Multistage/Multilevel Discrete Optimization

Ted Ralphs[1]
Joint work with Suresh Bolusani[1], Sahar Tahernejad[1]

[1]COR@L Lab, Department of Industrial and Systems Engineering, Lehigh University

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Multilevel and Multistage Games

- In game theory terminology, the problems we address are known as *finite extensive-form games*, sequential games involving n players.
- A *subgame* is the part of a game that remains after some moves have been made.

Multilevel Game

- A game in which n players alternate moves in a fixed sequence (the well-known case of two players is called a *Stackelberg game*).
- The goal is to find a *subgame perfect Nash equilibrium*, i.e., the move by each player that ensures that player's best outcome.

Multistage/Recourse Game

- A cooperative game in which play alternates between cooperating players and "chance" players.
- The goal is to find a *subgame perfect Markov equilibrium*, i.e., the move that ensures the best outcome in a probabilistic sense.

Two-Stage Mixed Integer Optimization

- We have the following general formulation:

2SMILP

$$z_{\text{2SMILP}} = \min_{x \in \mathcal{P}_1 \cap X} \{cx + \Xi(x)\}, \qquad \text{(2SMILP)}$$

where

$$\mathcal{P}_1 = \left\{x \in \mathbb{R}^{n_1} \mid A^1 x \geq b^1\right\}$$

is the *first-stage feasible region*, $X = \mathbb{Z}_+^{r_1} \times \mathbb{R}_+^{n_1 - r_1}$, $A^1 \in \mathbb{Q}^{m_1 \times n_1}$, and $b^1 \in \mathbb{R}^{m_1}$.

- $X = \mathbb{Z}_+^{r_1} \times \mathbb{R}_+^{n_1 - r_1}$ represents first-stage integrality requirements.
- Ξ is the *risk function* that represents the impact of future uncertainty.
- This uncertainty can arise either due to stochasticity or due to the fact that Ξ represents the reaction of a competitor (or both).
- This "risk function" is similar to that utilized in the finance literature.

Two-stage Mixed Integer Stochastic Bilevel Optimization

The risk function of 2SMISBLPs has the following form.

Canonical Risk Function

$$\Xi(x) = \mathbb{E}_{\omega\in\Omega}\left[\Xi_\omega(x)\right], \tag{RF}$$

Scenario Risk Function

$$\Xi_\omega(x) = \min\left\{d^1 y \mid y \in \operatorname{argmin}\{d^2 y \mid y \in \mathcal{P}_2(b^2_\omega - A^2_\omega x) \cap Y\}\right\} \tag{2LRF}$$

- ω is a random variable over a finite probability space $(\Omega, \mathscr{F}, \mathcal{P})$;
- Realizations of ω are called *scenarios*;
- $\mathcal{P}_2(\beta) = \{y \in \mathbb{R}^{n_2}_+ \mid Gy \geq \beta\}$;
- $Y = \mathbb{Z}^{r_2}_+ \times \mathbb{R}^{n_2-r_2}_+$; and
- $G \in \mathbb{Q}^{m_2\times n_2}$, $A^2_\omega \in \mathbb{Q}^{m_2\times n_1}$ for $\omega \in \Omega$.

Basic Assumptions

Linking Variables

Definition 1 *The set*

$$L = \bigcup_{\omega \in \Omega} \left(\{ i \in \{1, \ldots, n_1\} \mid (A^2_\omega)_i \neq 0 \} \right),$$

is the set of indices of the linking variables.

- In the above, $(A^2_\omega)_i$ denotes the i^{th} column of matrix A^2_ω.
- x_L will denote the sub-vector of $x \in \mathbb{R}^{n_1}$ corresponding to the linking variables.
- The linking variables are those with non-zero coefficients in the second-stage problem for at least one scenario.

Assumption 1 $L = \{1, \ldots, k_1\}$ *for* $k_1 \leq r_1$.

Assumption 2 $\mathcal{P}^\omega = \left\{ (x, y) \in \mathbb{R}_+^{n_1 \times n_2} \mid x \in \mathcal{P}_1, y \in \mathcal{P}_2(b^2_\omega - A^2_\omega x) \right\}$ *is bounded for* $\omega \in \Omega$.

Rational Reaction Sets

Corresponding to each $x \in X$, we have the rational reaction set for $\omega \in \Omega$.

Rational Reaction Set for Scenario ω

$$\mathcal{R}^{\omega}(x) = \operatorname{argmin}\{d^2 y \mid y \in \mathcal{P}_2(b^2_\omega - A^2_\omega x) \cap Y\}.$$

- For a given $x \in X$, $\mathcal{R}^{\omega}(x)$ set may be empty because either
  - $\mathcal{P}_2(b^2_\omega - A^2_\omega x) \cap Y$ is itself empty or
  - there exists $r \in \mathbb{R}^{n_2}_+$ such that $Gr \geq 0$ and $d^2 r < 0$.
- The latter case cannot occur, since Assumption 2 implies that $\left\{r \in \mathbb{R}^{n_2}_+ \setminus \{0\} \;\middle|\; Gr \geq 0\right\} = \emptyset$.

Feasible Regions

- The *bilevel feasible region* for scenario ω with respect to the first- and second-stage variables in (2LRF) is

$$\mathcal{F}^{\omega} = \{(x, y) \in X \times Y \mid x \in \mathcal{P}_1, y \in \mathcal{R}^{\omega}(x)\}.$$

- Members of $\mathcal{F}^{\omega}$ are called *bilevel feasible solutions* for scenario ω.
- The second-stage problem should be feasible for all scenarios, so the feasible region with respect to first-stage variables only is

$$\mathcal{F}_1 = \bigcap_{\omega \in \Omega} \operatorname{proj}_x(\mathcal{F}^{\omega}).$$

- For $x \in \mathbb{R}^{n_1}$, we have that

$$x \in \mathcal{F}_1 \Leftrightarrow x \in \bigcap_{\omega \in \Omega} \operatorname{proj}_x(\mathcal{F}^{\omega}) \Leftrightarrow \mathcal{R}^{\omega}(x) \neq \emptyset \ \forall \omega \in \Omega \Leftrightarrow \Xi(x) < \infty$$

and we say that $x \in \mathbb{R}^{n_1}$ is *feasible* if $x \in \mathcal{F}_1$.

Feasibility Conditions

The feasibility conditions for scenario ω with respect to the first- and second-stage variables in (2LRF) are

Feasibility Conditions

Feasibility Condition 1 $x \in \mathcal{F}_1$

Feasibility Condition 2 $y^\omega \in \mathcal{R}^\omega(x)$ *for all* $\omega \in \Omega$

Special Case I: Bilevel (Integer) Linear Optimization

In bilevel optimization, we have $|\Omega| = 1$, so 2SMISBLP can be re-written in the form

Mixed Integer Bilevel Linear Optimization Problem (MIBLP)

$$\min \left\{cx + d^1 y \mid x \in \mathcal{P}_1 \cap X, y \in \operatorname{argmin}\{d^2 y \mid y \in \mathcal{P}_2(b^2 - A^2 x) \cap Y\}\right\}. \tag{MIBLP}$$

Alternatively, this corresponds to

Bilevel Risk Function

$$\Xi(x) = \min \left\{d^1 y \mid y \in \operatorname{argmin}\{d^2 y \mid y \in \mathcal{P}_2(b^2 - A^2 x) \cap Y\}\right\}.$$

Note that we drop the subscripts associated with the scenario in this case, but notation is otherwise, the same.

Special Case II: Optimization with Recourse

- Recourse problems are a special case in which $d^1 = d^2$.
- In a two-stage stochastic mixed integer optimization problem, we have

Stochastic Risk Function

$$\Xi(x) = \mathbb{E}_{\omega \in \Omega}\left[\phi(b^2_\omega - A^2_\omega x)\right],$$

where ω is the random variable from probability space $(\Omega, \mathscr{F}, \mathcal{P})$ defined earlier.

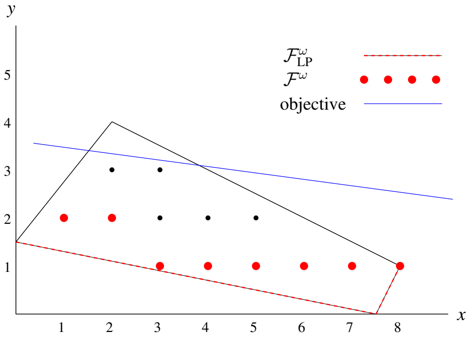
- For each $\omega \in \Omega$, $A^2_\omega \in \mathbb{Q}^{m_2 \times n_1}$ is the realization of the input to the second-stage problem for scenario ω.
- The function ϕ is the *value function* of the second-stage MILP.

Second-Stage Value Function

$$\phi(\beta) = \min\left\{d^2 y \;\middle|\; Gy \geq \beta, y \in Y\right\} \;\forall \beta \in \mathbb{R}^{m_2}. \tag{2S-VF}$$

Value Function Reformulation

When Ω represents a *discrete* and *finite* space, one way to reformulate 2SMISBLP as a bilevel problem is as follows.



$$\begin{aligned} \min \quad & cx + \sum_{\omega \in \Omega} p_\omega d^1 y^\omega \\ \text{subject to} \quad & A^1 x \geq b^1 \\ & G y^\omega \geq b^2_\omega - A^2_\omega x \ \forall \omega \in \Omega \\ & \sum_{\omega \in \Omega} d^2 y^\omega \leq \sum_{\omega \in \Omega} \phi(b^2_\omega - A^2_\omega x) \\ & x \in X \\ & y^\omega \in Y \ \forall \omega \in \Omega \end{aligned}$$

where $\sum_{\omega \in \Omega} \phi(b^2_\omega - A^2_\omega x)$ can be obtained by solving

$$\begin{aligned} \min \quad & \sum_{\omega \in \Omega} d^2 y^\omega \\ & G y^\omega \geq b^2_\omega - A^2_\omega x \quad \forall \omega \in \Omega \\ & y^\omega \in Y \quad \forall \omega \in \Omega \end{aligned}$$

Risk Function Reformulation ⇒ Generalized Benders'

A second reformulation can be obtained by projecting out the second-stage variables.

$$\begin{aligned} \min \quad & cx + \sum_{\omega \in \Omega} p_\omega z_\omega \\ \text{subject to} \quad & z_\omega \geq \Xi_\omega(x) \\ & x \in X \end{aligned}$$

- We can further simplify this reformulation by noting that Ξ_ω can itself be rewritten as

$$\Xi_\omega(x) = \rho(b^2_\omega - A^2_\omega x)$$

where

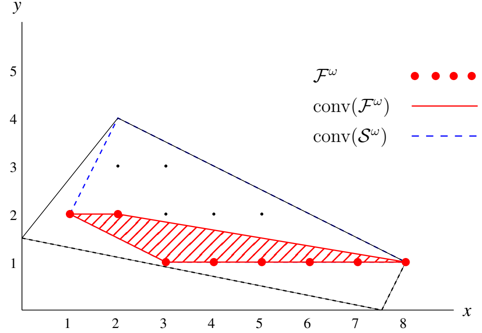
$$\rho(\beta) = \min \left\{ d^1 y \mid y \in \operatorname{argmin}\{d^2 y \mid y \in \mathcal{P}_2(\beta) \cap Y\} \right\}$$

is the *(optimistic) reaction function*.

- This leads to a generalized Benders algorithm obtained by constructing approximations of ρ dynamically (Benders "cuts").
- **We have two implementations of this algorithm** (Hassanzadeh and Ralphs [2014], Bolusani and Ralphs [2019])

Polyhedral Reformulation ⇒ Branch and Cut

Convexification considers the following conceptual reformulation.



$$\begin{aligned} \min \quad & cx + \sum_{\omega \in \Omega} p_\omega d^1 y \\ \text{s.t.} \quad & (x, y^\omega) \in \text{conv}(\mathcal{F}^\omega) \end{aligned}$$

where $\mathcal{S}^\omega = \mathcal{P}^\omega \cap (X \times Y)$ for $\mathcal{P}^\omega = \{(x, y) \in \mathbb{R}_+^{n_1 \times n_2} \mid A_1 x \geq b_1, G^2 y \geq b_\omega^2 - A_\omega^2 x\}$

- To get bounds, we'll optimize over a relaxed feasible region.
- We'll iteratively approximate the true feasible region with linear inequalities.

Branch-and-Cut Algorithm for 2SMISBLPs

- The algorithm is based on the framework originally described by DeNegre and Ralphs [2009], but with many additional enhancements.
- The algorithm has been implemented in the `MibS` framework, which is open source and available from COIN-OR.
- Details are contained in a forthcoming paper by Tahernejad et al. [2019] (preprint available).

Components

- Bounding
  - **Lower bound** ⇒ An LP relaxation strengthened with **valid inequalities**
  - Upper bound ⇒ Feasible solutions
- Feasibility checking
- Branching ⇒ Several schemes for branching
- Search strategies
- Preprocessing methods
- Primal heuristics

Lower Bound

Two possible relaxations

1. Removing the *optimality constraint of the second-stage problem*.

$$\mathcal{S}^{\omega} = \left\{(x, y) \in X \times Y \;\middle|\; A^1 x \geq b^1, Gy \geq b^2_{\omega} - A^2_{\omega} x\right\}$$

2. Removing the *optimality constraint of the second-stage problem* and the *integrality constraints*.

$$\mathcal{P}^{\omega} = \left\{(x, y) \in \mathbb{R}^{n_1 \times n_2}_{+} \;\middle|\; A^1 x \geq b^1, Gy \geq b^2_{\omega} - A^2_{\omega} x\right\}$$

- Let $(x^t, y^{1t}, \ldots, y^{|\Omega| t})$ be the optimal solution of the relaxation problem at node t.

Feasibility Checking

$(x^t, y^{1t}, \ldots, y^{|\Omega|t})$ *may* be bilevel feasible $\Rightarrow$ Feasibility check

- $(x^t, y^{1t}, \ldots, y^{|\Omega|t})$ does not satisfy integrality requirements $\Rightarrow$ *infeasible*
- $(x^t, y^{1t}, \ldots, y^{|\Omega|t})$ satisfies integrality requirements
    - Solve

$$\min \sum_{\omega \in \Omega} d^2 y^\omega$$
$$G y^\omega \geq b^2_\omega - A^2_\omega x^t \quad \forall \omega \in \Omega$$
$$y^\omega \in Y \quad \forall \omega \in \Omega$$

      to find $\sum_{\omega \in \Omega} \phi(b^2_\omega - A^2_\omega x^t)$
    - Let $(\hat{y}^1, \ldots, \hat{y}^{|\Omega|})$ be the optimal solution
        - $\sum_{\omega \in \Omega} d^2 \hat{y}^\omega = \sum_{\omega \in \Omega} d^2 y^{\omega t} \Rightarrow$ *feasible*
        - $\sum_{\omega \in \Omega} d^2 \hat{y}^\omega < \sum_{\omega \in \Omega} d^2 y^{\omega t} \Rightarrow$ *infeasible*

UB Problem

- The best bilevel feasible solution with $x_L = \gamma \in \mathbb{Z}^L$ can be obtained by solving just one MILP.

$$\min\{cx + \sum_{\omega \in \Omega} d^1 y^\omega \mid x \in X, A^1 x \geq b^1, G y^\omega \geq b^2_\omega - A^2_\omega x \ \forall \omega \in \Omega,$$

$$\sum_{\omega \in \Omega} d^2 y^\omega \leq \sum_{\omega \in \Omega} \phi(b^2_\omega - A^2_\omega x),$$

$$y_\omega \in Y \ \forall \omega \in \Omega, x_L = \gamma\}. \tag{UB}$$

- This can be employed to
  - find *heuristic bilevel feasible solutions*.
  - develop the *linking branching strategy*, which branches only on linking variables.

Deterministic Equivalent of 2SMISBLPs

- When Ω represents a *discrete* and *finite* space, 2SMISBLP can be converted to a *deterministic bilevel problem* in the usual way.

$$
\begin{aligned}
\min \ & cx + \sum_{\omega \in \Omega} p^{\omega} d^1 y^{\omega} \\
& A^1 x \geq b^1 \\
& G y^{\omega} \geq b^2_{\omega} - A^2_{\omega} x \quad \forall \omega \in \Omega \\
& \sum_{\omega \in \Omega} d^2 y^{\omega} \leq \sum_{\omega \in \Omega} \phi(b^2_{\omega} - A^2_{\omega} x) \\
& x \in X \\
& y^{\omega} \in Y \quad \forall \omega \in \Omega
\end{aligned}
$$

- The deterministic equivalent can be solved by *branch-and-cut*.
- As the *number of scenarios* increases, so does the difficulty.
- The majority of effort is in solving *large MIPs* required for checking the bilevel feasibility and for solving problem (UB)

Decomposition

- If $(x^t, y^{1t}, ..., y^{|\Omega|t})$ is an optimal solution of the relaxation problem at node t, checking its feasibility requires solving

$$\begin{aligned} \min \quad & \sum_{\omega \in \Omega} d^2 y^\omega \\ & G y^\omega \geq b^2_\omega - A^2_\omega x^t \quad \forall \omega \in \Omega \\ & y^\omega \in Y \quad \forall \omega \in \Omega \end{aligned} \tag{1}$$

- Due to the *block structure* of (1), it can be decomposed to $|\Omega|$ independent MIPs.

$$\begin{bmatrix} G & & \\ & \ddots & \\ & & G \end{bmatrix} \begin{bmatrix} y^1 \\ \vdots \\ y^{|\Omega|} \end{bmatrix} \geq \begin{bmatrix} b^2_1 - A^2_1 x^t \\ \vdots \\ b^2_{|\Omega|} - A^2_{|\Omega|} x^t \end{bmatrix}$$

- Since these small MIPs are *independent*, their solution can be *parallelized*.
- Problem (UB) can be similarly decomposed unless there are *non-linking first-stage variables*.

Progressive Hedging Heuristic

- Find a *heuristic* solution for 2SMISBLPs by employing the *Progressive Hedging (PH) Algorithm* [Rockafellar and Wets, 1991]
- It is used only for the 2SMISBLPs with *binary first-stage variables* to avoid the non-linear term in the objective of the PH subproblems
- The idea is
  1. Repeat the PH algorithm until reaching the iteration or time limit or consensus among all subproblems (PH subproblems are MIBLPs).
  2. Let V be the set of first-stage variables whose values are *consensus* for which consensus has been reached through the last iteration.
  3. Solve the *restricted* deterministic equivalent of 2SMISBLP obtained by fixing the values of first-stage variables belonging to the set V.

Sample Average Approximation (SAA)

- The difficulty of solving 2SMISBLPs as a deterministic bilevel problem is difficult/impossible when $|\Omega|$ is large (or infinite).
- SAA is a well-known Monte Carlo simulation-based approach in which
- N random samples are generated.
- The function value is approximated by solving a deterministic problem known as the *SAA problem* constructed by restricting to the generated scenarios.
- The SAA problem corresponding to 2SMISBLP is

SAAP

$$z_N = \min_{x \in \mathcal{P}_1 \cap X} \left\{ cx + \frac{1}{N} \sum_{i=1}^{N} \Xi_i(x) \right\}, \qquad \text{(SAAP)}$$

- This procedure is repeated to obtain statistical estimates.
- Problem (SAAP) can be solved as a deterministic bilevel problem.

Software Framework

- MibS is an open-source solver in C++ originally for MIBLPs, based on our branch-and-cut algorithm Tahernejad et al. [2019], DeNegre et al. [2019].
- It is built on top of the BLIS solver [Xu et al., 2009].
- It employs packages available from the Computational Infrastructure for Operations Research (COIN-OR) repository
    - COIN High Performance Parallel Search (CHiPPS): To manage the global branch-and-bound
    - SYMPHONY: To solve the required MIPs
    - COIN LP Solver (CLP): To solve the LPs arising in the branch and cut
    - Cut Generation Library (CGL): To generate cutting planes within both SYMPHONY and MibS
    - Open Solver Interface (OSI): To interface with other solvers
- MibS has been *generalized* to a solver for 2SMISBLPs by adding
    - function of reading the data files for 2SMISBLPs
    - option of decomposing the second-stage and UB problems
    - parallel solution of the decomposed second-stage and UB problems (OpenMP)
    - PH heuristic
    - SAA method

Data Set `MIBLP-XU`

- MIBLP instances from Xu and Wang [2014]:
  - includes 100 instances
  - all first-level variables are integer with upper bound 10
  - second-level variables are continuous with probability 0.5
  - number of first- and second-level variables are equal
  - n_1 and n_2 are in the range of 10-460 with an increments of 50
  - c, d^1 and d^2 are within $[-50, 50]$
  - All constraint coefficients are within $[0, 10]$
  - b^1 is within $[30, 130]$ and b^2 is within $[10, 110]$
- We changed the instances with $n_1 \in \{10, 60\}$ (20 instances) to 2SMISBLPs as follows.
  - The coefficients of the second-stage variables in the first-stage constraints are zero.
  - Elements of A^2_ω are discrete uniform random variable on the set $\{0, 0.5, ..., 10\}$.
  - Elements of b^2_ω are discrete uniform random variable on the set $\{10, 10.5, ..., 110\}$
- Instances `stocBmilplib-n-i` have n first-stage variables and i represents the instance index

Data Set `sslp`

- Stochastic Server Location Problem (sslp) [Ntaimo and Sen, 2005] from SIPLIB test library.
- Instances `sslp-n-m-k` have n locations, m customers and k scenarios.

Computational Results

- Computations were done on compute nodes running the Linux (Debian 8.7) operating system with dual AMD Opteron 6128 processors and 32 GB RAM.
- Time limit was 10 hours for all experiments
- All `MibS` parameters were set to the default values.

SAA Method

- Data set: MIBLP-XU
- $N \in \{20, 30, 40\}$, Evaluation sample size = 200 and Number of replications = 10

Instance	N = 20			N = 30			N = 40		
	Est UB	Est Gap	$\hat{\sigma}_{\text{Gap}}$	Est UB	Est Gap	$\hat{\sigma}_{\text{Gap}}$	Est UB	Est Gap	$\hat{\sigma}_{\text{Gap}}$
stocBmilplib_10_1	-646.59	35.42	43.72	-646.59	14.45	42.57	-646.59	-1.68	44.71
stocBmilplib_10_2	-99.55	36.36	10.54	-99.55	25.39	8.32	-99.55	22.54	6.78
stocBmilplib_10_3	-232.51	23.83	13.40	-232.51	10.49	13.50	-232.51	8.27	12.85
stocBmilplib_10_4	-181.32	0.96	13.65	-181.32	7.92	12.62	-181.32	13.97	11.07
stocBmilplib_10_5	∞	—-	—-	116.43	37.30	17.88	107.26	-2.35	17.15
stocBmilplib_10_6	-284.90	8.79	12.21	-284.90	0.96	11.09	-284.90	3.34	11.27
stocBmilplib_10_7	-184.71	26.53	29.30	-184.71	19.31	26.98	-184.71	8.58	21.44
stocBmilplib_10_8	-174.15	-4.13	14.15	-174.15	6.19	12.15	-174.15	11.53	11.33
stocBmilplib_10_9	-201.66	21.45	13.26	-201.66	6.78	11.25	-201.66	7.73	11.57
stocBmilplib_10_10	-98.14	19.45	6.79	-98.14	18.13	6.10	-98.14	10.13	4.85
stocBmilplib_60_1	-149.35	1.79	6.01	-149.35	1.80	5.68	-149.35	0.84	5.58
stocBmilplib_60_2	8.07	7.19	7.13	8.07	9.80	7.04	8.07	11.01	7.23
stocBmilplib_60_3	-11.45	19.21	7.61	-11.45	18.42	7.63	-11.45	16.77	7.53
stocBmilplib_60_4	-44.94	16.00	7.83	-42.47	13.40	6.93	-44.94	6.65	7.00
stocBmilplib_60_5	-39.02	7.27	7.83	-39.02	3.75	7.53	-39.02	0.16	7.14
stocBmilplib_60_6	-128.09	10.52	7.61	-128.09	10.28	7.58	-128.09	8.86	7.09
stocBmilplib_60_7	-44.60	-1.09	9.25	-39.53	7.73	8.25	-39.53	6.13	7.55
stocBmilplib_60_8	-82.80	5.41	9.16	-82.80	4.17	8.12	-82.80	1.88	8.08
stocBmilplib_60_9	13.23	5.82	6.87	13.23	3.62	5.70	13.23	1.63	5.62
stocBmilplib_60_10	-102.48	17.95	7.45	-102.48	15.59	7.22	-102.48	14.09	7.23

UB Problem Decomposition

- Data set: MIBLP-XU
- Number of scenarios: 40
- MIP solver: SYMPHONY

	Without UB Decomp		With UB Decomp	
Instance	Gap	Time	Gap	Time
stocBmilplib_60_1	0.0	223.43	0.0	82.50
stocBmilplib_60_2	0.0	624.63	0.0	140.5
stocBmilplib_60_3	0.0	659.91	0.0	164.37
stocBmilplib_60_4	0.0	487.44	0.0	128.18
stocBmilplib_60_5	0.0	372.65	0.0	137.48
stocBmilplib_60_6	0.0	985.46	0.0	142.28
stocBmilplib_60_7	0.0	1032.89	0.0	166.04
stocBmilplib_60_8	0.0	772.68	0.0	163.84
stocBmilplib_60_9	0.0	399.22	0.0	128.29
stocBmilplib_60_10	0.0	241.59	0.0	131.94

Comparing Alternatives (Stochastic Programming)

1. the D^2 algorithm [Ntaimo and Sen, 2005]
2. the PH based branch-and-bound algorithm (PH-BAB) [Atakan and Sen, 2018]

Instance	MibS-SYM	MibS-CPL	D^2	PH-BAB
sslp-5-25-50	1.89	1.80	0.53	0.60
sslp-5-25-100	7.78	7.30	1.03	1.20
sslp-10-50-50	181.98	59.53	295.95	9.40
sslp-10-50-100	533.39	169.66	480.46	19.50
sslp-10-50-500	3767.66	2590.51	1902.20	86.10
sslp-10-50-1000	14665.60	12152.36	5410.10	172.4
sslp-10-50-2000	10(h)	10(h)	9055.29	333.10
sslp-15-45-5	55.08	3.80	110.34	2.00
sslp-15-45-10	6.61	5.81	1494.89	10.90
sslp-15-45-15	2044.57	16.03	7210.63	5.400

PH Heuristic

- Data set: `sslp`
- Solving subproblems terminated after finding the first bilevel feasible solution.

Instance	Num Locations	Without PH	Iteration = 3				Iteration = 15				Iteration = 30			
			Num Fixed	Is Optimal	Time Subproblems	Total Time	Num Fixed	Is Optimal	Time Subproblems	Total Time	Num Fixed	Is Optimal	Time Subproblems	Total Time
sslp-5-25-50	5	1.80	1	yes	1.35	2.90	4	yes	5.43	6.13	5	yes	5.98	5.98
sslp-5-25-100	5	7.30	0	—	2.76	10.55	0	—	11.74	19.57	2	yes	20.90	25.18
sslp-10-50-50	10	59.53	0	—	18.59	79.21	0	—	82.92	143.38	0	—	170.92	231.47
sslp-10-50-100	10	169.66	0	—	33.70	208.92	0	—	149.11	324.38	0	—	297.68	473.54
sslp-10-50-500	10	2590.51	0	—	159.86	2936.14	0	—	726.88	3493.54	0	—	1433.61	4211.2
sslp-10-50-1000	10	12152.36	0	—	307.89	13280.50	0	—	1415.21	14743.00	0	—	2870.70	15901.40
sslp-10-50-2000	10	10(h)	0	—	619.36	10(h)	0	—	2765.15	10(h)	0	—	5461.21	10(h)
sslp-15-45-5	15	3.80	11	yes	2.53	4.86	7	no	9.37	10.73	10	No	42.69	50.80
sslp-15-45-10	15	5.81	9	yes	6.24	12.31	6	yes	35.10	43.40	10	yes	91.62	102.50
sslp-15-45-15	15	16.03	8	yes	8.16	19.36	7	yes	58.58	83.60	8	yes	138.12	150.66

References I

Semih Atakan and Suvrajeet Sen. A progressive hedging based branch-and-bound algorithm for mixed-integer stochastic programs. *Computational Management Science*, 15(3-4):501–540, 2018.

S. Bolusani and T.K. Ralphs. A Generalized Benders' Algorithm for Mixed Integer Bilevel Linear Optimization. Technical report, Working Paper, COR@L Laboratory, Lehigh University, 2019. URL `http://coral.ie.lehigh.edu/~ted/files/talks/BilevelBenders-MOPTA18.pdf`.

S. DeNegre and T.K. Ralphs. A Branch-and-Cut Algorithm for Bilevel Integer Programming. In *Proceedings of the Eleventh INFORMS Computing Society Meeting*, pages 65–78, 2009. doi: 10.1007/978-0-387-88843-9_4. URL `http://coral.ie.lehigh.edu/~ted/files/papers/BILEVEL08.pdf`.

S.T. DeNegre, T.K. Ralphs, and S. Tahernejad. MiBS version 1.1. 2019. doi: 10.5281/zenodo.1439384. URL `https://github.com/coin-or/MibS`.

References II

A. Hassanzadeh and T.K. Ralphs. A Generalized Benders' Algorithm for Two-Stage Stochastic Program with Mixed Integer Recourse. Technical report, COR@L Laboratory Report 14T-005, Lehigh University, 2014. URL http://coral.ie.lehigh.edu/~ted/files/papers/SMILPGenBenders14.pdf.

Lewis Ntaimo and Suvrajeet Sen. The million-variable march for stochastic combinatorial optimization. *Journal of Global Optimization*, 32(3):385–400, 2005.

R.T. Rockafellar and R.J.B. Wets. Scenarios and policy aggregation in optimization under uncertainty. *Mathematics of operations research*, pages 119–147, 1991.

S. Tahernejad, T.K. Ralphs, and S.T. DeNegre. A Branch-and-Cut Algorithm for Mixed Integer Bilevel Linear Optimization Problems and Its Implementation. *Mathematical Programming Computation (to appear)*, 2019. URL http://coral.ie.lehigh.edu/~ted/files/papers/MIBLP16.pdf.

Pan Xu and Lizhi Wang. An exact algorithm for the bilevel mixed integer linear programming problem under three simplifying assumptions. *Computers & operations research*, 41:309–318, 2014.

References III

Y. Xu, T.K. Ralphs, L. Ladányi, and M.J. Saltzman. Computational Experience with a Software Framework for Parallel Integer Programming. *The INFORMS Journal on Computing*, 21:383–397, 2009. doi: 10.1287/ijoc.1090.0347. URL http://coral.ie.lehigh.edu/~ted/files/papers/CHiPPS-Rev.pdf.