Branch, Cut, and Price for Capacitated Network Routing

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INFORMS Conference, Miami, November 5, 2001
Outline of Talk

- Introduction and Motivation
- Modeling
- Complexity and Special Cases
- Polyhedral Structure
- Implementation
- Computational Issues and Results
- Future Directions
The Vehicle Routing Problem

The VRP is a combinatorial problem whose ground set is the edges of a graph $G(V, E)$. Notation:

- $V$ is the set of customers and the depot (0).
- $d$ is a vector of the customer demands.
- $k$ is the number of routes.
- $C$ is the capacity of a truck.

A feasible solution is composed of:

- a partition \(\{R_1, \ldots, R_k\}\) of $V$ such that $\sum_{j \in R_i} d_j \leq C$, $1 \leq i \leq k$;
- a permutation $\sigma_i$ of $R_i \cup \{0\}$ specifying the order of the customers on route $i$. 
Standard IP Formulation for the VRP

IP Formulation:

\[
\sum_{j=1}^{n} x_{0j} = 2k \\
\sum_{j=1}^{n} x_{ij} = 2 \quad \forall i \in V \setminus \{0\} \\
\sum_{\substack{i \in S \atop j \not\in S}} x_{ij} \geq 2b(S) \quad \forall S \subset V \setminus \{0\}, \ |S| > 1.
\]

\(b(S) = \text{lower bound on the number of trucks required to service } S\) (normally \(\lceil \left( \sum_{i \in S} d_i \right)/C \rceil\)).

If \(C = \sum_{i \in S} d_i\), then we have the Multiple Traveling Salesman Problem.

Alternatively, if the edge costs are all zero, then we have the Bin Packing Problem.
How hard is the VRP?

- **Test Set**
  - TSPLIB/VRPLIB
  - Augerat’s repository
  - Available at [BranchAndCut.org/VRP](http://BranchAndCut.org/VRP)

- Largest VRP instance solved: F-n135-k7

- Smallest VRP instance unsolved: B-n50-k8

- Largest TSP instance solved: usa13509

- Time to solve B-n50-k8 as an MTSP: .1 sec

- Why the gap?
Motivation

• Why is the Vehicle Routing Problem difficult?

• It is the intersection of two difficult problems.
  – Traveling Salesman Problem (Routing)
  – Bin Packing Problem (Packing)

• We don’t have an effective, polynomially sized relaxation.

• Current approaches treat it as a routing problem.

• We know very little about the packing aspect.

• Idea: Consider flow-based formulations.
Node Routing

- We are given an undirected graph $G = (V, E)$.
  - The nodes represent supply/demand points.
- We consider problems with one supply point (the depot).
- A node routing is a directed subgraph $G'$ of $G$ satisfying the following properties:
  - $G'$ is (weakly) connected.
  - The in-degree of each non-depot node is 1.
Capacitated Routing

- A *capacitated node routing* is one in which the demand in each component of $G' \setminus \{0\}$ is $\leq C$.
- Feasible solutions are bin packings.
- This restriction is easily modeled using a flow-based formulation.
- With capacities, we can model the VRP and the Capacitated Spanning Tree Problem (CSTP).
Optimal Node Routing

• Properties of a node routing.
  – It is a spanning arborescence plus (possibly) some edges returning to the depot.
  – There is a unique path from the depot to each demand point.

• We wish to construct a least cost routing.

• Cost Measures
  – Lengths of all edges in $G'$.  
  – Length of all paths from the depot. 
  – Linear combination of these two.
IP Formulation

IP formulation for this routing problem:

\[
\text{Min} \quad \sum_{(i,j) \in A} \gamma c_{ij} x_{ij} + \tau c_{ij} f_{ij}
\]

s.t.

\[
x(\delta(V \setminus \{i\})) = 1 \quad \forall i \in V \setminus \{0\}
\]

\[
f(\delta(V \setminus \{i\})) - f(\delta(\{i\})) = d_i \quad \forall i \in V \setminus \{0\}
\]

\[
f_{ij} \leq C x_{ij} \quad \forall (i, j) \in A
\]

\[
f_{ij} \geq 0 \quad \forall (i, j) \in A
\]

\[
x_{ij} \in \{0, 1\} \quad \forall (i, j) \in A
\]

where:

- \(x_{ij}, x_{ji}\) (fixed-charge variables) indicate whether \(\{i, j\}\) is in the routing and its orientation.

- \(f_{ij}\) (flow variable) represents demand flow from \(i\) to \(j\).
Complexity

• This node routing problem is **NP-complete** even in the uncapacitated case.

• Polynomially solvable special cases.
  
  – $\tau = 0 \Rightarrow$ Minimum Spanning Tree Problem.
  
  – $\gamma = 0 \Rightarrow$ Shortest Paths Tree Problem.
  
  – Note that demands are irrelevant.

• Other special cases.

  – $\tau, \gamma > 0 \Rightarrow$ Cable-Trench Problem.
  
  – $\tau = 0$ and $x(\delta(\{i\})) = 1 \Rightarrow$ Traveling Salesman Problem.
  
  – $\tau > 0$ and $x(\delta(\{i\})) = 1 \Rightarrow$ Variable Cost TSP.
  
  – $x(\delta(V \setminus \{0\})) = x(\delta(\{0\})) = k \Rightarrow$ VRP.
Figure 1: Optimal Uncpacitated Spanning Trees
Figure 2: Uncapacitated and Capacitated Spanning Tree ($\tau = 0$)
Connection to Other Models

- There are connections to many well-studied models.
- The basic model can be seen as an instance of the Fixed-charge Network Flow Problem.
- Removing the upper bounds on the fixed-charge variables yields the Capacitated Network Design Problem.
- We have already mentioned several other related combinatorial models.
- We are looking to make stronger connections among these varied areas of the literature.
Valid Inequalities

• Note that any inequalities valid for the TSP, VRP, or CSTP have counterparts here.

• Many can be strengthened by taking advantage of the directed formulation.

• Fractional Capacity Constraints

\[
\sum_{i \notin S, j \in S} x_{ij} \geq \frac{d(S)}{C}, \ 0 \notin S
\]

• Multi-star Inequalities

\[
\sum_{i \notin S, j \in S} x_{ij} \geq \frac{d(S)}{C} + \frac{\sum_{i \notin S, j \in S} x_{ji}d_i}{C}, \ 0 \notin S
\]
Valid Inequalities

- Rounded Capacity Constraints
  \[ \sum_{i \notin S, j \in S} x_{ij} \geq \lceil d(S)/C \rceil \]

- Generalized, framed capacity constraints
- Combs, Hypo-tours, Clique Clusters
- Path-bin inequalities
Flow Linking

• Note that only the edge variables are required to be integral.

• We use the flow variables to force integrality of the edge variables through \textit{flow linking constraints}.

• Flow Linking Constraints

\[ f_{ij} \leq (C - d_i)x_{ij} \iff x_{ij} \geq \frac{f_{ij}}{C - d_i} \]

\[ f_{ij} - \sum_{k \neq j} f_{jk} \leq x_{ij}d_j \]

• Edge Cuts

\[ x_{ij} + x_{ji} \leq 1 \]
Separation

- The **fractional capacity constraints** and **multi-star inequalities** are automatically satisfied.
- **Flow linking constraints** and **edge cuts** can be included explicitly or separated in **polynomial time**.
- Separating **rounded capacity constraints** is **NP-complete**, but can be done effectively.
- **Heuristic procedures** for other classes have not yet been implemented.
Solver Implementation

- The implementation uses SYMPHONY, a parallel framework for branch, cut, and price (relative of COIN/BCP).

- In SYMPHONY, the user supplies:
  - the initial LP relaxation,
  - separation subroutines,
  - feasibility checker, and
  - other optional subroutines.

- SYMPHONY handles everything else.

- The source code and documentation are available from www.BranchAndCut.org

- For more information, see Workshop TB42.
Preliminary Computation: Formulation Issues

- The new formulation is *polynomial* and yields *stronger relaxations* initially, but there are drawbacks.
- For the VRP, the formulation creates *symmetry*.
- It also seems to make branching less effective.
- There is a related “undirected” formulation which uses one fixed-charge variable per edge.
  - This formulation is smaller and performs much better for the VRP.
  - For the CSTP and CTP, however, the undirected formulation is extremely weak.
Preliminary Computation

• So far, the presence of the flow variables does not seem to help.

• Capacitating the model does increase difficulty significantly.

• Consider relaxations of the VRP.
  – The TSP is very easy relative to the VRP.
  – The CSTP is not much easier than the VRP.

• Versions of these models with positive variable (flow) costs are extremely difficult.
  – Is this due to the upper bound or lower bound?
  – The flow linking constraints are important for these models.
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Conclusions and Future Directions

- So far, this formulation has not proven better than the classical one for the VRP.
- However, we have yet to take advantage of the information provided by the flow variables.
- We need to know much more about polyhedral structure.
- Better flow linking seems to be the key.
- We also need some new branching rules.
- The connection to the network design literature needs to be explored.
- It is easy to generalize the model even further.
  - Pickup and delivery problems.
  - General degree constraints.
MTSP Polytope

BPP/VRP Polytope

Feasible MTSP/Infeasible BPP