# (Duality), Warm Starting, and Sensitivity Analysis for MILP

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# Outline of Talk

- A little bit of theory
  - Duality
  - Sensitivity analysis
  - Warm starting
- A little bit of implementation
  - SYMPHONY 5.0
  - Examples

# Introduction to Duality

- For an optimization problem

$$z = \min\{f(x) \mid x \in X\},$$

called the *primal problem*, an optimization problem

$$w = \max\{g(u) \mid u \in U\}$$

such that $w \leq z$ is called a *dual problem*.

- It is a *strong dual* if $w = z$.
- Uses for the dual problem
  - Bounding
  - Deriving optimality conditions
  - Sensitivity analysis
  - Warm starting

# Some Previous Work

- R. Gomory (and W. Baumol) ('60–'73)
- G. Roodman ('72)
- E. Johnson (and Burdet) ('72–'81)
- R. Jeroslow (and C. Blair) ('77-'85)
- A. Geoffrion and R. Nauss ('77)
- D. Klein and S. Holm ('79–'84)
- L. Wolsey (and L. Schrage) ('81–'84)
- ...
- D. Klabjan ('02)

# Duals for ILP

- Let $\mathcal{P} = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ nonempty for $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$.
- We consider the (bounded) pure integer linear program $\min_{x \in \mathcal{P} \cap \mathbb{Z}^n} c^\top x$ for $c \in \mathbb{R}^n$.
- The most common dual for this ILP is the well-known *Lagrangian dual*.
  - The Lagrangian dual is not generally strong.
  - Blair and Jeroslow discussed how to make the Lagrangian dual strong by for ILP by introducing a quadratic penalty term.
- How do we derive a strong dual? Consider the following more formal notion of dual (Wolsey).

$$w^g_{IP} = \max_{g:\mathbb{R}^m \to \mathbb{R}} \{g(b) \mid g(Ax) \leq c^\top x, x \geq 0\} \tag{1}$$

$$= \max_{g:\mathbb{R}^m \to \mathbb{R}} \{g(b) \mid g(d) \leq z_{IP}(d), d \in \mathbb{R}^m\}, \tag{2}$$

where $z_{IP}(d) = \min_{x \in \mathcal{P}^I(d)} c^\top x$ is the *value function* and $\mathcal{P}^I(d) = \{x \in \mathbb{Z}^n \mid Ax = d, x \geq 0\}$

# Subadditive Duality

- Solutions to the dual (2) bound the value function from below.
- Any function that agrees with the value function at $b$, including the value function itself, is optimal.
- This shows the dual (2) is strong.
- Question: Under what restrictions on the function $g$ does this remain a strong dual?
  - $g$ linear results in the dual of the LP relaxation $\Rightarrow$ not strong.
  - $g$ convex also results in the dual of the LP relaxation $\Rightarrow$ not strong. (Jeroslow)
  - $g$ subadditive $\Rightarrow$ strong (Gomory; Johnson; Jeroslow).
  - In this case, the dual simplifies to

$$w_{IP}^{s} = \max\{f(b) \mid f(a^i) \leq c_i, f \text{ subadditive}\},$$

  - This is called the *subadditive dual*

# Optimal Solutions to the Subadditive Dual

- The subadditive dual has most of the nice properties of the LP dual.
- With an optimal solution, we can calculate reduced costs, perform local sensitivity analysis, etc.
- Again, the value function is subadditive and hence optimal.
- Blair and Jeroslow showed the value function has a closed form.
- One can produce the value function from a cutting plane proof of optimality obtained using the Gomory procedure.
- What about other optimal solutions?
  - Different solutions estimate the value function differently.
  - One would like to produce such a solution as a by-product of an efficient algorithm.
  - Wolsey discusses how to do this with several classes of algorithms.
  - We will focus here on branch and bound.

# Dual Solutions from Primal Algorithms

- The approach is to consider the implicit optimality conditions associated with an algorithm such as branch and bound.
- Let $\mathcal{P}_1, \ldots, \mathcal{P}_s$ be a partition of $\mathcal{P}$ into (nonempty) subpolyhedra.
- Let $LP_i$ be the linear program $\min_{x^i \in \mathcal{P}_i} c^\top x^i$ associated with the subpolyhedron $\mathcal{P}_i$.
- Let $B^i$ be an optimal basis for $LP_i$.
- Then the following is a valid lower bound

$$L = \min\{c_{B^i}(B^i)^{-1}b + \gamma_i \mid 1 \leq i \leq s\},$$

  where $\gamma_i$ is the constant factor associated with the nonbasic variables fixed at nonzero bounds.
- A similar function yields an upper bound.
- We call a partition that yields lower and upper bounds equal is called an *optimal partition*.
- Note the similarity to LP duality.

# Sensitivity Analysis

- The function

$$L(d) = \min\{c_{B^i}(B^i)^{-1}d + \gamma_i \mid 1 \le i \le s\},$$

  is not subadditive, but provides an optimal solution to (2).

- Here is the corresponding upper bounding function

$$U(c) = \min\{c_{B^i}(B^i)^{-1}b + \beta_i \mid 1 \le i \le s, \hat{x}^i \in \mathcal{P}^I\}$$

- These functions can be used for local sensitivity analysis, just as one would do in linear programming.
  - For changes in the right-hand side, the lower bound remains valid.
  - For changes in the objective function, the upper bound remains valid.
  - One can also add cuts and variables.

# Some Details

- Note that we've swept some things under the carpet:
  - The "allowable range" is the intersection of the ranges for each member of the partition, so it may be very small or empty.
  - The method presented only applies to branch and bound.
  - Cut generation complicates matters.
  - Fixing by reduced cost also complicates matters.
  - Have to deal with infeasibility of subproblems.
- Question: What happens outside the allowable range?
- Answers:
  - Continue solving from a "warm start."
  - Perform a parametric analysis.

# Warm Starting

- Question: What is "warm starting"?
- Question: Why are we interested in it?
- There are many examples of algorithms that solve a sequence of related ILPs.
  - Decomposition algorithms
  - Stochastic ILP
  - Parametric/Multicriteria ILP
  - Determining irreducible inconsistent subsystem
  - 
- For such problems, warm starting can potentially yield big improvements.
- Warm starting is also important for performing sensitivity analysis outside of the allowable range.

# Warm Starting Information

- Question: What is "warm starting information"?
- Many optimization algorithms can be viewed as iterative procedures for satisfying a set of optimality conditions, often based on duality.
- These conditions provide a measure of "distance from optimality."
- Warm starting information can be seen as additional input data that allows an algorithm to quickly get "close to optimality."
- In linear and integer linear programming, the *duality gap* is the usual measure.
- A starting basis can reduce the initial duality gap in LP.
- The corresponding concept in ILP is a *starting partition*.
- It is not at all obvious what makes a good starting partition.
- The most obvious choice for a starting partition is to use the optimal partition from a previous computation.

# Parametric Analysis

- For global sensitivity analysis, we need to solve parametric programs.
- Along with Saltzman and Wiecek, we have developed an algorithm for determining all Pareto outcomes for a bicriteria MILP.
- The algorithm consists of solving a sequence of related ILPs and is asymptotically optimal.
- Such an algorithm can be used to perform global sensitivity analysis by constructing a "slice" of the value function.
- Warm starting can be used to improve efficiency.

# SYMPHONY 5.0

- Overview
  - A callable library for solving mixed-integer linear programs with a wide variety of customization options.
  - Core solution methodology is branch, cut, and price.
  - Outfitted as a generic MILP solver.
  - Fully integrated with the Computational Infrastructure for Operations Research (COIN-OR) libraries.
  - Extensive documentation available.
  - Stable versions available from `www.branchandcut.org` or current source at CVS on `www.coin-or.org`.
  - All of the methods discussed in this talk are in SYMPHONY 5.0.
- SYMPHONY Solvers
  - Generic MILP
  - Multicriteria MILP
  - Traveling Salesman Problem
  - Vehicle Routing Problem
  - Mixed Postman Problem
  - Set Partitioning Problem
  - Matching Problem
  - Network Routing

# Basic Sensitivity Analysis

- SYMPHONY will calculate bounds after changing the objective or right-hand side vectors.

```
int main(int argc, char **argv)
{
   OsiSymSolverInterface si;
   si.parseCommandLine(argc, argv);
   si.loadProblem();
   si.setSymParam(OsiSymSensitivityAnalysis, true);
   si.initialSolve();
   int ind[2];
   double val[2];
   ind[0] = 4;    val[0] = 7000;
   ind[1] = 7;    val[1] = 6000;
   lb = si.getLbForNewRhs(2, ind, val);
}
```

# Warm Starts for MILP

- To allow resolving from a warm start, we have defined a SYMPHONY warm start class, which is derived from `CoinWarmStart`.
- The class stores a snapshot of the search tree, with node descriptions including:
  - lists of active cuts and variables,
  - branching information,
  - warm start information, and
  - current status (candidate, fathomed, etc.).
- The tree is stored in a compact form by storing the node descriptions as differences from the parent.
- Other auxiliary information is also stored, such as the current incumbent.
- A warm start can be saved at any time and then reloaded later.
- The warm starts can also be written to and read from disk.

# Warm Starting Procedure

- After modifying parameters
  - If only parameters have been modified, then the candidate list is recreated and the algorithm proceeds as if left off.
  - This allows parameters to be tuned as the algorithm progresses if desired.
- After modifying problem data
  - Currently, we only allow modification of rim vectors.
  - After modification, all leaf nodes must be added to the candidate list.
  - After constructing the candidate list, we can continue the algorithm as before.
- There are many opportunities for improving the basic scheme, especially when solving a known family of instances (Geoffrion and Nauss)

# Warm Starting Example (Parameter Modification)

- The following example shows a simple use of warm starting to create a dynamic algorithm.

```
int main(int argc, char **argv)
{
   OsiSymSolverInterface si;
   si.parseCommandLine(argc, argv);
   si.loadProblem();
   si.setSymParam(OsiSymFindFirstFeasible, true);
   si.setSymParam(OsiSymSearchStrategy, DEPTH_FIRST_SEARCH);
   si.initialSolve();
   si.setSymParam(OsiSymFindFirstFeasible, false);
   si.setSymParam(OsiSymSearchStrategy, BEST_FIRST_SEARCH);
   si.resolve();
}
```

# Warm Starting Example (Problem Modification)

- The following example shows how to warm start after problem modification.

```
int main(int argc, char **argv)
{
   OsiSymSolverInterface si;
   CoinWarmStart ws;
   si.parseCommandLine(argc, argv);
   si.loadProblem();
   si.setSymParam(OsiSymNodeLimit, 100);
   si.initialSolve();
   ws = si.getWarmStart();
   si.resolve();
   si.setObjCoeff(0, 1);
   si.setObjCoeff(200, 150);
   si.setWarmStart(ws);
   si.resolve();
}
```

# Example: Warm Starting

- Consider the simple warm-starting code from earlier in the talk.
- Applying this code to the MIPLIB 3 problem p0201, we obtain the results below.
- Note that the warm start doesn't reduce the number of nodes generated, but does reduce the solve time dramatically.

| | CPU Time | Tree Nodes |
|---|---|---|
| Generate warm start | 28 | 100 |
| Solve orig problem (from warm start) | 3 | 118 |
| Solve mod problem (from scratch) | 24 | 122 |
| Solve mod problem (from warm start) | 6 | 198 |

# Using Warm Starting: Network Routing

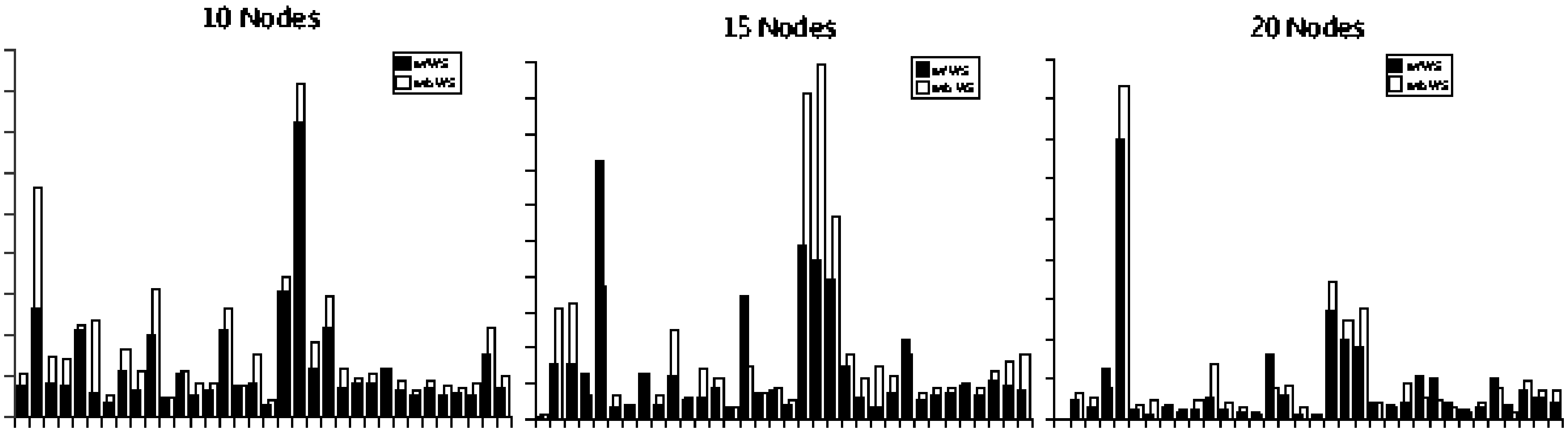



Table 1: Results of using warm starting to solve multi-criteria optimization problems.

# Using Warm Starting: Stochastic Integer Programming

| Problem | Tree Size Without WS | Tree Size With WS | % Gap Without WS | % Gap With WS | CPU Without WS | CPU With WS |
|---|---|---|---|---|---|---|
| storm8 | 1 | 1 | - | - | 14.75 | 8.71 |
| storm27 | 5 | 5 | - | - | 69.48 | 48.99 |
| storm125 | 3 | 3 | - | - | 322.58 | 176.88 |
| LandS27 | 71 | 69 | - | - | 6.50 | 4.99 |
| LandS125 | 37 | 29 | - | - | 15.72 | 12.72 |
| LandS216 | 39 | 35 | - | - | 30.59 | 24.80 |
| dcap233_200 | 39 | 61 | - | - | 256.19 | 120.86 |
| dcap233_300 | 111 | 89 | 0.387 | - | 1672.48 | 498.14 |
| dcap233_500 | 21 | 36 | 24.701 | 14.831 | 1003 | 1004 |
| dcap243_200 | 37 | 53 | 0.622 | 0.485 | 1244.17 | 1202.75 |
| dcap243_300 | 64 | 220 | 0.0691 | 0.0461 | 1140.12 | 1150.35 |
| dcap243_500 | 29 | 113 | 0.357 | 0.186 | 1219.17 | 1200.57 |
| sizes3 | 225 | 165 | - | - | 789.71 | 219.92 |
| sizes5 | 345 | 241 | - | - | 964.60 | 691.98 |
| sizes10 | 241 | 429 | 0.104 | 0.0436 | 1671.25 | 1666.75 |

# Example: Bicriteria ILP

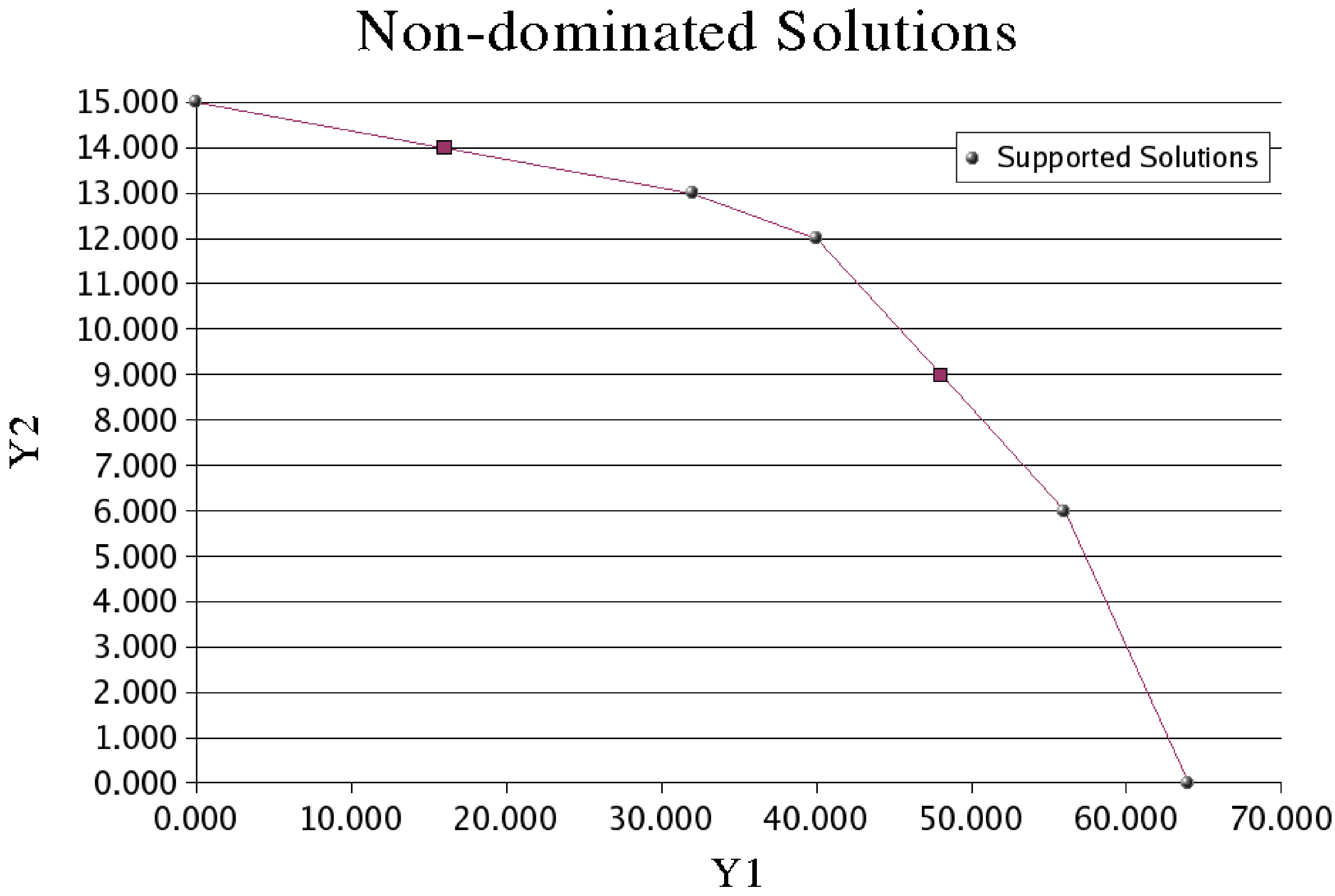
- Consider the following bicriteria ILP:

$$\begin{aligned} \text{vmax } & [8x_1 + x_2, x_2] \\ \text{s.t. } & 7x_1 + x_2 \leq 56 \\ & 28x_1 + 9x_2 \leq 252 \\ & 3x_1 + 7x_2 \leq 105 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- We get the set of Pareto outcomes with the following code.

```
int main(int argc, char **argv)
{
   OsiSymSolverInterface si;
   si.parseCommandLine(argc, argv);
   si.loadProblem();
   si.setObj2Coeff(0, 1);
   si.setSymParam(OsiSymMCFindSupportedSolutions, true);
   si.multiCriteriaBranchAndBound();
}
```

# Example: Pareto and Supported Outcomes for Example

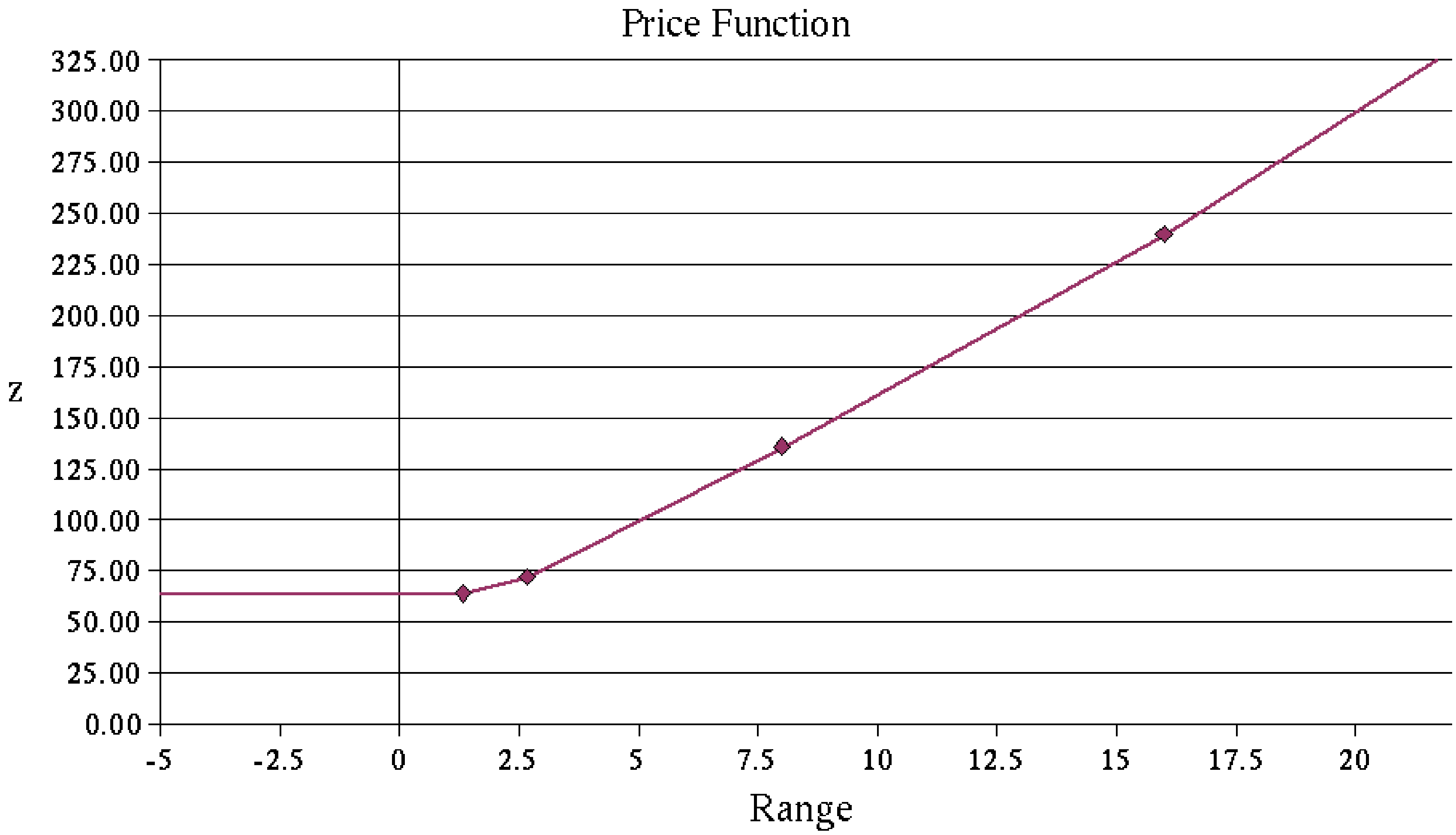


# Example: Bicriteria Solver

- By examining the supported solutions and break points, we can easily determine $p(\theta)$, the optimal solution to the ILP with objective $8x_1 + \theta$.

| $\theta$ range | $p(\theta)$ | $x_1^*$ | $x_2^*$ |
|---|---|---|---|
| $(-\infty, 1.333)$ | 64 | 8 | 0 |
| $(1.333, 2.667)$ | $56 + 6\theta$ | 7 | 6 |
| $(2.667, 8.000)$ | $40 + 12\theta$ | 5 | 12 |
| $(8.000, 16.000)$ | $32 + 13\theta$ | 4 | 13 |
| $(16.000, \infty)$ | $15\theta$ | 0 | 15 |

# Example: Graph of Price Function



# Conclusion

- We have briefly introduced the issues surrounding warm starting and sensitivity analysis for integer programming.
- An examination of early literature has yielded some ideas that can be useful in today's computational environment.
- We presented a new version of the SYMPHONY solver supporting warm starting and sensitivity analysis for MILPs.
- We have also demonstrated SYMPHONY's multicriteria optimization capabilities.
- This work has only scratched the surface of what can be done.
- In future work, we plan on refining SYMPHONY's warm start and sensitivity analysis capabilities.
- We will also provide more extensive computational results.