{\pi}}^t)$ is an optimal solution to the dual (3).

Outline

The Risk Function

Recall problem (MSMILP) from earlier. The general class of functions we consider are of the following form.

Canonical Risk Function

$$\Xi(x) = \mathbb{E}_{\omega \in \Omega}\left[\Xi_\omega(x)\right], \tag{RF}$$

Scenario Risk Function

$$\Xi_\omega(x) = \min\left\{d^1 y \mid y \in \operatorname{argmin}\{d^2 y \mid y \in \mathcal{P}_2(A^2_\omega x) \cap Y\}\right\} \tag{2LRF}$$

- ω is a random variable over a finite probability space $(\Omega, \mathscr{F}, \mathcal{P})$;
- Realizations of ω are called *scenarios*;
- $\mathcal{P}_2(\beta) = \{y \in \mathbb{R}^{n_2}_+ \mid G^2 y \geq A^2_\omega x\}$;
- $Y = \mathbb{Z}^{r_2}_+ \times \mathbb{R}^{n_2 - r_2}_+$; and
- $G^2 \in \mathbb{Q}^{m_2 \times n_2}$, $A^2_\omega \in \mathbb{Q}^{m_2 \times n_1}$ for $\omega \in \Omega$.

Value Function Reformulation

One way to reformulate (MSMILP) as a single-level problem is as follows.

$$
\begin{aligned}
\min \quad & c^1 x + \sum_{\omega \in \Omega} p_\omega d^1 y^\omega && (5)\\
\text{subject to} \quad & A^1 x \geq b^1 && (6)\\
& G^2 y^\omega \geq A^2_\omega x \;\forall \omega \in \Omega && (7)\\
& d^2 y^\omega \leq \phi(A^2_\omega x) \;\forall \omega \in \Omega && (8)\\
& x \in X && (9)\\
& y^\omega \in Y \;\forall \omega \in \Omega && (10)
\end{aligned}
$$

where ϕ is the value function of the second-stage problem. This is, in principle, a standard mathematical optimization problem.

Bound Inequalities

- One relaxation is to drop the value function constraint (8) and replace it with the linear inequality

$$d^2 y \leq \eta,$$

where η is such that

$$\eta \geq \max_{(x,y)\in\mathcal{F}} \quad d^2 y, \tag{11}$$

where F is the feasible region of the bilevel problem.

- One way to derive η is to solve the problem on the left-hand side of (11) by branch-and-bound.
- We can stop the branch and bound after a given time limit and extract a dual function from the tree.
- With the dual function, we may even be able to tighten the inequality after branching.

Conclusions

- The concept of a parametric inequality has a wide range of applications, of which we only scraped the surface in this talk.
- Parametric inequalities can also be defined using similarly defined *primal functions*, which arise from *restrictions*.
- They have already been applied successfully in some limited contexts.
- If we can generate them effectively, this will open up many interesting lines of research.