Branch, Cut, and Price
for
Capacitated Network Routing

Ted Ralphs
Joe Hartman
Matt Galati

Industrial and Systems Engineering
Lehigh University

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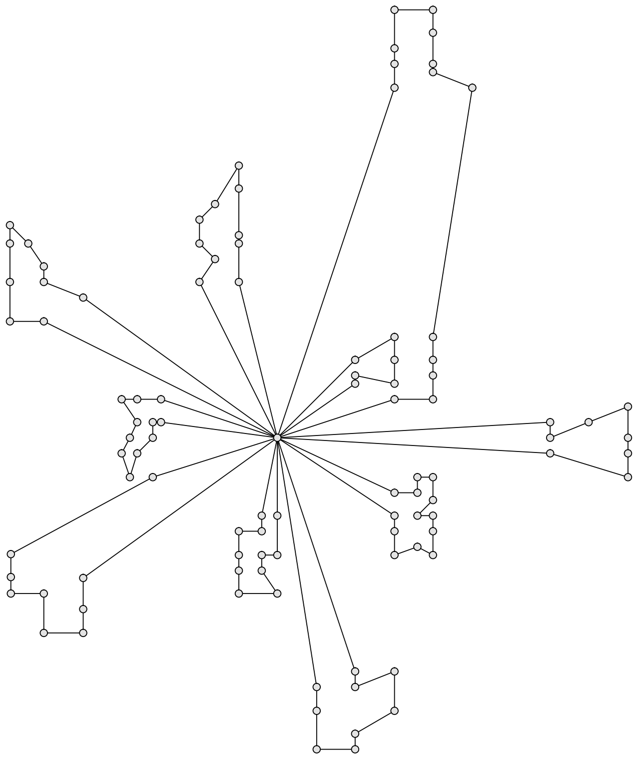
Outline of Talk

- Introduction and Motivation
- Modeling
- Complexity and Special Cases
- Polyhedral Structure
- Implementation
- Computational Issues and Results
- Future Directions



The Vehicle Routing Problem

The VRP is a combinatorial problem whose *ground set* is the edges of a graph $G(V, E)$. Notation:

- V is the set of customers and the depot (0).
- d is a vector of the customer demands.
- k is the number of routes.
- C is the capacity of a truck.

A feasible solution is composed of:

- a partition $\{R_1, \ldots, R_k\}$ of V such that $\sum_{j \in R_i} d_j \leq C,\ 1 \leq i \leq k$;
- a permutation σ_i of $R_i \cup \{0\}$ specifying the order of the customers on route i.

Standard IP Formulation for the VRP

IP Formulation:

$$
\begin{array}{rcll}
\sum_{j=1}^{n} x_{0j} & = & 2k & \\
\sum_{j=1}^{n} x_{ij} & = & 2 & \forall i \in V \setminus \{0\} \\
\sum_{\substack{i \in S \\ j \notin S}} x_{ij} & \geq & 2b(S) & \forall S \subset V \setminus \{0\},\ |S| > 1.
\end{array}
$$

$b(S)$ = lower bound on the number of trucks required to service S (normally $\lceil (\sum_{i \in S} d_i)/C \rceil$).

If $C = \sum_{i \in S} d_i$, then we have the Multiple Traveling Salesman Problem.

Alternatively, if the edge costs are all zero, then we have the Bin Packing Problem.

How hard is the VRP?

- Test Set
  - TSPLIB/VRPLIB
  - Augerat's repository
  - Available at `BranchAndCut.org/VRP`
- Largest VRP instance solved: F-n135-k7
- Smallest VRP instance unsolved: B-n50-k8
- Largest TSP instance solved: usa13509
- Time to solve B-n50-k8 as an MTSP: .1 sec
- Why the gap?

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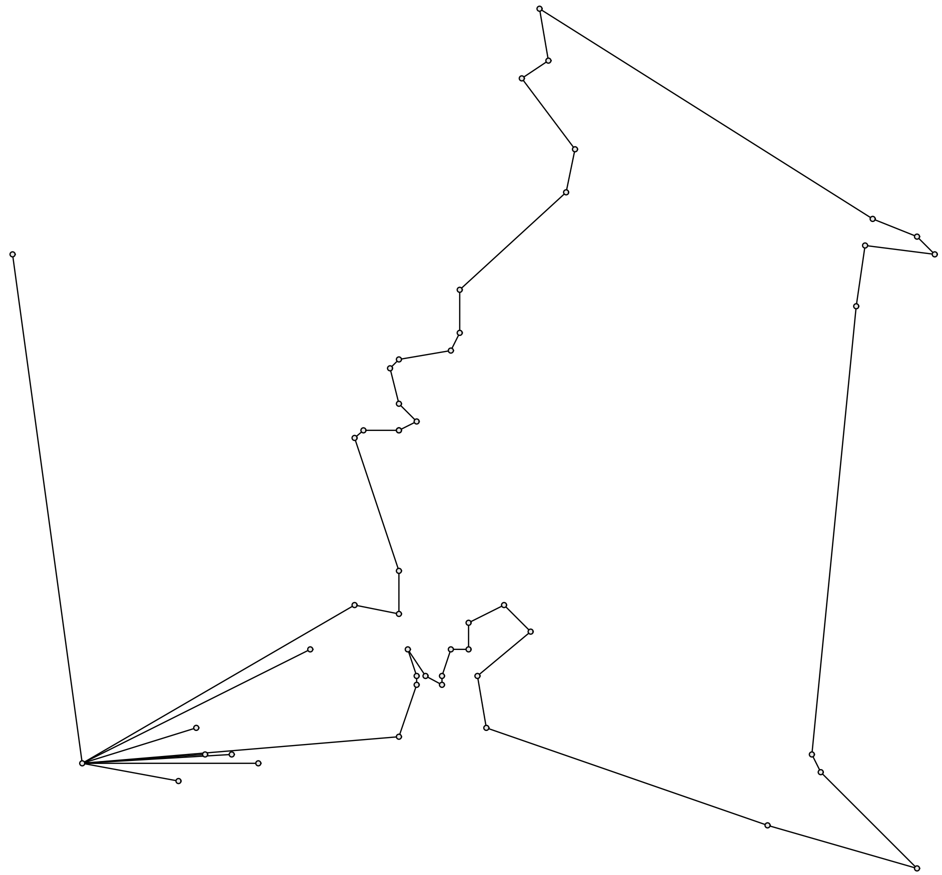
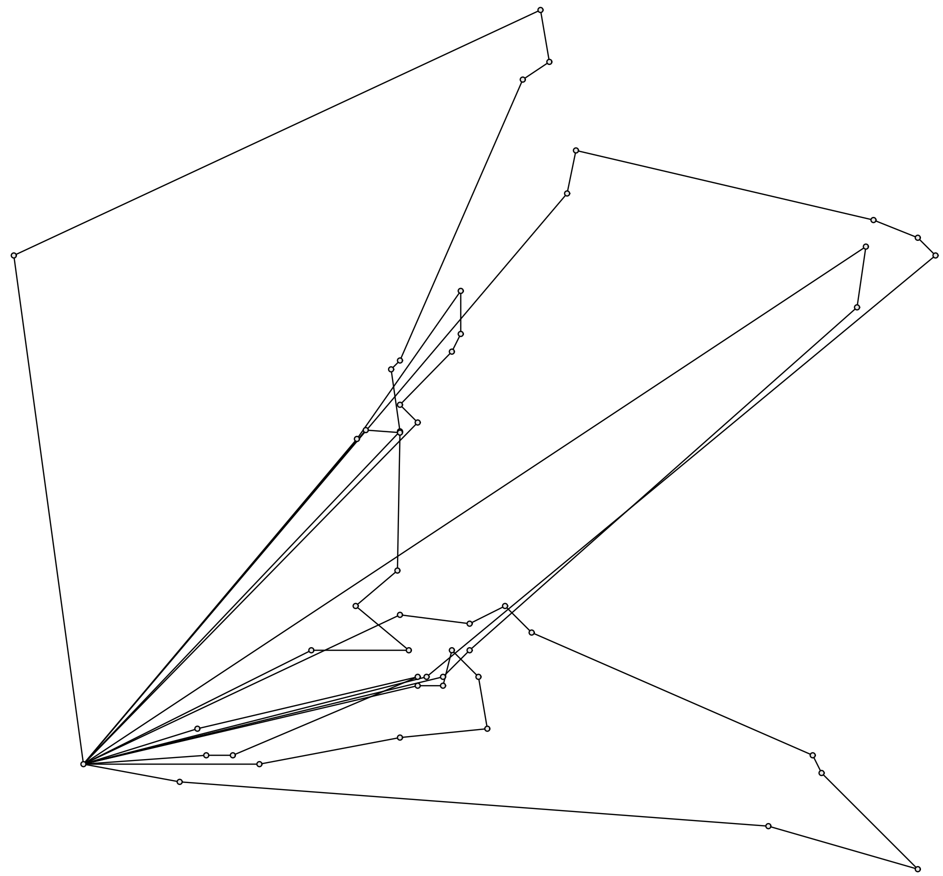
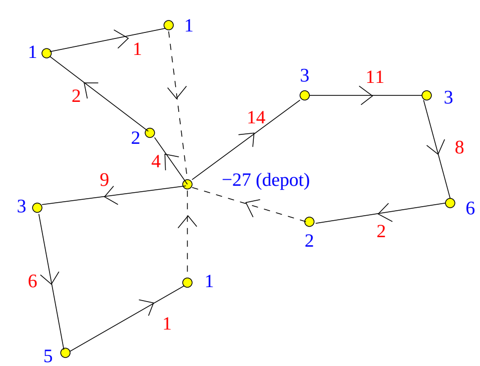




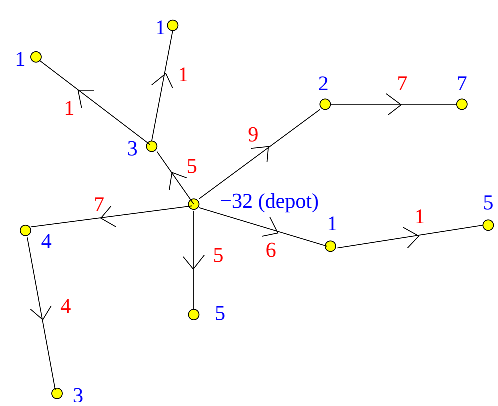
Motivation

- Why is the Vehicle Routing Problem difficult?
- It is the intersection of two difficult problems.
  - Traveling Salesman Problem (Routing)
  - Bin Packing Problem (Packing)
- We don't have an effective, polynomially sized relaxation.
- Current approaches treat it as a routing problem.
- We know very little about the packing aspect.
- Idea: Consider flow-based formulations.



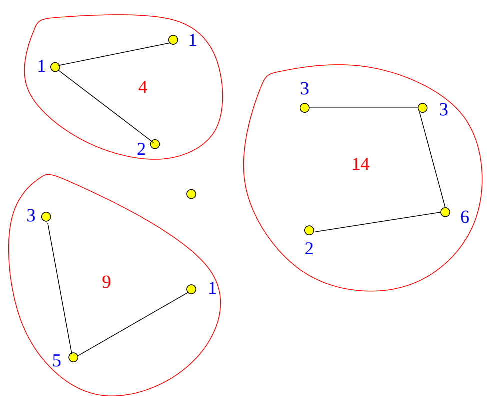
Node Routing

- We are given an undirected graph $G = (V, E)$.
  - The nodes represent supply/demand points.
- We consider problems with one supply point (the *depot*).
- A *node routing* is a directed subgraph G' of G satisfying the following properties:
  - G' is (weakly) connected.
  - The in-degree of each non-depot node is 1.



Capacitated Routing

- A *capacitated node routing* is one in which the demand in each component of $G' \setminus \{0\}$ is $\leq C$.
- Feasible solutions are bin packings.
- This restriction is easily modeled using a flow-based formulation.
- With capacities, we can model the VRP and the Capacitated Spanning Tree Problem (CSTP).

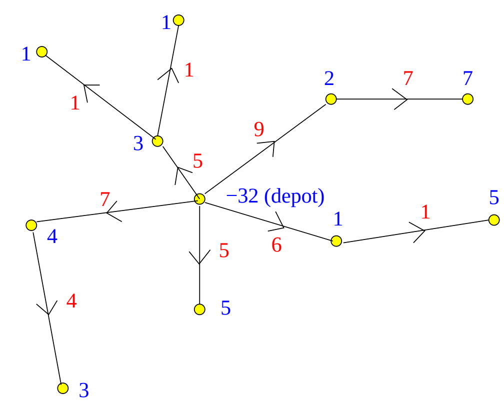


Optimal Node Routing

- Properties of a node routing.
  - It is a spanning arborescence plus (possibly) some edges returning to the depot.
  - There is a unique path from the depot to each demand point.

- We wish to construct a least cost routing.

- Cost Measures
  - Lengths of all edges in G'.
  - Length of all paths from the depot.
  - Linear combination of these two.



IP Formulation

IP formulation for this routing problem:

$$
\begin{aligned}
Min \quad & \sum_{(i,j)\in A} \gamma\, c_{ij}x_{ij} + \tau\, c_{ij}f_{ij} \\
s.t. \quad & x(\delta(V\setminus\{i\})) = 1 && \forall i \in V\setminus\{0\} \\
& f(\delta(V\setminus\{i\})) - f(\delta(\{i\})) = d_i && \forall i \in V\setminus\{0\} \\
& f_{ij} \le C x_{ij} && \forall (i,j) \in A \\
& f_{ij} \ge 0 && \forall (i,j) \in A \\
& x_{ij} \in \{0,1\} && \forall (i,j) \in A
\end{aligned}
$$

where:

- x_{ij}, x_{ji} (*fixed-charge variables*) indicate whether $\{i,j\}$ is in the routing *and its orientation*.
- f_{ij} (*flow variable*) represents demand flow from i to j.

Complexity

- This node routing problem is NP-complete even in the uncapacitated case.
- Polynomially solvable special cases.
  - $\tau = 0 \Rightarrow$ Minimum Spanning Tree Problem.
  - $\gamma = 0 \Rightarrow$ Shortest Paths Tree Problem.
  - Note that demands are irrelevant.
- Other special cases.
  - $\tau, \gamma > 0 \Rightarrow$ Cable-Trench Problem.
  - $\tau = 0$ and $x(\delta(\{i\})) = 1 \Rightarrow$ Traveling Salesman Problem.
  - $\tau > 0$ and $x(\delta(\{i\})) = 1 \Rightarrow$ Variable Cost TSP.
  - $x(\delta(V \setminus \{0\})) = x(\delta(\{0\})) = k \Rightarrow$VRP.

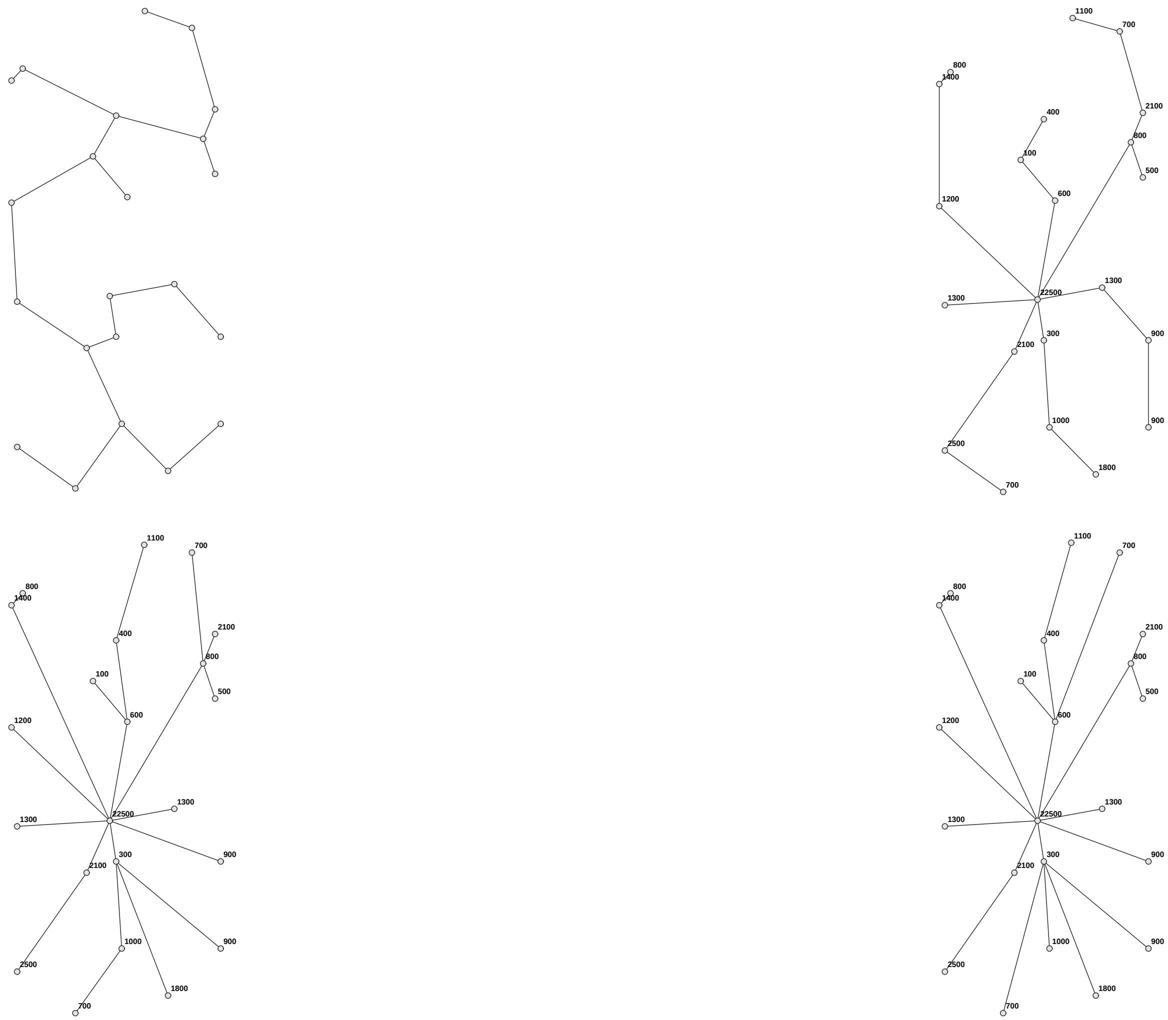


Figure 1: Optimal Uncpacitated Spanning Trees

Figure 2: Uncapacitated and Capacitated Spanning Tree ($\tau = 0$)

Connection to Other Models

- There are connections to many well-studied models.
- The basic model can be seen as an instance of the Fixed-charge Network Flow Problem.
- Removing the upper bounds on the fixed-charge variables yields the Capacitated Network Design Problem.
- We have already mentioned several other related combinatorial models.
- We are looking to make stronger connections among these varied areas of the literature.

Valid Inequalities

- Note that any inequalities valid for the TSP, VRP, or CSTP have counterparts here.
- Many can be strengthened by taking advantage of the directed formulation.
- Fractional Capacity Constraints

$$\sum_{i \notin S,\ j \in S} x_{ij} \geq d(S)/C,\ 0 \notin S$$

- Multi-star Inequalities

$$\sum_{i \notin S,\ j \in S} x_{ij} \geq d(S)/C + \frac{\sum_{i \notin S,\ j \in S} x_{ji} d_i}{C},\ 0 \notin S$$

Valid Inequalities

- Rounded Capacity Constraints

$$\sum_{i \notin S,\ j \in S} x_{ij} \geq \lceil d(S)/C \rceil$$

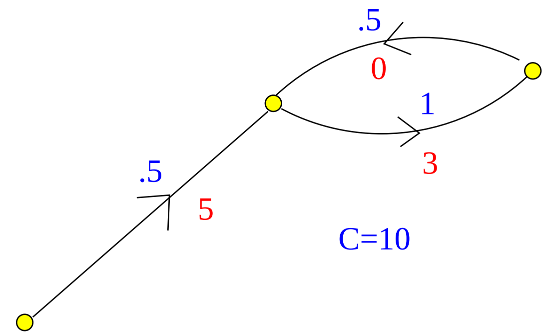
- Generalized, framed capacity constraints
- Combs, Hypo-tours, Clique Clusters
- Path-bin inequalities

Flow Linking

- Note that only the edge variables are required to be integral.
- We use the flow variables to force integrality of the edge variables through *flow linking constraints*.
  - Flow Linking Constraints

$$f_{ij} \le (C - d_i)x_{ij} \Leftrightarrow x_{ij} \ge \frac{f_{ij}}{C - d_i}$$

$$f_{ij} - \sum_{k \ne j} f_{jk} \le x_{ij} d_j$$

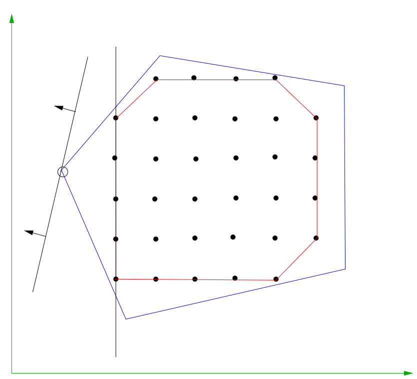


  - Edge Cuts

$$x_{ij} + x_{ji} \le 1$$

Separation

- The fractional capacity constraints and multi-star inequalities are automatically satisfied.
- Flow linking constraints and edge cuts can be included explicitly or separated in polynomial time.
- Separating rounded capacity constraints is NP-complete, but can be done effectively.
- Heuristic procedures for other classes have not yet been implemented.



Solver Implementation

- The implementation uses SYMPHONY, a parallel framework for branch, cut, and price (relative of COIN/BCP).
- In SYMPHONY, the user supplies:
  - the initial LP relaxation,
  - separation subroutines,
  - feasibility checker, and
  - other optional subroutines.
- SYMPHONY handles everything else.
- The source code and documentation are available from `www.BranchAndCut.org`
- For more information, see Workshop TB42.

Preliminary Computation: Formulation Issues

- The new formulation is polynomial and yields stronger relaxations initially, but there are drawbacks.
- For the VRP, the formulation creates symmetry.
- It also seems to make branching less effective.
- There is a related "undirected" formulation which uses one fixed-charge variable per edge.
  - This formulation is smaller and performs much better for the VRP.
  - For the CSTP and CTP, however, the undirected formulation is extremely weak.

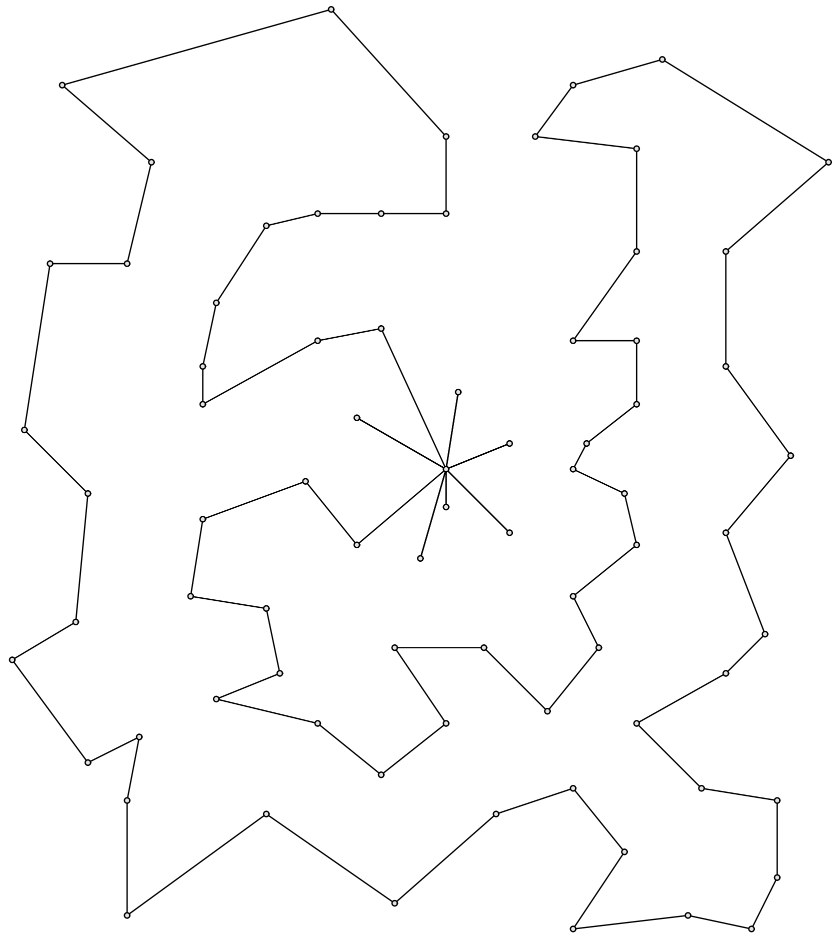
Preliminary Computation

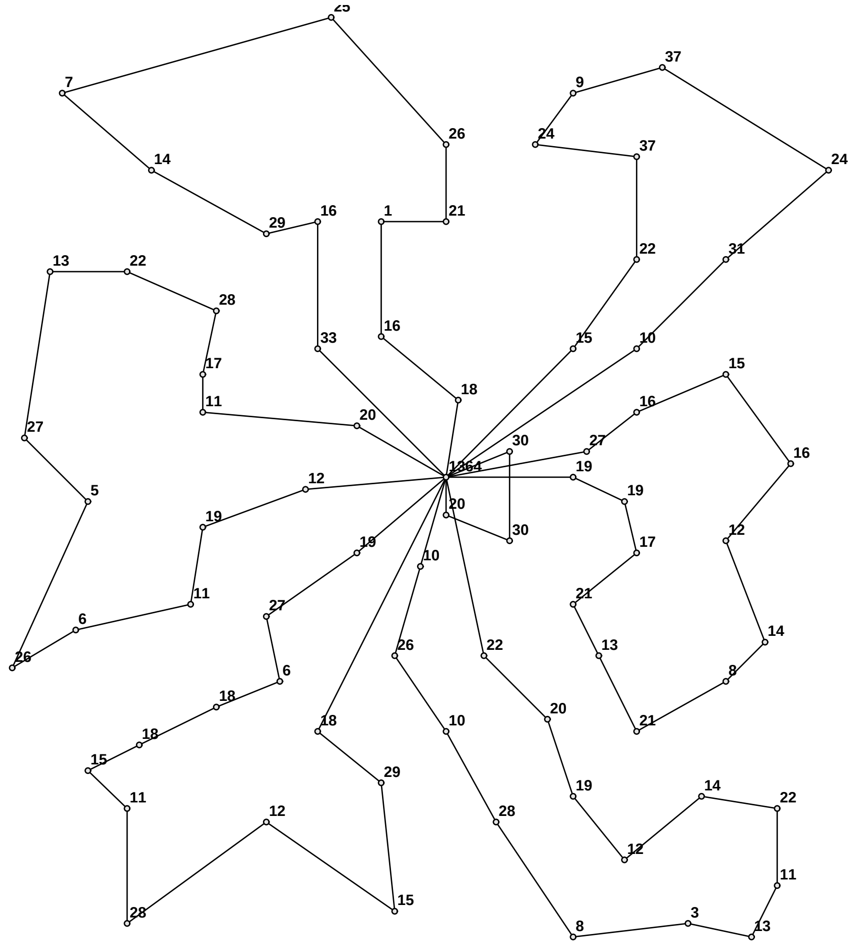
- So far, the presence of the flow variables does not seem to help.
- Capacitating the model does increase difficulty significantly.
- Consider relaxations of the VRP.
  - The TSP is very easy relative to the VRP.
  - The CSTP is not much easier than the VRP.
- Versions of these models with positive variable (flow) costs are extremely difficult.
  - Is this due to the upper bound or lower bound?
  - The flow linking constraints are important for these models.

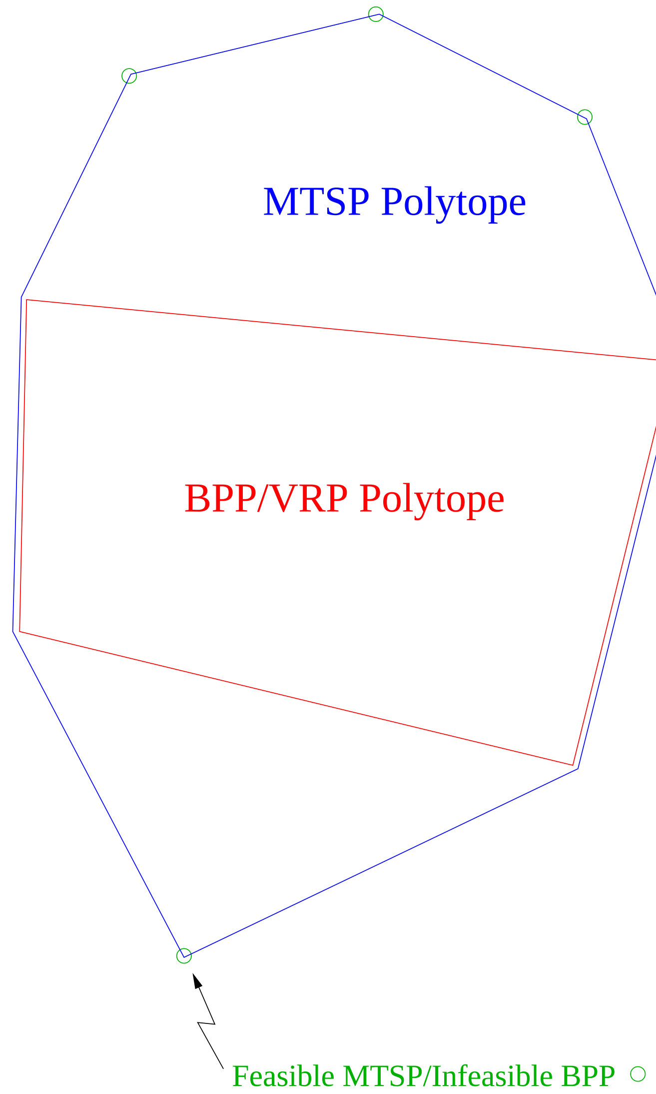
	TSP		*CSTP*		*VRP*	
problem	Tree Size	CPU sec	Tree Size	CPU sec	Tree Size	CPU sec
eil13	1	0.00	13	0.09	1	0.00
eil22	1	0.11	2	0.10	1	0.02
eil33	1	0.02	69	3.97	2	0.44
bayg29	1	0.12	1	0.04	4	0.32
bays29	1	0.17	15	1.12	5	0.55
ulysses16.tsp	1	0.00	1	0.03	1	0.01
ulysses22.tsp	1	0.00	1	0.06	1	0.03
gr17	1	0.01	5	0.05	1	0.01
gr21	1	0.00	1	0.02	1	0.03
gr24	1	0.02	5	0.27	4	0.40
fri26	1	0.02	1	0.07	8	0.39
swiss42	1	0.02	35	3.66	10	2.45
att48	2	0.30	92	5.04	193	30.10
gr48	2	1.38	1	0.07	16	4.17
hk48	1	0.19	209	22.88	45	21.19
eil51	1	0.16	77	15.11	11	10.79
A − n32 − k5	1	0.02	1	0.07	2	0.20
A − n33 − k5	3	0.81	3	0.21	7	0.90
A − n34 − k5	6	2.06	4	0.40	9	2.63
A − n36 − k5	1	0.03	52	5.17	51	7.95
A − n37 − k5	1	0.03	5	0.22	11	0.97
A − n38 − k5	1	0.10	1	0.13	111	21.80
A − n39 − k5	1	0.30	11	0.99	480	310.92
A − n44 − k6	3	1.72	586	84.08	1185	1525.78
A − n45 − k6	2	0.27	47	6.19	133	145.59
A − n46 − k7	1	1.25	3	0.20	2	1.95
A − n48 − k7	2	2.01	775	507.41	1949	1620.57
A − n53 − k7	1	0.62	115	19.99	619	881.05
B − n31 − k5	1	0.01	3	0.63	1	0.08
B − n38 − k6	1	0.04	5	0.56	14	1.73
B − n39 − k5	1	0.03	188	9.67	1	0.05
B − n41 − k6	1	0.08	216	18.96	20	2.89
B − n43 − k6	1	0.09	1	0.36	138	34.92
B − n45 − k5	1	0.09	22	1.13	18	5.81
B − n51 − k7	1	0.36	1	0.13	[illegible]	32.48

Conclusions and Future Directions

- So far, this formulation has not proven better than the classical one for the VRP.
- However, we have yet to take advantage of the information provided by the flow variables.
- We need to know much more about polyhedral structure.
- Better flow linking seems to be the key.
- We also need some new branching rules.
- The connection to the network design literature needs to be explored.
- It is easy to generalize the model even further.
  - Pickup and delivery problems.
  - General degree constraints.






MTSP Polytope
BPP/VRP Polytope
Feasible MTSP/Infeasible BPP