Decomposition and Dynamic Cut Generation in Integer Programming

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Outline

- Preliminaries, Traditional Decomposition Methods
  - Dantzig-Wolfe Decomposition
  - Lagrangian Relaxation
  - Cutting Plane Method
- Dynamic Decomposition Methods
  - Price and Cut
  - Relax and Cut
  - Decompose and Cut
- Applications/Examples
- DECOMP Framework

Preliminaries

- We consider the following pure integer linear program:

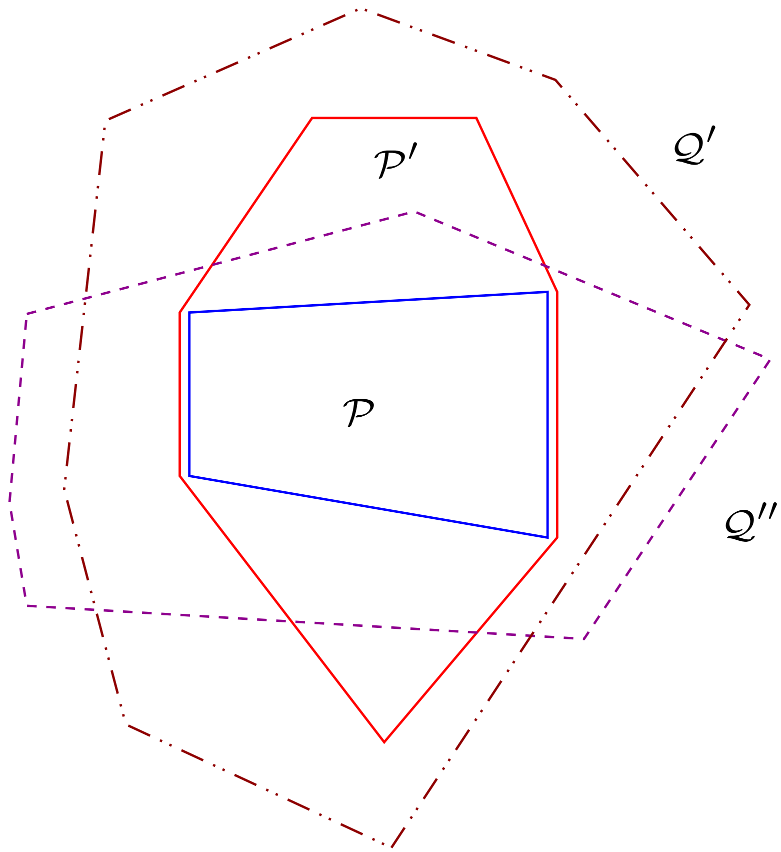
$$z_{IP} = \min_{x \in \mathcal{F}} \{c^\top x\} = \min_{x \in \mathcal{P}} \{c^\top x\}$$

where

$$\begin{aligned} \mathcal{F} &= \{x \in \mathbb{Z}^n : A'x \geq b', A''x \geq b''\} & \mathcal{Q} &= \{x \in \mathbb{R}^n : A'x \geq b', A''x \geq b''\} \\ \mathcal{F}' &= \{x \in \mathbb{Z}^n : A'x \geq b'\} & \mathcal{Q}' &= \{x \in \mathbb{R}^n : A'x \geq b'\} \\ & & \mathcal{Q}'' &= \{x \in \mathbb{R}^n : A''x \geq b''\} \end{aligned}$$

- We will consider $\mathcal{P} = conv(\mathcal{F})$ and $\mathcal{P}' = conv(\mathcal{F}')$.
- Assumptions
  - All input data are rational.
  - $\mathcal{P}$ is bounded.
  - Optimization/separation over $\mathcal{P}$ is "difficult."
  - Optimization/separation over $\mathcal{P}'$ is "easy."

Polyhedra



$\mathcal{P} = conv(\{x \in \mathbb{Z}^n : Ax \geq b\})$

$\mathcal{P}' = conv(\{x \in \mathbb{Z}^n : A'x \geq b'\})$

$\mathcal{Q}' = \{x \in \mathbb{R}^n : A'x \geq b'\}$

$\mathcal{Q}'' = \{x \in \mathbb{R}^n : A''x \geq b''\}$

Bounding

- Goal: Compute a lower bound on z_{IP} by solving a *bounding problem*.
- The most commonly used bounding problem is the initial LP relaxation.

$$\min_{x \in \mathcal{Q}} \{c^\top x\}$$

- Decomposition approaches attempt to improve on this bound by utilizing implicit knowledge of $\mathcal{P}'$.
  - Enforce membership in $\mathcal{Q}''$ *explicitly*.
  - Enforce membership in $\mathcal{P}'$ *implicitly* through solution of a subproblem.
- Decomposition algorithms
  - Dantzig-Wolfe decomposition
  - Lagrangian relaxation
  - Cutting plane method

Dantzig-Wolfe Decomposition (DW)

- The bounding problem is the *Dantzig-Wolfe LP*:

$$z_{DW} = \min\{c(\sum_{s \in \mathcal{F}'} s\lambda_s) : A''(\sum_{s \in \mathcal{F}'} s\lambda_s) \geq b'', \sum_{s \in \mathcal{F}'} \lambda_s = 1, \lambda_s \geq 0 \; \forall s \in \mathcal{F}'\} \tag{1}$$

- Solution method: Simplex algorithm with dynamic column generation.
- Subproblem: Optimization over $\mathcal{P}'$.
- Let $\hat{\lambda}$ be an optimal solution to (1) (the *optimal decomposition*) and

$$\hat{x} = \sum_{s \in \mathcal{F}'} s\hat{\lambda}_s \in \mathcal{P}' \tag{2}$$

Then, $z_{IP} \geq z_{DW} = c^\top \hat{x} \geq z_{LP}$.

Lagrangian Relaxation (LD)

- The bounding problem is the *Lagrangian dual*:

$$z_{LR}(u) = \min_{s \in \mathcal{F}'} \{(c^\top - u^\top A'')s + u^\top b''\} \tag{3}$$

$$z_{LD} = \max_{u \in \mathbb{R}_+^{m''}} \{z_{LR}(u)\} \tag{4}$$

- Solution method: Subgradient optimization.
- Subproblem: Optimization over $\mathcal{P}'$.
- Rewriting (4) as a linear program, we see it is dual to the DW LP.

$$z_{LD} = \max_{\eta \in \mathbb{R}, u \in \mathbb{R}_+^{m''}} \{\eta : \eta \leq (c - uA'')s + ub'' \ \forall s \in \mathcal{F}'\} \tag{5}$$

- So we have $z_{IP} \geq z_{LD} = z_{DW} \geq z_{LP}$.
- We denote by $\hat{u}$ an optimal (dual) solution to (4).

Cutting Plane Method (CP)

- The bounding problem is the initial LP relaxation augmented with facet-defining inequalities from $\mathcal{P}'$:

$$z_{CP} = \min_{x \in \mathcal{P}'} \{cx : A''x \geq b''\} \tag{6}$$

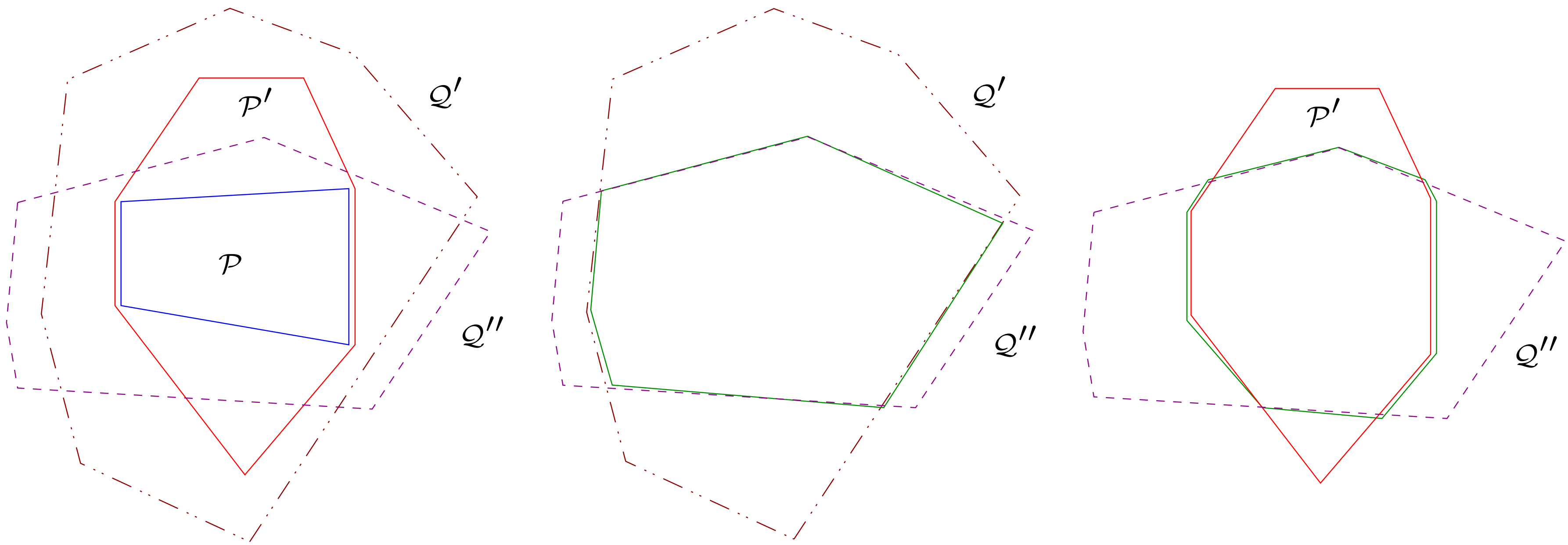
.

- Solution method: Simplex with dynamic cut generation.
- Subproblem: Separation from $\mathcal{P}'$.
- We assumed that separation over $\mathcal{P}'$ was also "easy."
- Note that $\hat{x}$ from (2) is an optimal solution to (6), so $z_{IP} \geq z_{CP} = z_{DW} \geq z_{LP}$.

A Common Framework

- The three methods compute the same bound [Geoffrion74].
- The basic ingredients are the same:
  - the *original polyhedron* ($\mathcal{P}$),
  - an *implicit polyhedron* ($\mathcal{P}'$), and
  - an *explicit polyhedron* ($\mathcal{Q}''$).
- The essential difference is how the implicit polyhedron is represented:
  - CP: as the intersection of half-spaces (the *outer representation*), or
  - DW/LD: as the convex hull of a finite set (the *inner representation*).

Polyhedra, LP Bound, LD/DW/CP Bound



$\mathcal{P} = conv(\{x \in \mathbb{Z}^n : Ax \geq b\})$

$\mathcal{P}' = conv(\{x \in \mathbb{Z}^n : A'x \geq b'\})$

$\mathcal{Q}' = \{x \in \mathbb{R}^n : A'x \geq b'\}$

$\mathcal{Q}'' = \{x \in \mathbb{R}^n : A''x \geq b''\}$

$\mathcal{Q} = \mathcal{Q}' \cap \mathcal{Q}''$ (LP Bound)

$\mathcal{P}' \cap \mathcal{Q}''$ (LD/DW/CP Bound)

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Improving the Bound

- With traditional methods, we achieve the bound $\min_{x \in \mathcal{P}'}\{cx : A''x \geq b''\}$.
- With the cutting plane method, this bound can be improved.

Cutting Plane Method

1. Construct the initial LP relaxation LP^0 and set $i \leftarrow 0$.

$$z_{LP} = \min_{x \in \mathbb{R}^n}\{c^\top x : A'x \geq b', A''x \geq b''\}$$

2. Solve LP^i to obtain an optimal solution $\hat{x}^i$ and valid lower bound $z^i = c^\top \hat{x}^i$.
3. Attempt to separate $\hat{x}^i$ from $\mathcal{P}$, generating a set $[D^i, d^i]$ of valid inequalities violated by $\hat{x}^i$.
4. If valid inequalities were found in Step 3, form the augmented LP relaxation LP^{i+1} by setting $[A'', b''] \leftarrow \left[\begin{smallmatrix} A'' & b'' \\ D^i & d^i \end{smallmatrix}\right]$. Then, set $i \leftarrow i + 1$ and go to Step 2.
5. If no valid inequalities were found in Step 3, then output z^i.

Improving Inequalities

- The challenge is in performing Step 3.
- An inequality found in Step 3 that improves the current bound when added to the bounding problem is an *improving inequality*.
- A necessary and sufficient condition for an inequality to be improving is that it is violated by all optimal primal solutions to (6).
- This condition is difficult to verify.
- As a surrogate, we use the necessary condition that the generated inequality be violated by $\hat{x}$.
- This process can be thought of as a dynamic tightening of the explicit polyhedron.
- In principle, there are analogs of the cutting plane method for Dantzig-Wolfe decomposition and Lagrangian relaxation.
- We call these *dynamic decomposition methods*.

Dynamic Decomposition Methods

Dynamic Decomposition Method

1. Construct the initial bounding problem P^0 and set $i \leftarrow 0$.

$$\begin{aligned} z_{CP} &= \min_{x \in \mathcal{P}'}\{c^\top x : A''x \geq b''\} \\ z_{LD} &= \max_{u \in \mathbb{R}^n_+} \min_{x \in \mathcal{P}'}\{(c^\top - u^\top A'')x + u^\top b''\} \\ z_{DW} &= \min_{\lambda \in \mathbb{R}^{\mathcal{F}'}_+}\{c^\top(\textstyle\sum_{s \in \mathcal{F}'} s\lambda_s) : A''(\textstyle\sum_{s \in \mathcal{F}'} s\lambda_s) \geq b'', \\ &\qquad \textstyle\sum_{s \in \mathcal{F}'} \lambda_s = 1\} \end{aligned}$$

2. Solve P^i to obtain a valid lower bound z^i.

3. Try to generate a set of improving inequalities $[D^i, d^i]$ valid for $\mathcal{P}$.

4. If valid inequalities were found in Step 3, form the bounding problem P^{i+1} by setting $[A'', b''] \leftarrow \left[\begin{smallmatrix} A'' & b'' \\ D^i & d^i \end{smallmatrix}\right]$. Then, set $i \leftarrow i+1$ and go to Step 2.

5. If no valid inequalities were found in Step 3, then output z^i.

Price and Cut (PC)

Price and Cut: Use DW as the bounding problem.

$$z_{DW} = \min_{\lambda \in \mathbb{R}_+^{\mathcal{F}'}} \{c^\top (\sum_{s \in \mathcal{F}'} s\lambda_s) : A''(\sum_{s \in \mathcal{F}'} s\lambda_s) \geq b'', \sum_{s \in \mathcal{F}'} \lambda_s = 1\}$$

and attempt to separate $\hat{x} = \sum_{s \in \mathcal{F}'} s\hat{\lambda}_s$.

- Cut generation takes place in original space, maintaining the structure of the column generation subproblem.
- Both PC and CP try to separate $\hat{x}$ from $\mathcal{P}$.
- With PC, however, we get additional information, i.e., the optimal decomposition $\hat{\lambda}$.
- Question: Can we take advantage of this information?

Relax and Cut (RC)

Relax and Cut: Use LD as the bounding problem.

$$z_{LD} = \max_{u \in \mathbb{R}^n_+} \min_{s \in \mathcal{F}'} \{(c^\top - u^\top A'')s + u^\top b''\}$$

and attempt to separate $\hat{s} \in \mathcal{F}'$, a solution to $z_{LR}(\hat{u})$.

- It is often much easier to separate a member of $\mathcal{F}'$ from $\mathcal{P}$ than an arbitrary real vector, such as $\hat{x}$.
- However, there is no way to know whether the generated inequalities are improving or are violated by $\hat{x}$.
- Questions:
  - Can we improve our chances of generating an improving inequality?
  - Can we characterize the relationship between $\hat{s}$ and $\hat{x}$?

Improving Inequalities (cont.)

Observation 1. *The set of alternative optimal solutions to* $z_{LR}(\hat{u})$ *is*

$$\mathcal{S} = \{s \in \mathcal{F}' : (c^\top - \hat{u}^\top A'')s = (c^\top - \hat{u}^\top A'')\hat{s}\}.$$

Theorem 1. *The convex hull of* $\mathcal{S}$ *is a face of* $\mathcal{P}'$ *and the optimal face* F *of* $\min_{x \in \mathcal{P}'}\{c^\top x : A''x \geq b''\}$ *is contained in* $conv(S)$.

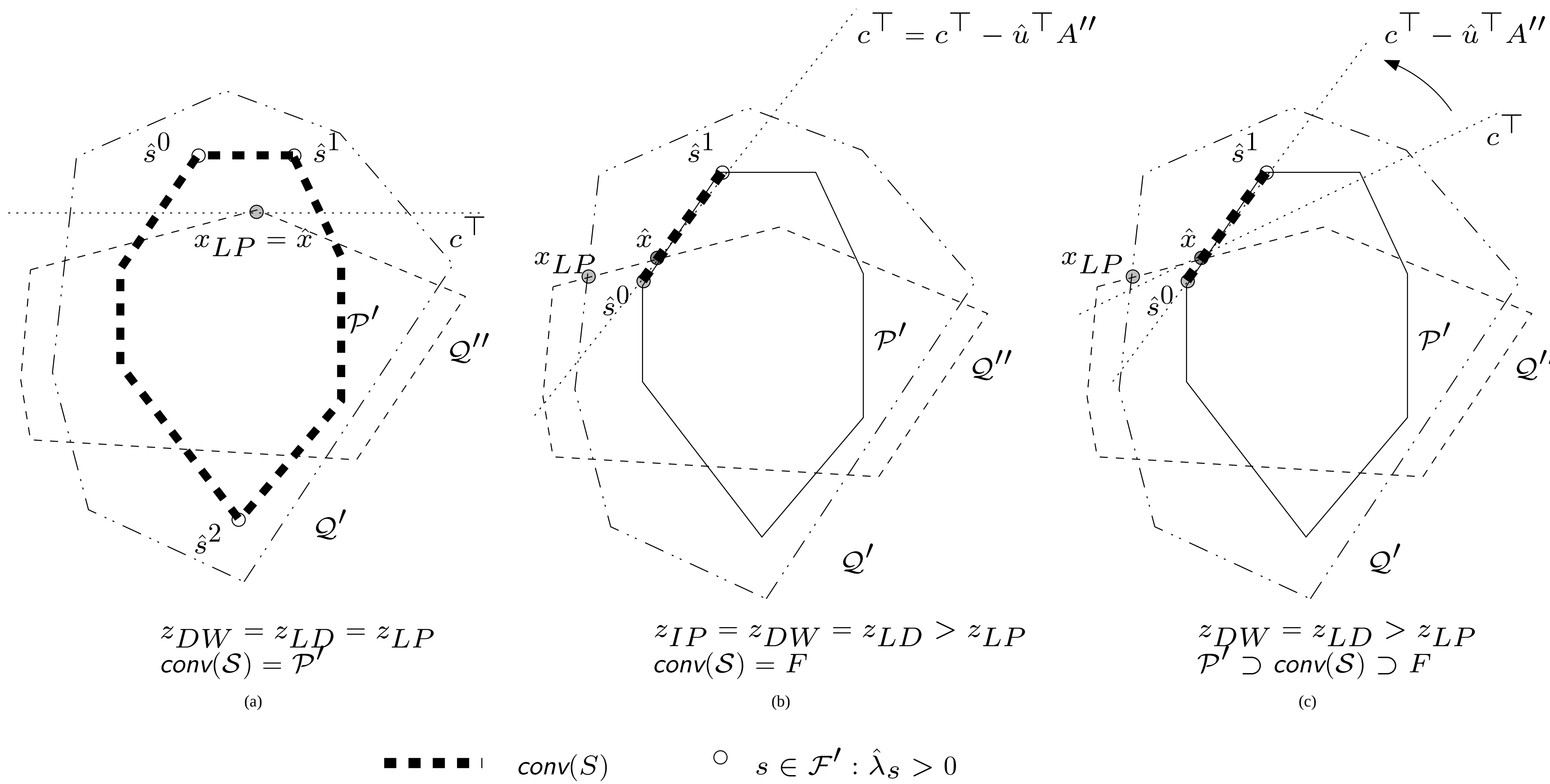
Theorem 2. $D = \{s \in \mathcal{F}' : \hat{\lambda}_s > 0\} \subseteq \mathcal{S}$

Theorem 3. *If* $(a, \beta) \in \mathbb{R}^{(n+1)}$ *is an improving inequality, then there must exist an* $s \in D$ *such that* $a^\top s < \beta$.

- Hence, any improving inequality must be violated by
  - $\hat{x}$,
  - at least one alternative optimal solution to $z_{LR}(\hat{u})$, and
  - at least one $s \in \mathcal{F}'$ such that $\hat{\lambda}_s > 0$.

Price and Cut (revisited)

- Idea: Use the optimal decomposition to help generate improving inequalities.
- Rather than (or in addition to) separating $\hat{x}$, separate each $s \in D$.
- As with RC, it is often much easier to separate a member of $\mathcal{F}'$ from $\mathcal{P}$ than an arbitrary real vector, such as $\hat{x}$.
- RC only gives us one member of S to separate, while PC gives us a set $D \subseteq \S$, one of which must be violated.



$\hat{s}^0$
$\hat{s}^1$
$\hat{s}^2$
$x_{LP} = \hat{x}$
$c^\top$
$\mathcal{P}'$
$\mathcal{Q}''$
$\mathcal{Q}'$
$z_{DW} = z_{LD} = z_{LP}$
$conv(\mathcal{S}) = \mathcal{P}'$
(a)
$c^\top = c^\top - \hat{u}^\top A''$
$\hat{s}^1$
$\hat{x}$
x_{LP}
$\hat{s}^0$
$\mathcal{P}'$
$\mathcal{Q}''$
$\mathcal{Q}'$
$z_{IP} = z_{DW} = z_{LD} > z_{LP}$
$conv(\mathcal{S}) = F$
(b)
$c^\top - \hat{u}^\top A''$
$c^\top$
$\hat{s}^1$
$\hat{x}$
x_{LP}
$\hat{s}^0$
$\mathcal{P}'$
$\mathcal{Q}''$
$\mathcal{Q}'$
$z_{DW} = z_{LD} > z_{LP}$
$\mathcal{P}' \supset conv(\mathcal{S}) \supset F$
(c)
$conv(S)$
$s \in \mathcal{F}' : \hat{\lambda}_s > 0$

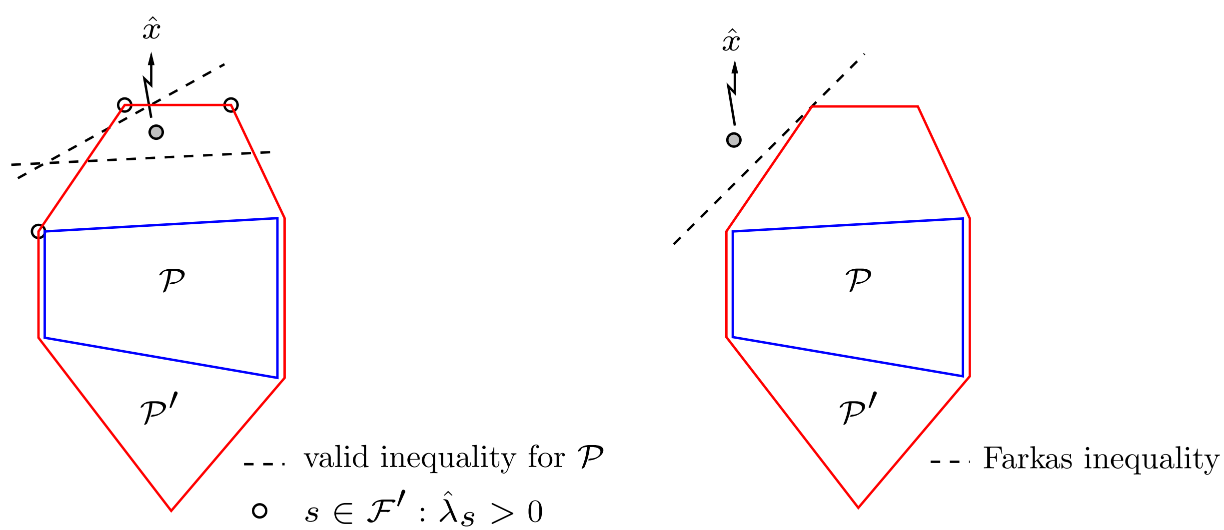
Decompose and Cut (DC)

Decompose and Cut: Use CP as the bounding problem.

$$z_{CP} = \min_{x \in \mathcal{P}'} \{c^\top x : A'' x \geq b''\}$$

Compute the decomposition $\hat{\lambda}$ of $\hat{x}$, then separate each $s \in D$, as in PC.

- Both DC and PC separate the members of a decomposition of $\hat{x}$.
- DC may be more efficient than PC, since we only need to compute the decomposition when standard separation fails.



(a) $\hat{x} \in \mathcal{P}'$ (b) $\hat{x} \notin \mathcal{P}'$

Decompose and Cut (details)

- Separation in Decompose and Cut

  1. Attempt to decompose $\hat{x}$ into a convex combination of members of $\mathcal{F}'$ by solving the LP:

$$\max_{\lambda \in \mathbb{R}_+^{\mathcal{F}'}} \{\mathbf{0}^\top \lambda : \sum_{s \in \mathcal{F}'} s\lambda_s = \hat{x}, \; \sum_{s \in \mathcal{F}'} \lambda_s = 1\}, \tag{7}$$

  2.1 If (7) is feasible, set $D = \{s \in \mathcal{F}' : \hat{\lambda}_s > 0\}$
  2.2 Else, return a *Farkas Cut* (a, β) valid for $\mathcal{P}' \subseteq \mathcal{P}$ which violates $\hat{x}$.
  3. Separate each $s \in D$ and return any cuts that also violate $\hat{x}$.

- Column Generation in Decompose and Cut

  1.0 Generate an initial subset $\mathcal{G}$ of $\mathcal{F}'$.
  1.1 Solve (7) over $\mathcal{G}$ using the dual simplex algorithm.
  1.2a If (7) is feasible, return $D = \{s \in \mathcal{F}' : \hat{\lambda}_s > 0\}$.
  1.2b Else, optimize over $\mathcal{P}'$ using the resulting Farkas inequality (row of B^{-1}). If the result has negative reduced cost, add it to $\mathcal{G}$ and go to Step 1.1, else return the Farkas inequality.

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Separating the Members of $\mathcal{F}'$

- All three dynamic decomposition methods rely on the existence of a class of valid inequalities for $\mathcal{P}$ for which it is
  - difficult to separate an arbitrary fractional solution, but
  - easy to separate members of $\mathcal{F}'$.
- Does this occur in practice? Yes.

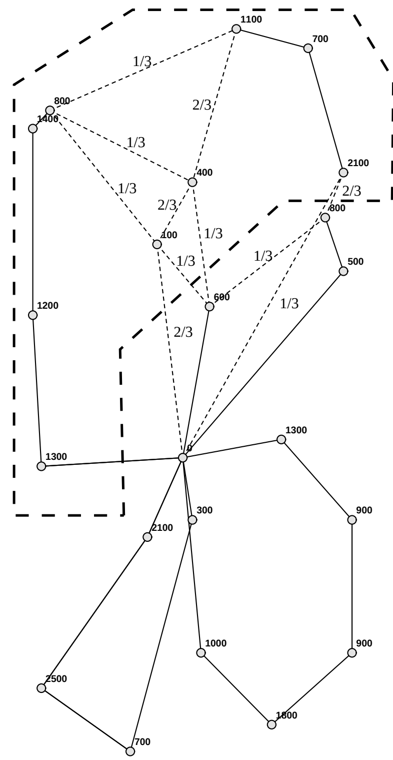
The Vehicle Routing Problem

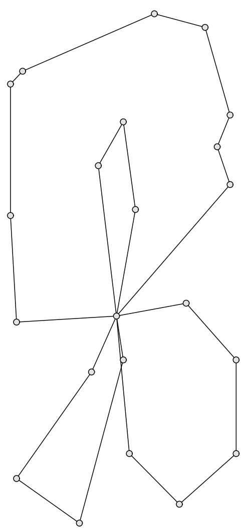
ILP Formulation:

$$
\begin{array}{rcll}
\sum_{e \in \delta(0)} x_e & = & 2k & & (1)\\
\sum_{e \in \delta(i)} x_e & = & 2 & \forall i \in V \setminus \{0\} & (2)\\
\sum_{e \in \delta(S)} x_e & \geq & 2b(S) & \forall S \subset V \setminus \{0\},\ |S| > 1 & (3)
\end{array}
$$

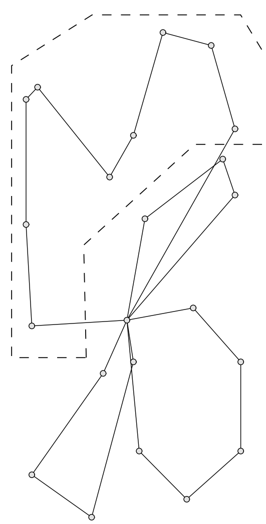
$$
\begin{array}{rcl}
b(S) & = & \text{lower bound on the number of trucks required to service } S\\
 & = & \lceil (\sum_{i \in S} d_i)/C \rceil \text{ (normally)}
\end{array}
$$

- Relaxations:
  - Multiple Traveling Salesman Problem: Set $C = \sum_{i \in S} d_i$.
  - k-Tree: Set $C = \sum_{i \in S} d_i$. Relax (2) but leave $\sum_{e \in E} x_e = n + k$.
- Facets of VRP (under certain conditions): GSECs (3), Combs, Multistars
- *Decompose and Cut* - VRP/kTSP for GSECs [Ralphs, et al. *On the Capacitated Vehicle Routing Problem*, Mathematical Programming 03]
- *Relax and Cut* - VRP/kTree for GSECs, Combs, Multistars [Martinhon, Lucena, Maculan, *Stronger K-Tree Relaxations for the VRP*, unpublished 01]

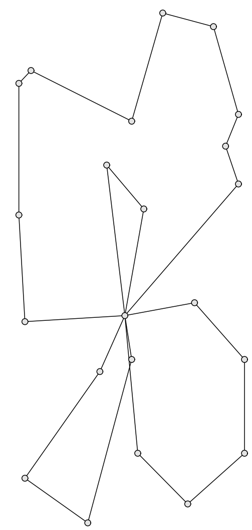




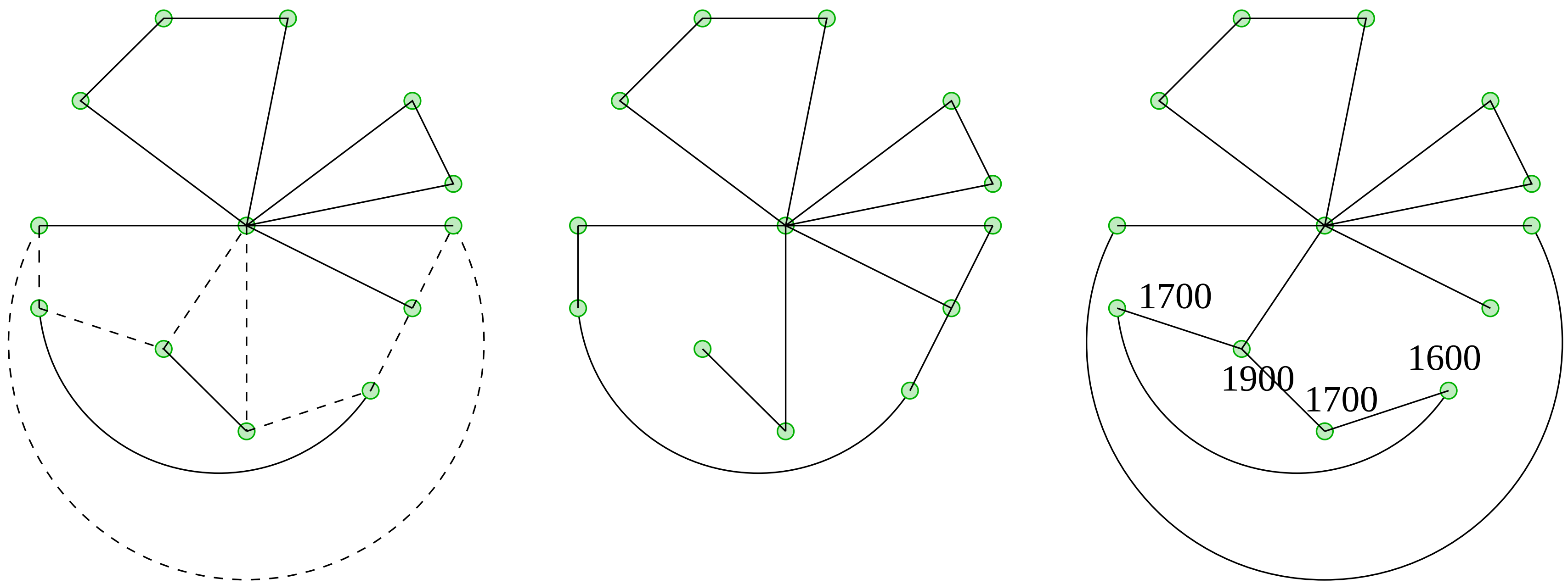
(a) $\hat{\lambda}^1 = \frac{1}{3}$



(b) $\hat{\lambda}^2 = \frac{1}{3}$



(c) $\hat{\lambda}^3 = \frac{1}{3}$



(a) $\hat{x}$ (b) $\hat{\lambda}^1 = \frac{1}{2}$ (c) $\hat{\lambda} = \frac{1}{2}$

Axial Assignment Problem

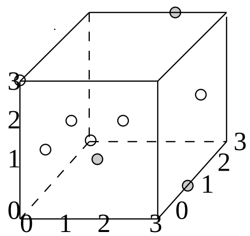
ILP Formulation:

$$
\begin{array}{rllll}
\min & \sum_{(i,j,k) \in I \times J \times K} c_{ijk} x_{ijk} & & & \\
\text{s.t.} & \sum_{(j,k) \in J \times K} x_{ijk} & = & 1 & \forall i \in I \\
& \sum_{(i,k) \in I \times K} x_{ijk} & = & 1 & \forall j \in J \\
& \sum_{(i,j) \in I \times J} x_{ijk} & = & 1 & \forall k \in K \\
& x_{ijk} & \in & \{0,1\} & \forall (i,j,k) \in I \times J \times K
\end{array}
$$

- Relaxation: Assignment Problem: Relax first set of constraints.
- Facets of AAP: $Q_1(u)$ and $P_1(u,v)$ - cliques of the intersection graph $K_{n,n,n}$
- Let $C(u) = \{w \in T : |u \cap w| = 2\}$, $C(u,v) = \{w \in T : |u \cap w| = 1, |w \cap v| = 2\}$

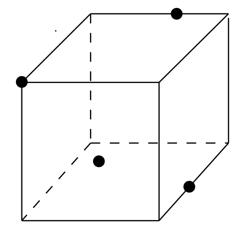
$$
\begin{array}{lcll}
x_u + \sum_{w \in C(u)} x_w & \leq & 1 & \forall u \in T \\
x_u + \sum_{w \in C(u,v)} x_w & \leq & 1 & \forall u, v \in T, u \cap v = \emptyset
\end{array}
$$

- *Relax and Cut* - AP3/AP for Q_1 [Balas and Saltzman, *An Algorithm for the Three-Index Assignment Problem* Operations Research 91]



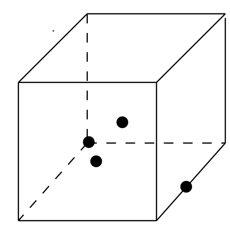
$(0,0,3)$	$1/3$	$(0,3,1)$	$2/3$
$(1,0,1)$	$1/3$	$(1,1,2)$	$2/3$
$(2,1,0)$	$1/3$	$(2,2,0)$	$1/3$
$(2,3,2)$	$1/3$	$(3,0,0)$	$1/3$
$(3,2,3)$	$2/3$		

(a) $\hat{x}$



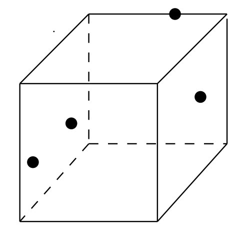
$(3,0,0)$
$(0,3,1)$
$(1,1,2)$
$(3,2,3)$

(b) $\hat{\lambda}_1 = \frac{1}{3}$



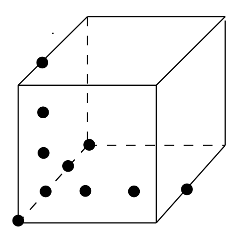
$(2,2,0)$
$(0,3,1)$
$(1,1,2)$
$(0,0,3)$

(c) $\hat{\lambda}_2 = \frac{1}{3}$



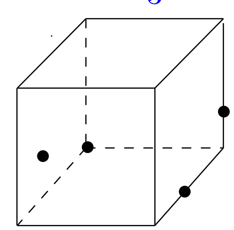
$(2,1,0)$
$(1,0,1)$
$(2,3,2)$
$(3,2,3)$

(d) $\hat{\lambda}_3 = \frac{1}{3}$



$\sum_{w \in C(0,0,1)} \hat{x}w = 1\ 1/3 > 1$

(e) $Q_1(0,0,1)$



$\sum_{w \in C((0,0,3),(1,3,1))} \hat{x}w = 1\ 1/3 > 1$

(f) $P_1((0,0,3),(1,3,1))$

Applications Under Development

- Vehicle Routing Problem
  - k-Traveling Salesman Problem : GSECs
  - k-Tree : GSECs, Combs, Multistars
- Axial Assignment Problem
  - Assignment Problem : Clique-Facets
- Steiner Tree Problem
  - Minimum Spanning Tree : Lifted SECs, Partition Inequalities
- Knapsack Constrained Circuit Problem
  - Knapsack Problem : Cycle Cover, Maximal-Set Inequalities
- Edge-Weighted Clique Problem
  - Tree Relaxation : Trees, Cliques
- Subtour Elimination Problem [G. Benoit / S. Boyd]
  - Fractional 2-Factor Problem : SECs

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DECOMP Framework

- Goal: Framework to allow for direct comparison of all three dynamic decomposition methods.
- COIN-OR: **COmputational INfrastructure for Operations Research**
- BCP: Parallel Branch, Price and Cut (LP-based Bounding)
- ALPs: Abstract Library for Parallel Search
  - BiCePS: Branch, Constrain and Price Software (Generic Bounding)
  - BLIS: BiCePS Linear Integer Solver
- DECOMP will provide
  - CGL-like full implementation of *Decompose and Cut*
  - BiCePS *plug-and-play* for *Price and Cut* and *Relax and Cut*
- DECOMP user implements two methods:
  - `solve_relaxed_problem` (includes several built-in solvers)
  - `separate_relaxed_point`

Future Work

- There are many, many computational questions to be answered to make these methods practical.
  - How do the cuts generated by separating members of $\mathcal{F}'$ compare to those generated by separating $\hat{x}$?
  - Which members of $\mathcal{F}'$ should we separate if we have a choice?
  - Of the cuts that are violated by a given member of $\mathcal{F}'$, which are the most likely to be improving?
  - Can we "warmstart" the process of finding a decomposition in DC?
  - Can we use a global pool of members of $\mathcal{F}'$ similar to a cut pool?
- One can also envision numerous extensions to this basic framework.
  - Explore uses outside of integer programming.
  - Consider tightening the implicit polyhedron instead of the explicit.
  - Try to generate more Farkas inequalities and possibly tighten them.
  - Restrict column generation when finding the decomposition in DC.
- More theoretical issues also need to be thought about, such as complexity and convergence.

Conclusions

- There are many variants on this basic theme.
- Very little is known about these methods computationally.
- There is a large gap between theory and practice.
- We are currently in the process of implementing these methods and working through the computational issues.
- Our very preliminary results indicate that these methods have potential, but there are many tradeoffs.
- Our software framework will end up in the COIN-OR repository, so that others may use it.
- Stay tuned, a computational paper will be coming soon.