Multistage Integer Programming: Algorithms and Complexity

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Outline

1. Introduction

2. Value Function

3. Algorithm

4. Conclusions

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2 Value Function

3 Algorithm

4 Conclusions

A Bit of Game Theory

- The optimization problems we address can be conceptualized as *finite extensive-form games*, which are sequential games involving n players.

Loose Definition

- The game is specified on a tree with each node corresponding to a move and the outgoing arcs specifying possible choices.
- The leaves of the tree have associated payoffs.
- Each player's goal is to maximize payoff.
- There may be *chance* players who play randomly according to a probability distribution and do not have payoffs (*stochastic games*).

- All players are rational and have perfect information.
- The problem faced by a player in determining the next move is a *multistage* optimization problem.
- The move must be determined by taking into account the *uncertainty about future stages*.

Multilevel and Multistage Games

- In the literature, the term *multilevel* is used for competitive games in which there is no chance player.
- *Multistage* is used for cooperative games in which all players receive the same payoff, but there are chance players.
- A *subgame* is the part of a game that remains after some moves have been made.

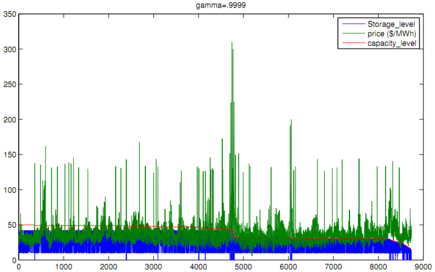
Stackelberg Game

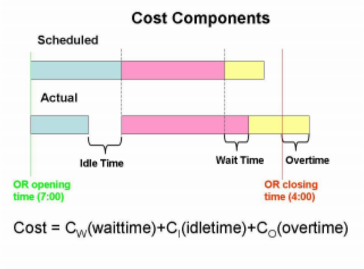
- A Stackelberg game is a game with two players who make one move each.
- The goal is to find a *subgame perfect Nash equilibrium*, i.e., the move by each player that ensures that player's best outcome.

Recourse Game

- A cooperative game in which play alternates between cooperating players and chance players.
- The goal is to find a *subgame perfect Markov equilibrium*, i.e., the move that ensures the best outcome in a probabilistic sense.

Quick Examples





Multistage Optimization

- A standard mathematical program models a (set of) decision(s) to be made *simultaneously* by a *single* decision-maker (i.e., with a *single* objective).
- Decision problems arising in sequential games and other real-world applications involve
  - multiple, independent decision-makers (DMs),
  - sequential/multi-stage decision processes, and/or
  - multiple, possibly conflicting objectives.
- Modeling frameworks
  - Multiobjective Programming $\Leftarrow$ multiple objectives, single DM
  - Mathematical Programming with Recourse $\Leftarrow$ multiple stages, single DM
  - Multilevel Programming $\Leftarrow$ multiple stages, multiple objectives, multiple DMs
- *Multilevel programming* generalizes standard mathematical programming by modeling hierarchical decision problems, such as finite extensive-form games.
- Such models arises in a remarkably wide array of applications.

Brief Overview of Practical Applications

- Hierarchical decision systems
  - Government agencies
  - Large corporations with multiple subsidiaries
  - Markets with a single "market-maker."
  - Decision problems with recourse
- Parties in direct conflict
  - Zero sum games
  - Interdiction problems
- Modeling "robustness": Chance player is external phenomena that cannot be controlled.
  - Weather
  - External market conditions
- Controlling optimized systems: One of the players is a system that is optimized by its nature.
  - Electrical networks
  - Biological systems

Two-Stage Mixed Integer Linear Optimization

- With two stages, we hav the following general formulation:

$$z_{\mathrm{2SMILP}} = \min_{x \in \mathcal{P}_1} \Psi(x) = \min_{x \in \mathcal{P}_1} \left\{ c^\top x + \Xi(x) \right\}, \tag{1}$$

  where

$$\mathcal{P}_1 = \{x \in X \mid Ax = b, x \geq 0\} \tag{2}$$

  is the *first-stage feasible region* with $X = \mathbb{Z}_+^{r_1} \times \mathbb{R}_+^{n_1 - r_1}$.

- Ξ represents the leader's expectation of the impact of future uncertainty.
- The canonical form employed in stochastic programming with recourse is

$$\Xi(x) = \mathbb{E}_{\omega \in \Omega} \left[\phi(h_\omega - T_\omega x) \right], \tag{3}$$

- ϕ is the second-stage *value function* to be defined shortly.
- $T_\omega \in \mathbb{Q}^{m_2 \times n_1}$ and $h_\omega \in \mathbb{Q}^{m_2}$ represent the input to the second-stage problem for scenario $\omega \in \Omega$.

The Second-Stage Value Function

- The structure of the objective function *Psi* depends primarily on the structure of the *value function*

$$\phi(\beta) = \min \left\{ d^\top y \mid y \in \operatorname{argmin}_{y \in \mathcal{P}_L(\beta)} q^\top y \right\}. \tag{4}$$

where

$$\mathcal{P}_2(\beta) = \{ y \in Y \mid Wy = \beta \} \tag{5}$$

is the *second-stage feasible region* with respect to a given right-hand side β and $Y = \mathbb{Z}_+^{r_2} \times \mathbb{R}_+^{n_2 - r_2}$.

- The second-stage problem is parameterized on the unknown value β of the right-hand side.
- This value is determined jointly by the realized value of ω and the values of the first-stage decision variables.
- The second-stage solution is evaluated with respect to two objective vectors, q and d, that represent the (possibly) differing valuations of the two players.

Two-Stage Stochastic Program with Recourse

For the remainder of the talk, we consider the simpler case of two-stage stochastic programming:

$$\min \Psi(x) = \min_{x \in \mathcal{P}_1} c^\top x + \sum_{\omega \in \Omega} p_\omega \phi(h_\omega - T_\omega x) \tag{SP}$$

where

$$\phi(\beta) = \min_{y \in \mathcal{P}_2(\beta)} q^\top y \tag{RP}$$

In this talk, we assume

- ω follows a discrete distribution with a finite support,
- W and q are fixed,
- $\mathcal{P}_1$ is compact, and
- $\mathbb{E}_{w \in \Omega}[\phi(h_\omega - T_\omega x)]$ is finite for all $x \in X$.

Unless otherwise indicated, all probability distributions will be uniform.

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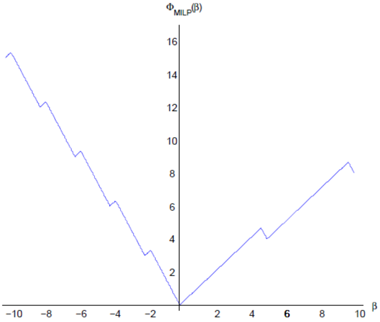
Illustrating the Value Function

Example 1

$$\begin{aligned}\phi(\beta) = \min\ & 6y_1 + 4y_2 + 3y_3 + 4y_4 + 5y_5 + 7y_6 \\ s.t.\ & 2y_1 + 5y_2 - 2y_3 - 2y_4 + 5y_5 + 5y_6 = \beta \\ & y_1, y_2, y_3 \in \mathbb{Z}_+, y_4, y_5, y_6 \in \mathbb{R}_+.\end{aligned} \qquad \text{(Ex.MILP)}$$

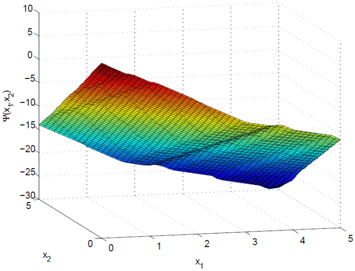


Illustrating the Objective Function

Example 2

$$\Psi(x) = -3x_1 - 4x_2 + \sum_{\omega \in \Omega} \phi(h_\omega - 2x_1 - 0.5x_2) \tag{Ex.SMP}$$

and $\Omega = \{1, 2\}, h_1 = 6, h_2 = 12$.



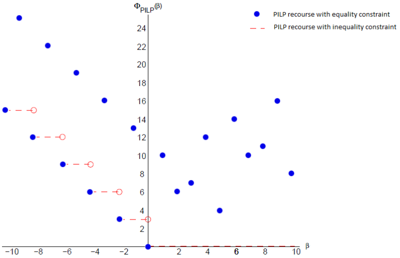
Note the similarity in structure of the objective function to the value function.

MILP Value Function (Pure Integer)

MILP value function is non-convex, discontinuous,and piecewise polyhedral in general.

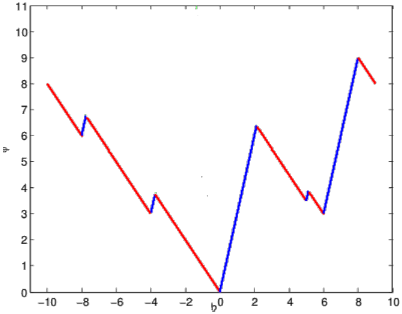
Example 3

$$\phi(b) = \min 3x_1 + \frac{7}{2}x_2 + 3x_3 + 6x_4 + 7x_5 + 5x_6$$
$$s.t.\ 6x_1 + 5x_2 - 4x_3 + 2x_4 - 7x_5 + x_6 = b$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \in \mathbb{Z}_+$$



Example 4

$$\phi(b) = \min\ 3x_1 + \frac{7}{2}x_2 + 3x_3 + 6x_4 + 7x_5 + 5x_6$$
$$s.t.\ 6x_1 + 5x_2 - 4x_3 + 2x_4 - 7x_5 + x_6 = b$$
$$x_1, x_2, x_3 \in \mathbb{Z}_+,\ x_4, x_5, x_6 \in \mathbb{R}_+$$



Continuous and Integer Restriction of an MILP

Consider the general form of the second-stage value function

$$\begin{aligned} \phi(\beta) = \min\, & q_I^\top y_I + q_C^\top y_C \\ \text{s.t. } & W_I y_I + W_C y_C = b, \\ & y \in \mathbb{Z}_+^{r_2} \times \mathbb{R}_+^{n_2 - r_2} \end{aligned} \tag{MILP}$$

The structure is inherited from that of the *continuous restriction*:

$$\begin{aligned} \phi_C(\beta) = \min\, & q_C^\top y_C \\ \text{s.t. } & W_C y_C = \beta, \\ & y_c \in \mathbb{R}_+^{n_2 - r_2} \end{aligned} \tag{CR}$$

and the similarly defined *integer restriction*:

$$\begin{aligned} \phi_I(\beta) = \min\, & q_I^\top y_I \\ \text{s.t. } & W_I y_I = \beta \\ & y_I \in \mathbb{Z}_+^{r_2} \end{aligned} \tag{IR}$$

Discrete Representation of the Value Function

For $\beta \in \mathbb{R}^{m_2}$, we have that

$$\begin{aligned} \phi(\beta) = \min\ & q_I^\top y_I + \phi_C(\beta - W_I y_I) \\ \text{s.t. } & y_I \in \mathbb{Z}_+^{r_2} \end{aligned} \tag{6}$$

- From this we see that the value function is comprised of the minimum of a set of shifted copies of ϕ_C.
- The set of shifts, along with ϕ_C describe the value function exactly.
- For $\hat{y}_I \in \mathbb{Z}_+^{r_2}$, let

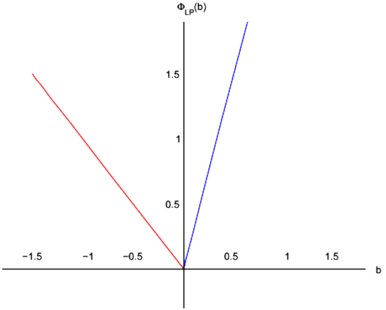
$$\phi_C(\beta, \hat{y}_I) = q_I^\top \hat{y}_I + \phi_C(\beta - W_I \hat{y}_I) \ \ \forall \beta \in \mathbb{R}^{m_2}. \tag{7}$$

- Then we have that $\phi(\beta) = \min_{y_I \in \mathbb{Z}_+^{r_2}} \phi_C(\beta, \hat{y}_I)$.

Illustrating the Continuous Restriction

Example 5

$$\phi_C(\beta) = \min 6y_1 + 7y_2 + 5y_3$$
$$s.t.\ 2y_1 - 7y_2 + y_3 = \beta$$
$$y_1, y_2, y_3 \in \mathbb{R}_+$$



Value Function of the Continuous Restriction

Recall the previously defined continuous restriction.

$$\begin{aligned} \phi_C(\beta) = \min\ & q_C^\top y_C \\ \text{s.t. } & W_C y_C = \beta \\ & y_C \in \mathbb{R}^n_+ \end{aligned} \tag{CR}$$

When the dual of (CR) is feasible, the epigraph of ϕ_C is the convex cone

$$\mathcal{L} := cone\{(W_1, q_1), (W_2, q_2), \ldots, (W_n, q_n), (0, 1)\} \tag{8}$$

Let $u_1, \ldots, u_k$ be extreme points of the feasible region of the dual of (CR) and $d_1, \ldots, d_p$ be its extreme directions. Then

$$\mathcal{L} := \{(\beta, z) : z \geq u_i^\top \beta, i = 1, \ldots, k, d_j^\top \beta \leq 0, j = 1, \ldots, p\}. \tag{9}$$

Properties of MILP Value Function

- We can improve on the previous representation by deriving a *minimal* discrete set that suffices to describe ϕ.

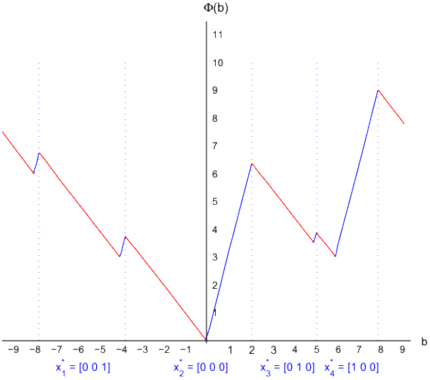
Theorem 1 *[Hassanzadeh et al., 2014]*
Under the assumption that $\{\beta \in \mathbb{R}^{m_2} \mid \phi_I(\beta) < \infty\}$ *is finite, there exists a finite set* $\mathcal{S} \subseteq Y$ *such that*

$$\phi(\beta) = \min_{y_I \in \mathcal{S}} \{q_I^\top y_I + \phi_C(\beta - W_I y_I)\}. \tag{10}$$

- The points in $\mathcal{S}$ are the points of *strict local convexity* of the value function.
- Associated with each of these points is a region (the *local stability set*) over which the integer part of the optimal solution remains constant.
- The value function of the MILP, when restricted to that region, is a translation of the value function of the continuous restriction (and thus convex).
- In [Hassanzadeh et al., 2014], we describe an algorithm for constructing a superset of $\mathcal{S}$ that is easy to implement.

Points of Strict Local Conexity

Example 6



The figure above shows the points of strict local convexity and the associated local stability sets for the previous example.

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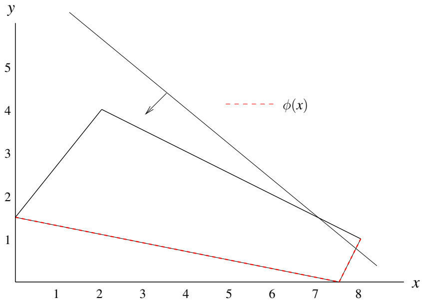
Benders' Principle (Linear Programming)

$$z_{\text{LP}} = \min_{(x,y)\in\mathbb{R}^n} \left\{ c'x + c''y \;\middle|\; A'x + A''y \geq b \right\}$$
$$= \min_{x\in\mathbb{R}^{n'}} \left\{ c'x + \phi(b - A'x) \right\},$$

where

$$\phi(d) = \min c''y \quad \text{s.t. } A''y \geq d \quad y \in \mathbb{R}^{n''}$$

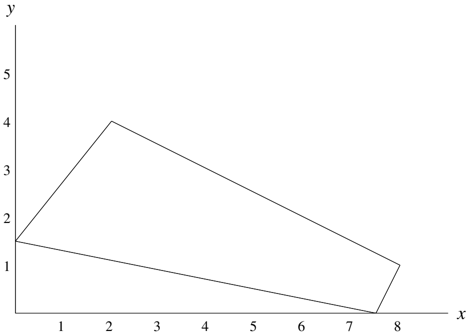


Basic Strategy:

- The function ϕ is the *value function* of a linear program.
- The value function is piecewise linear and convex.
- We iteratively generate a lower approximation by sampling the domain.

Example

$$
\begin{aligned}
z_{LP} \quad = \quad \min \quad & x + y \\
\text{s.t.} \quad & 25x - 20y \geq -30 \\
& -x - 2y \geq -10 \\
& -2x + y \geq -15 \\
& 2x + 10y \geq 15 \\
& x, y \in \mathbb{R}
\end{aligned}
$$

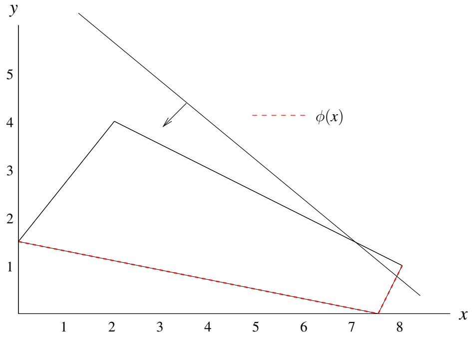


Value Function Reformulation

$$z_{LP} \quad = \quad \min_{x \in \mathbb{R}} x + \phi(x),$$

where

$$\begin{aligned} \phi(x) \quad = \quad \min \quad & y \\ \text{s.t.} \quad -20y & \geq -30 - 25x \\ -2y & \geq -10 + x \\ y & \geq -15 + 2x \\ 10y & \geq 15 - 2x \\ y & \in \mathbb{R} \end{aligned}$$

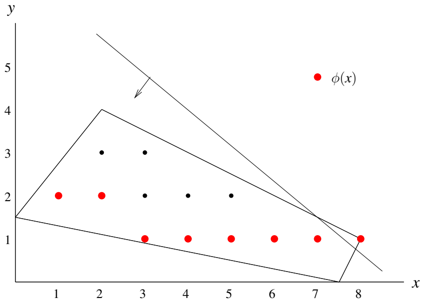


Benders' Principle (Integer Programming)

$$z_{\mathrm{IP}} = \min_{(x,y)\in\mathbb{Z}^n} \left\{ c'x + c''y \;\middle|\; A'x + A''y \geq b \right\}$$
$$= \min_{x\in\mathbb{R}^{n'}} \left\{ c'x + \phi(b - A'x) \right\},$$

where

$$\phi(d) = \min c''y \quad \text{s.t. } A''y \geq d \quad y \in \mathbb{Z}^{n''}$$

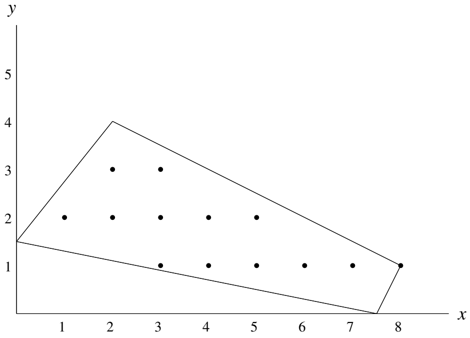


Basic Strategy:

- Here, ϕ is the value function of an *integer program*.
- In the general case, the function ϕ is piecewise linear but not convex.
- Here, we also iteratively generate a lower approximation by evaluating ϕ.

Example

$$
\begin{aligned}
z_{IP} \quad = \quad \min \quad & x + y \\
\text{s.t.} \quad & 25x - 20y \geq -30 \\
& -x - 2y \geq -10 \\
& -2x + y \geq -15 \\
& 2x + 10y \geq 15 \\
& x, y \in \mathbb{Z}
\end{aligned}
$$

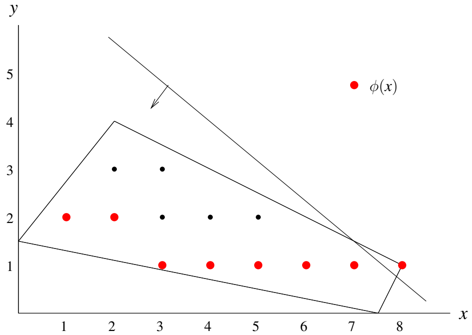


Value Function Reformulation

$$z_{IP} = \min_{x\in\mathbb{Z}} x + \phi(x),$$

where

$$\begin{aligned}\phi(x) = \min \quad & y \\ \text{s.t.} \quad & -20y \geq -30 - 25x \\ & -2y \geq -10 + x \\ & y \geq -15 + 2x \\ & 10y \geq 15 - 2x \\ & y \in \mathbb{Z}\end{aligned}$$



Related Algorithms

The algorithmic framework we utilize builds on a number of previous works.

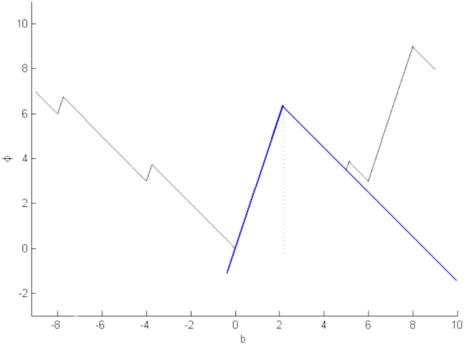
- Modification to the L-shaped framework [Laporte and Louveaux, 1993, Carøe and Tind, 1998, Sen and Higle, 2005]
  - Linear cuts in first stage for binary first stage
  - Optimality cuts from B&B and cutting plane, applied to pure integer second stage
  - Disjunctive programming approaches and cuts in the second stage
- Value function approaches: Pure integer case [Ahmed et al., 2004, Kong et al., 2006]
- Scenario decomposition [Carøe and Schultz, 1998]
- Enumeration/Gröbner basis reduction [Schultz et al., 1998]

Summary of Related Work

	First Stage			Second Stage			Stochasticity			
	$\mathbb{R}$	$\mathbb{Z}$	$\mathbb{B}$	$\mathbb{R}$	$\mathbb{Z}$	$\mathbb{B}$	$\mathbf{W}$	$\mathbf{T}$	$\mathbf{h}$	$\mathbf{q}$
[Laporte and Louveaux, 1993]			*	*	*	*	*	*	*	
[Carøe and Tind, 1997]	*		*	*		*	*	*	*	*
[Carøe and Tind, 1998]	*	*	*		*	*		*	*	
[Carøe and Schultz, 1998]	*	*	*	*	*	*		*	*	*
[Schultz et al., 1998]	*				*	*			*	
[Sherali and Fraticelli, 2002]			*	*		*	*	*	*	*
[Ahmed et al., 2004]	*	*	*		*	*	*		*	*
[Sen and Higle, 2005]			*	*		*		*	*	
[Sen and Sherali, 2006]			*	*	*	*		*	*	
[Sherali and Zhu, 2006]	*		*	*		*	*	*	*	
[Kong et al., 2006]		*	*		*	*	*	*	*	*
[Sherali and Smith, 2009]			*	*		*	*	*	*	*
[Yuan and Sen, 2009]			*	*		*		*	*	*
[Ntaimo, 2010]			*	*		*	*			*
[Gade et al., 2012]			*		*	*	*	*	*	*
[Trapp et al., 2013]		*	*		*	*			*	
Current work	*	*	*	*	*	*		*	*	

Lower Bounds on the Value Function

We already observed that for an effective integer Benders' method, we need effective lower *bounding functions* to *approximate* the MILP value function.



Dual Functions

A dual function $\varphi : \mathbb{R}^{m_2} \to \mathbb{R} \cup \{\pm\infty\}$ is

$$\varphi(\beta) \leq \phi(\beta) \quad \forall \boldsymbol{b} \in \mathbb{R}^{m_2} \tag{11}$$

For a particular instance $\hat{\beta}$, the dual problem is

$$\phi_D = \max\{\varphi(\hat{\beta}) : \varphi(\beta) \leq \phi(\beta) \quad \forall \beta \in \mathbb{R}^{m_2},\ \varphi : \mathbb{R}^{m_2} \to \mathbb{R} \cup \{\pm\infty\}\} \tag{12}$$

Let $\mathcal{F}$ be a set of dual functions generated so far. Then Benders' master problem is

$$\begin{aligned} \min\ & c^\top x + \theta \\ & \theta \geq \sum_{\omega \in \Omega} \max_{f \in \mathcal{F}} f(h_\omega - T_\omega x)] \\ & x \in \mathcal{P}_1 \end{aligned} \tag{MP}$$

MILP Duals from Branch-and-Bound

Let T be set of the terminating nodes of the tree. Then in a terminating node $t \in T$ we solve:

$$\begin{aligned} \min\ & c^\top x \\ \text{s.t. } & Ax = b, \\ & l^t \le x \le u^t, x \ge 0 \end{aligned} \tag{13}$$

The dual at node t:

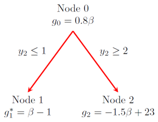
$$\begin{aligned} \max\ & \{\pi^t b + \underline{\pi}^t l^t + \bar{\pi}^t u^t\} \\ \text{s.t. } & \pi^t A + \underline{\pi}^t + \bar{\pi}^t \le c^\top \\ & \underline{\pi} \ge 0, \bar{\pi} \le 0 \end{aligned} \tag{14}$$

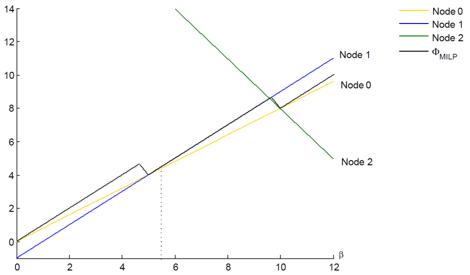
We obtain the following strong dual function:

$$\min_{t \in T} \{\pi^t b + \underline{\pi}^t l^t + \bar{\pi}^t u^t\} \tag{15}$$

Warm Starting the Solution Process

- Here, we illustrate the procedure.
- We can improve on the basic scheme by warm starting the solution of each subproblem from the tree generated during solution of the previous subproblem.





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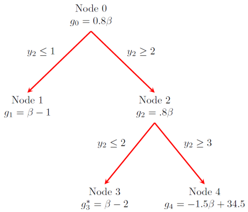

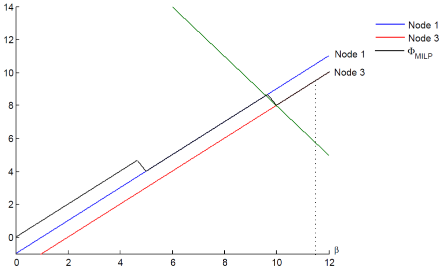

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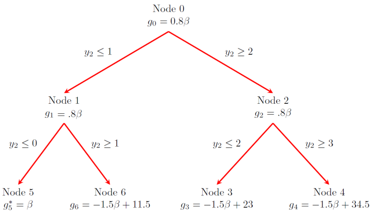
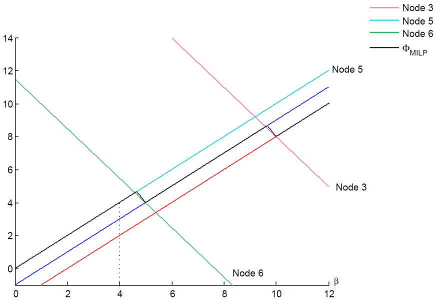
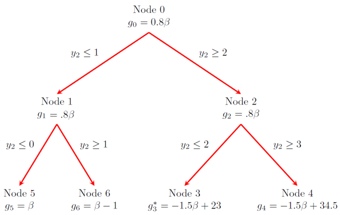
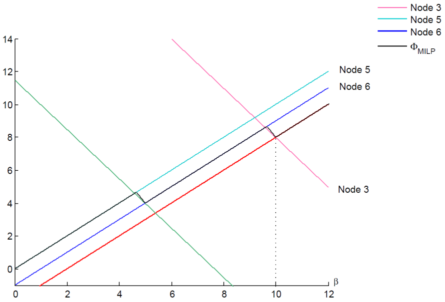




- Here, we illustrate the procedure.
- We can improve on the basic scheme by warm starting the solution of each subproblem from the tree generated during solution of the previous subproblem.





Generating the Value Function in a Single Tree

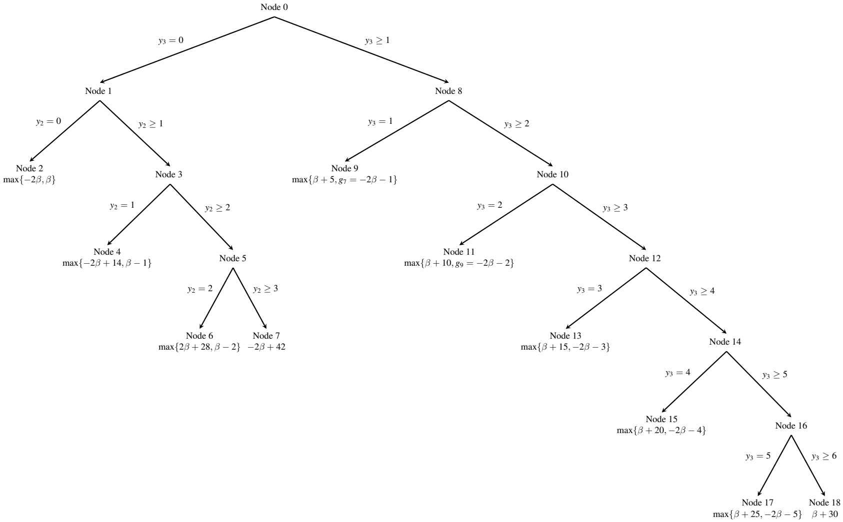
- Continuing the process, we eventually generate the entire value function.
- Consider the strengthened dual

$$\underline{\phi}^*(\beta) = \min_{t \in T} q_{I_t}^\top y_{I_t}^t + \phi_{N \setminus I_t}(\beta - W_{I_t} y_{I_t}^t), \tag{16}$$

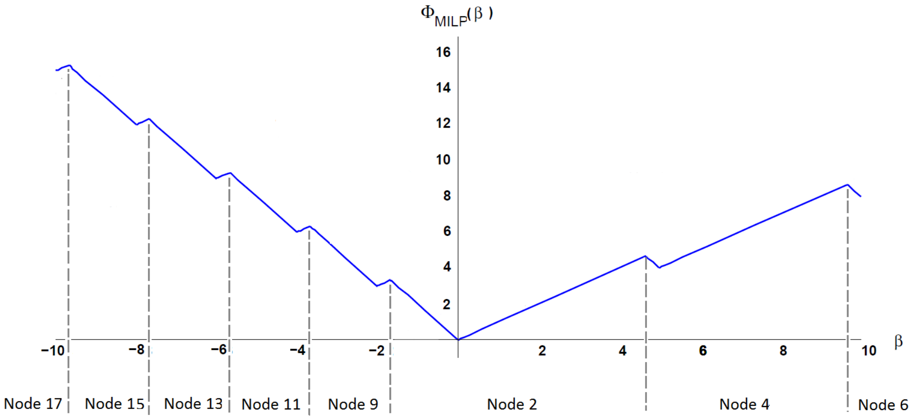
- I_t is the set of indices of fixed variables, $y_{I_t}^t$ are the values of the corresponding variables in node t.
- $\phi_{N \setminus I_t}$ is the value function of the linear program including only the unfixed variables.

Theorem 2 *Under the assumption that* $\{\beta \in \mathbb{R}^{m_2} \mid \phi_I(\beta) < \infty\}$ *is finite, there exists a branch-and-bound tree with respect to which* $\underline{\phi}^* = \phi$.

Example of Value Function Tree



Example of Value Function Tree



Master Problem Formulation

Notation:

- $s, r \in \{1, \ldots, S\}$ where S is the number of scenarios
- $p \in \{1, \ldots, k\}$ where k is the iteration number
- $n \in \{1, \ldots, N(p, r)\}$ where $N(p, r)$ is the number of terminating nodes in the B&B tree solved for scenario r at iteration p.
- $\theta_s = \mathcal{F}(h(s) - \beta)$
- $t_{spr} = F_r^p(h(s) - \beta)$ the approximation of scenario s's recourse obtained from the optimal dual function of iteration p and scenario r.
- ν_{prn}, a_{prn} respectively, the dual vector and intercept obtained from node n of the B&B tree solved for scenario r in iteration p.
- p_s probability of scenario s
- $M > 0$ an appropriate large number

Master Problem Formulation

$$
\begin{aligned}
f^k = \min\ & c^\top x + \sum_{s=1}^{S} p_s \theta_s \\
\text{s.t. } & \theta_s \geq t_{spr} && \forall s, p, r \\
& t_{spr} \leq a_{prn} + \nu_{prn}^\top (h(s) - T(s)x) && \forall s, r, p, n \\
& t_{spr} \geq a_{prn} + \nu_{prn}^\top (h(s) - T(s)x) - M u_{sprn} && \forall s, p, r, n \\
& \sum_{n=1}^{N} u_{sprn} = N(p, r) - 1 && \forall s, p, r \\
& x \in X, u_{sprn} \in \mathbb{B} && \forall s, p, r, n
\end{aligned}
\tag{master}
$$

Example

Consider

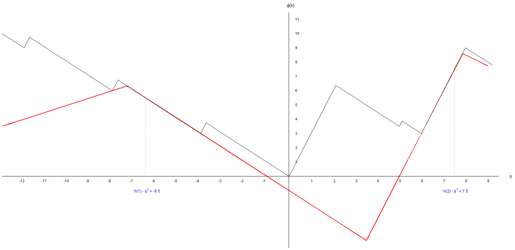
$$\min f(x) = \min \; -3x_1 - 4x_2 + \sum_{s=1}^{2} 0.5Q(x,s) \quad \text{s.t. } x_1 + x_2 \leq 5, \quad x \in \mathbb{Z}_+ \tag{17}$$

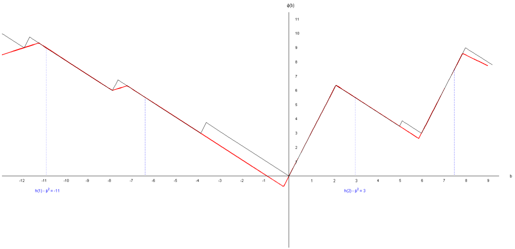
where

$$Q(x,s) = \min 3y_1 + \frac{7}{2}y_2 + 3y_3 + 6y_4 + 7y_5 \quad \text{s.t. } 6y_1 + 5y_2 - 4y_3 + 2y_4 - 7y_5 = h(s) - 2x_1 - \frac{1}{2}x_2, \quad y_1, y_2, y_3 \in \mathbb{Z}_+,\; y_4, y_5 \in \mathbb{R}_+ \tag{18}$$

with $h(s) \in \{-4, 10\}$.

Example



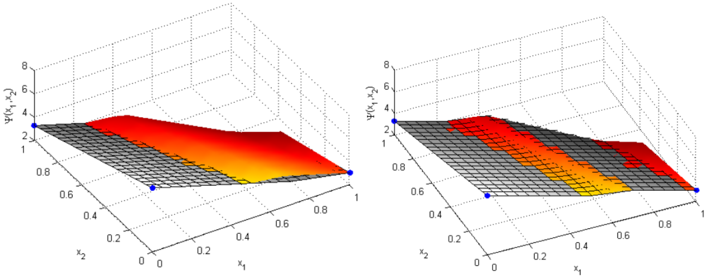


Outline

1. Introduction
2. Value Function
3. Algorithm
4. Conclusions

Conclusions

Non-convex optimality cuts are ugly. But they may be worthwhile!



- We have developed an algorithm for the two-stage problem with general mixed integer in both stages.
- The algorithm uses the Benders' framework with B&B dual functions as the optimality cuts.
- Such cuts have computationally desirable properties such as warm-starting.
- We need to keep the size of approximations small. This can be done through warm-starting trees and scenario bunching.

Future Work

- We have implemented the algorithm using SYMPHONY as our mixed-integer linear optimization solver.
- Warm-starting a B&B tree is possible in the solver.
- We so far have a fairly “naive” implementation and anticipate much improvment is possible.
- In particular, we shuld be able to exploit parallelism much more easily here than in the traditional MILP case.
- We also need to develop a scenario bunching scheme. Doing this, we decide on the local area of the tree to examine.
- Finally, we hope to move on soon to the more general case of multilevel programming.

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