DIP with CHiPPS: Decomposition Methods for Integer Linear Programming

Ted Ralphs
Lehigh University
Matthew Galati
SAS Institute

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Outline

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(Very) Brief Introduction to Mathematical Programming

The general form of a *mathematical programming model* is

$$\begin{array}{lrcl} \textbf{min} & f(x) & & \\ s.t. & g_i(x) & \left\{ \begin{array}{c} \leq \\ = \\ \geq \end{array} \right\} & b_i \\ & x & \in & X \end{array}$$

where $X \subseteq \mathbb{R}^n$ is an (implicitly defined) set that may be discrete.

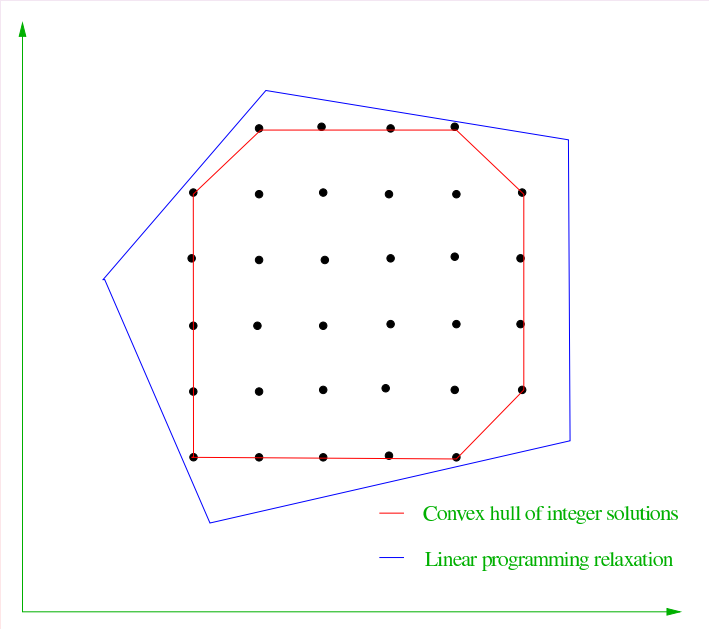
- A *mathematical programming problem* is a problem that can be expressed using a mathematical programming model (called the *formulation*).
- A single mathematical programming problem can be represented using many different formulations (important!).

Our Basic Setting

Integer Linear Program: Minimize/Maximize a linear *objective function* over a (discrete) set of *solutions* satisfying specified *linear constraints*.

$$z_{\mathrm{IP}} = \min_{x \in \mathbb{Z}^n} \left\{ c^\top x \mid A'x \geq b', A''x \geq b'' \right\}$$

$$z_{\mathrm{LP}} = \min_{x \in \mathbb{R}^n} \left\{ c^\top x \mid A'x \geq b', A''x \geq b'' \right\}$$



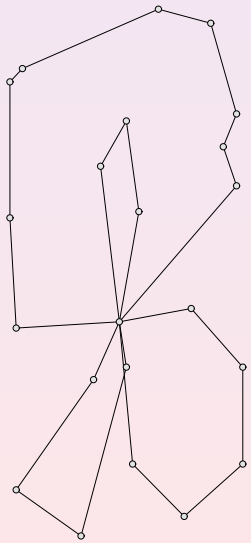
Combinatorial Optimization

Combinatorial Optimization Problem

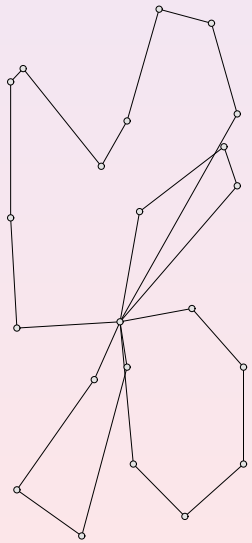
$CP = (E, \mathcal{F})$ consists of

- A *ground set* E,
- A set $\mathcal{F} \subseteq 2^E$ of *feasible solutions*, and
- A *cost function* $c \in \mathbb{Z}^E$ (optional).

The *cost* of $S \in \mathcal{F}$ is $c(S) = \sum_{e \in S} c_e$ and the problem is to find a least cost member of $\mathcal{F}$.



Cost 1100



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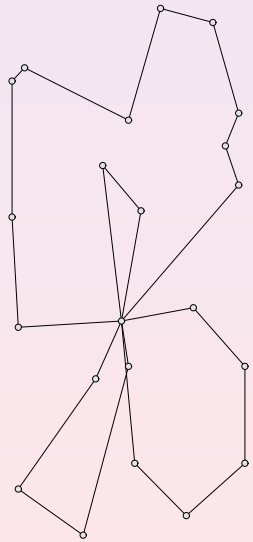


Cost 1107

Solving Integer Programs

- *Implicit enumeration* techniques try to enumerate the solution space in an intelligent way.
- The most common algorithm of this type is *branch and bound*.
- Suppose F is the set of feasible solutions for a given MILP. We wish to solve $\min_{x \in F} c^\top x$.

Divide and Conquer

Consider a partition of F into subsets $F_1, \dots F_k$. Then

$$\min_{x \in F} c^\top x = \min_{1 \leq i \leq k} \{ \min_{x \in F_i} c^\top x \}.$$

We can then solve the resulting *subproblems* recursively.

- Dividing the original problem into subproblems is called *branching*.
- Taken to the extreme, this scheme is equivalent to complete enumeration.
- We avoid complete enumeration primarily by deriving bounds on the value of an optimal solution to each subproblem.

Branch and Bound

- A *relaxation* of an ILP is an auxiliary mathematical program for which
  - the feasible region contains the feasible region for the original ILP, and
  - the objective function value of each solution to the original ILP is not increased.
  - Relaxations can be used to efficiently get bounds on the value of the original integer program.
- Types of Relaxations
  - Continuous relaxation
  - Combinatorial relaxation
  - Lagrangian relaxations

Branch and Bound

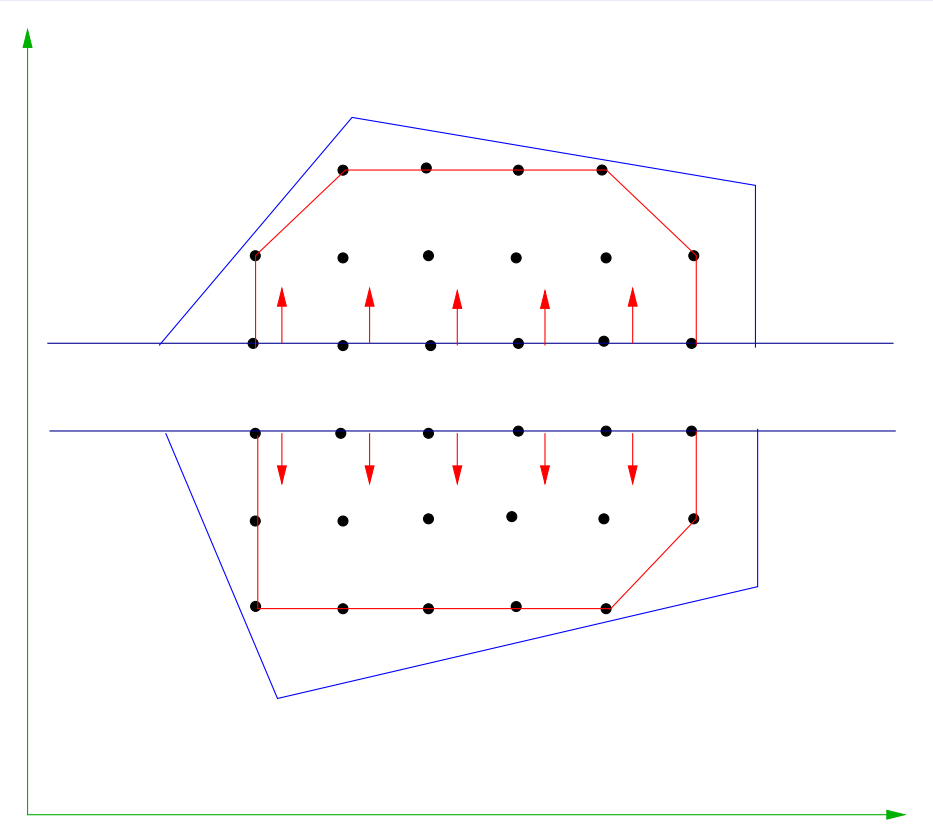
Initialize the queue with F. While there are subproblems in the queue, do

1. Remove a subproblem and solve its relaxation.
2. The relaxation is infeasible $\Rightarrow$ subproblem is infeasible and can be pruned.
3. Solution is feasible for the MILP $\Rightarrow$ subproblem solved (update upper bound).
4. Solution is not feasible for the MILP $\Rightarrow$ lower bound.
   - If the lower bound exceeds the global upper bound, we can *prune the node*.
   - Otherwise, we *branch* and add the resulting subproblems to the queue.

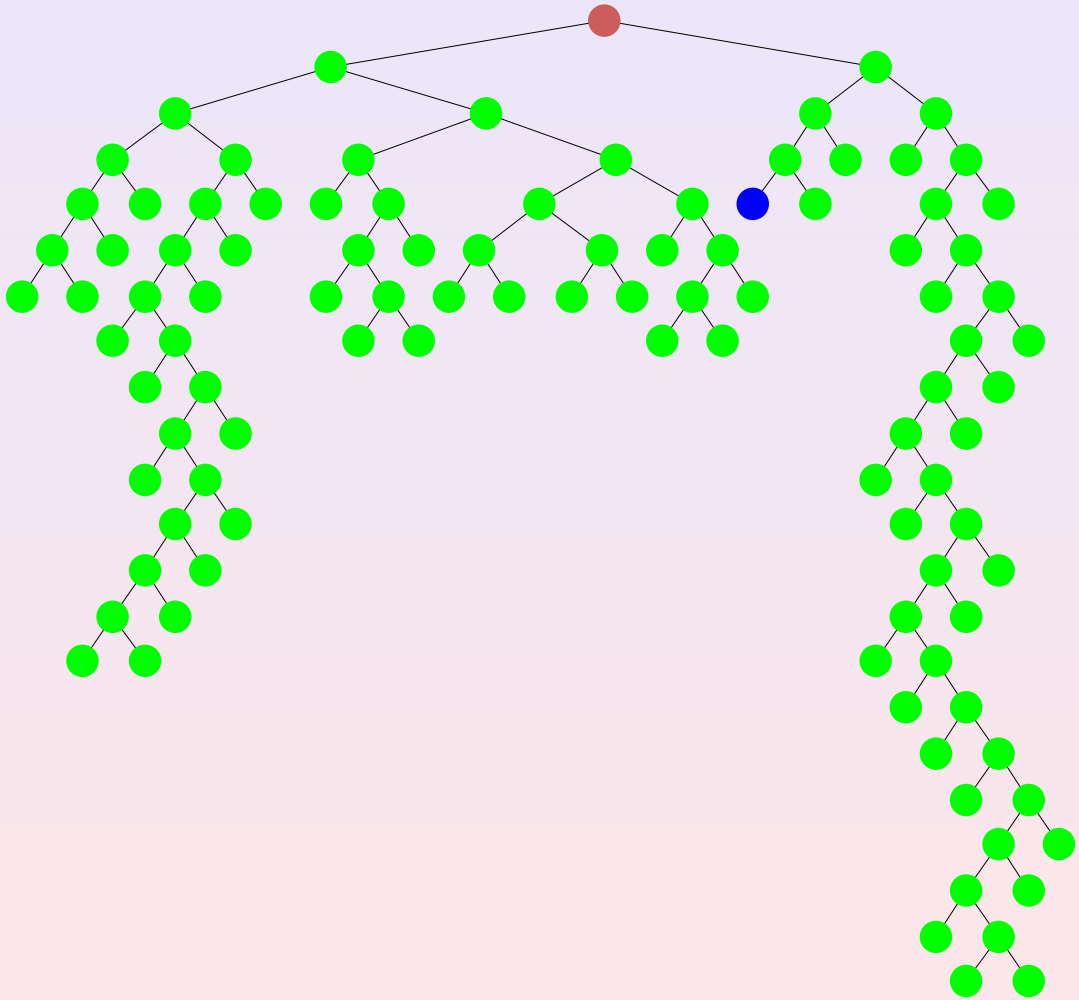
Branching

Branching involves partitioning the feasible region by imposing a *valid disjunction* such that:

- All optimal solutions are in one of the members of the partition.
- The solution to the current relaxation is not in any of the members of the partition.

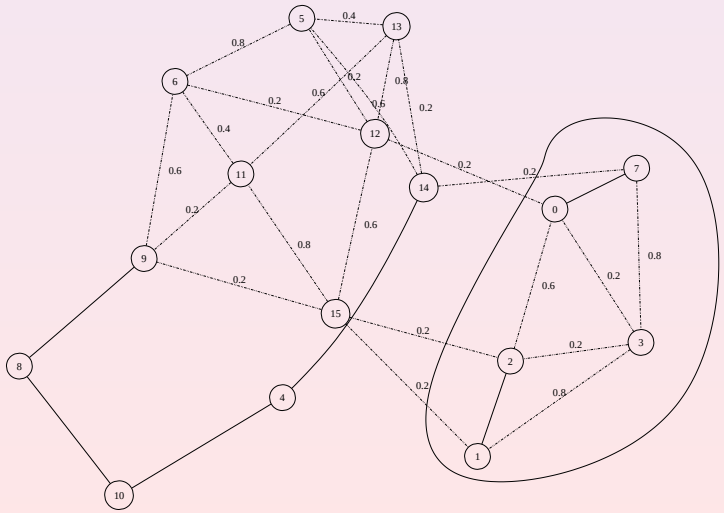


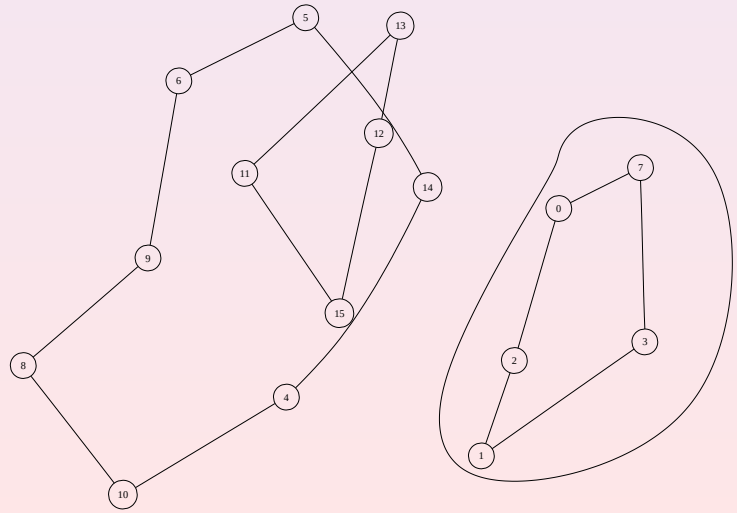
Branch and Bound Tree



What is the Goal of Decomposition?

- Basic Idea: Exploit knowledge of underlying structural components of the model to improve the bound by developing a stronger relaxation.
- Many complex models are built up from multiple underlying substructures.
  - Subsystems linked by global constraints.
  - Complex combinatorial structures obtained by combining simple ones.
- We want to exploit knowledge of efficient, customized methodology for substructures.
- This can be done in two primary ways (with many variants).
  - Identify independent subsystems.
  - Identify subsets of constraints that can be dealt with efficiently.





Example: Exposing Combinatorial Structure

Traveling Salesman Problem Formulation

$$
\begin{array}{lcll}
x(\delta(\{u\})) & = & 2 & \forall u \in V \\
x(E(S)) & \leq & |S| - 1 & \forall S \subset V,\ 3 \leq |S| \leq |V| - 1 \\
x_e \in \{0, 1\} & & & \forall e \in E
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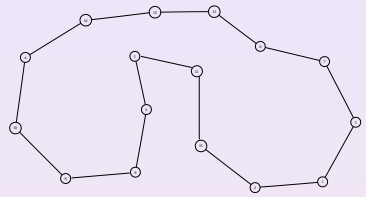

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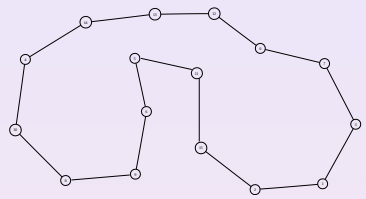

Two relaxations

Find a spanning subgraph with $|V|$ edges ($\mathcal{P}' = $ 1-Tree)

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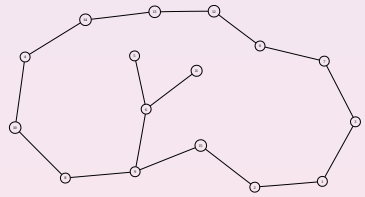

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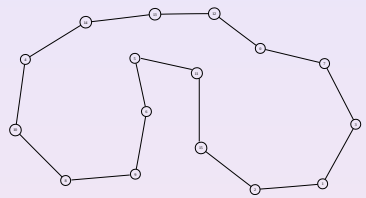

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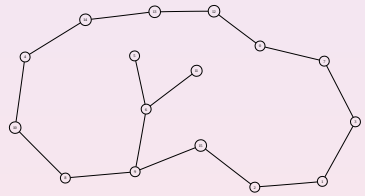

Find a 2-matching that satisfies the subtour constraints ($\mathcal{P}' =$ 2-Matching)

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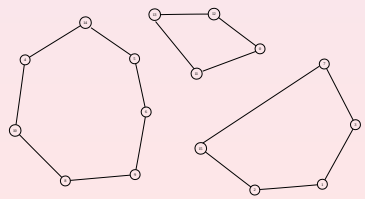

Example: Exposing Block Structure

- A motivation for decomposition is to expose *independent subsystems*.
- The key is to identify *block structure* in the constraint matrix.
- The separability lends itself nicely to parallel implementation.

$$\begin{pmatrix} A''_1 & A''_2 & \cdots & A''_\kappa \\ \hline A'_1 & & & \\ & A'_2 & & \\ & & \ddots & \\ & & & A'_\kappa \end{pmatrix}$$

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Generalized Assignment Problem (GAP)

- The problem is to assign m tasks to n machines subject to capacity constraints.
- An IP formulation of this problem is

$$\begin{array}{lllll} \min & \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} & & & \\ & \sum_{j \in N} w_{ij} x_{ij} & \leq & b_i & \forall i \in M \\ & \sum_{i \in M} x_{ij} & = & 1 & \forall j \in N \\ & x_{ij} & \in & \{0,1\} & \forall i,j \in M \times N \end{array}$$

- The variable x_{ij} is one if task i is assigned to machine j.
- The "profit" associated with assigning task i to machine j is c_{ij}.

Example: Eliminating Symmetry

- In some cases, the identified blocks are *identical*.
- In such cases, the original formulation will often be highly symmetric.
- The decomposition eliminates the symmetry by collapsing the identical blocks.

Vehicle Routing Problem (VRP)

$$
\begin{aligned}
\min \quad & \sum_{k \in M} \sum_{(i,j) \in A} c_{ij} x_{ijk} & & & \\
& \sum_{k \in M} \sum_{j \in N} x_{ijk} & = & 1 & \forall i \in V \\
& \sum_{i \in V} \sum_{j \in N} d_i x_{ijk} & \leq & C & \forall k \in M \\
& \sum_{j \in N} x_{0jk} & = & 1 & \forall k \in M \\
& \sum_{i \in N} x_{ihk} - \sum_{j \in N} x_{hjk} & = & 0 & \forall h \in V, k \in M \\
& \sum_{i \in N} x_{i,n+1,k} & = & 1 & \forall k \in M \\
& x_{ijk} \in \{0,1\} & & & \forall (i,j) \in A, k \in M
\end{aligned}
$$

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The Decomposition Principle in Integer Programming

Basic Idea: By leveraging our ability to solve the optimization/separation problem for a relaxation, we can improve the bound yielded by the LP relaxation.

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$$z_{\mathrm{D}} = \min_{x \in \mathcal{P}'} \left\{ c^\top x \mid A''x \geq b'' \right\}$$

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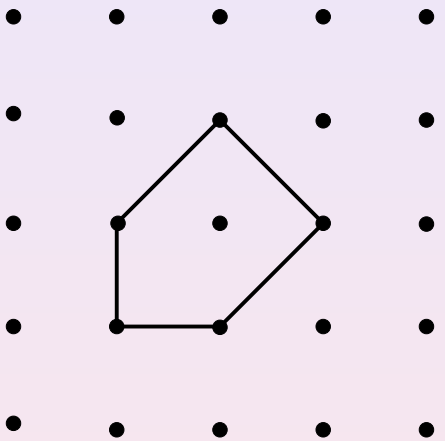

——— $\mathcal{P} = \mathrm{conv}\{x \in \mathbb{Z}^n \mid A'x \geq b', A''x \geq b''\}$

Assumptions:

- $\mathrm{OPT}(\mathcal{P}, c)$ and $\mathrm{SEP}(\mathcal{P}, x)$ are *"hard"*
- $\mathrm{OPT}(\mathcal{P}', c)$ and $\mathrm{SEP}(\mathcal{P}', x)$ are *"easy"*
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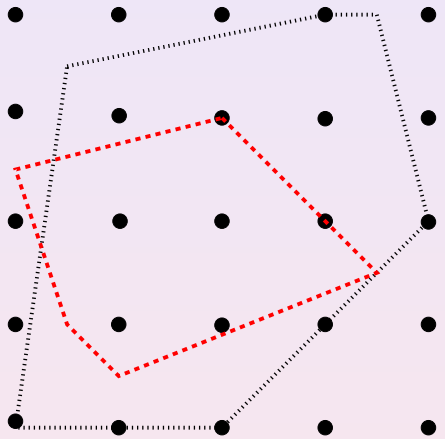

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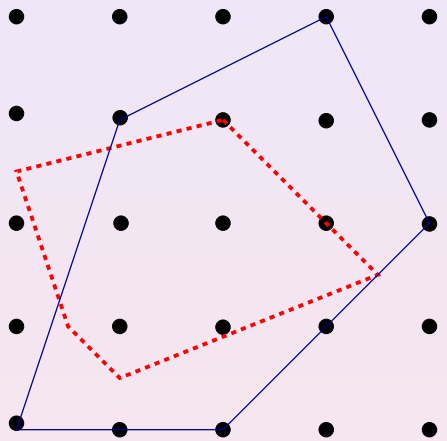

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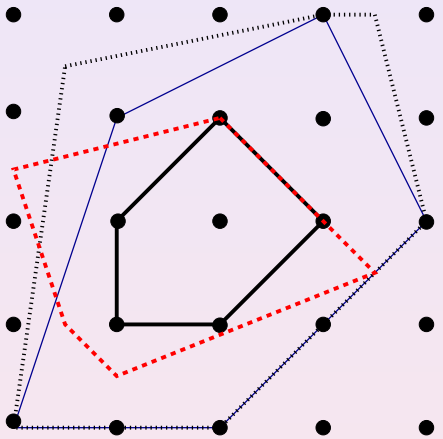

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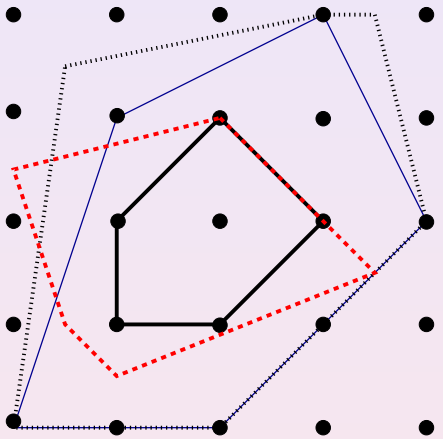

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Cutting Plane Method (CPM)

CPM combines an *outer* approximation of $\mathcal{P}'$ with an explicit description of $\mathcal{Q}''$

- **Master**: $z_{\text{CP}} = \min_{x \in \mathbb{R}^n} \left\{ c^\top x \mid Dx \geq d, A''x \geq b'' \right\}$
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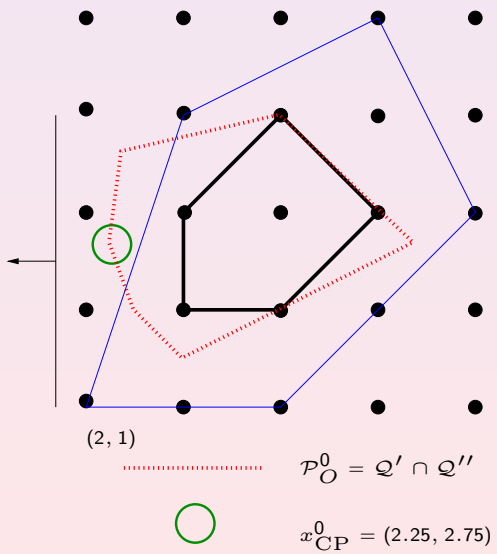
Exponential number of constraints



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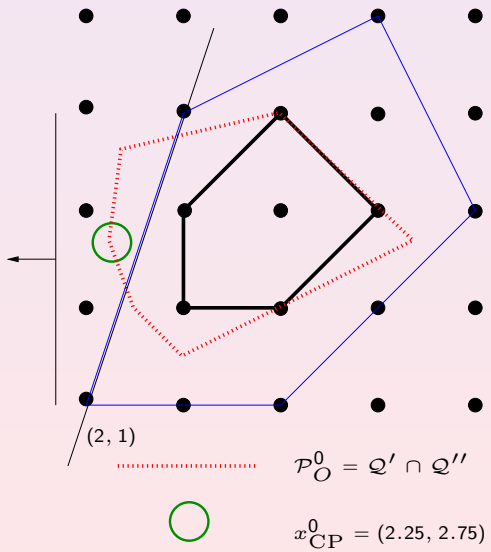

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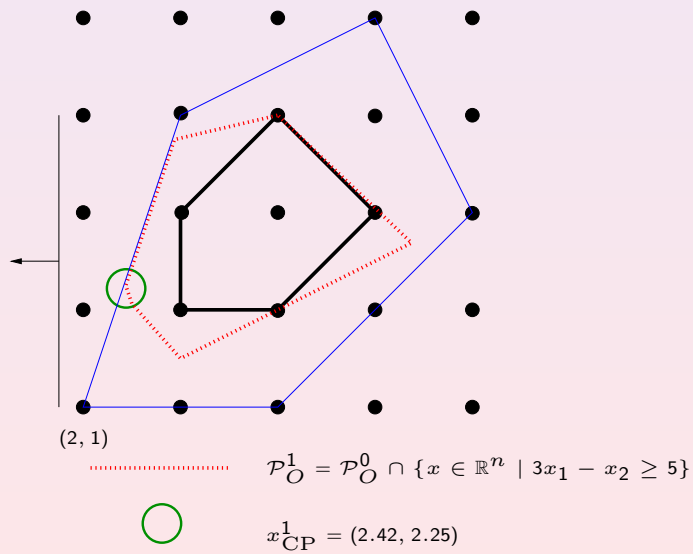

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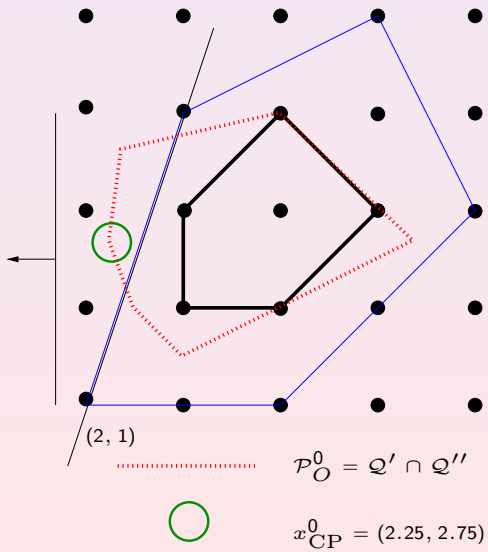
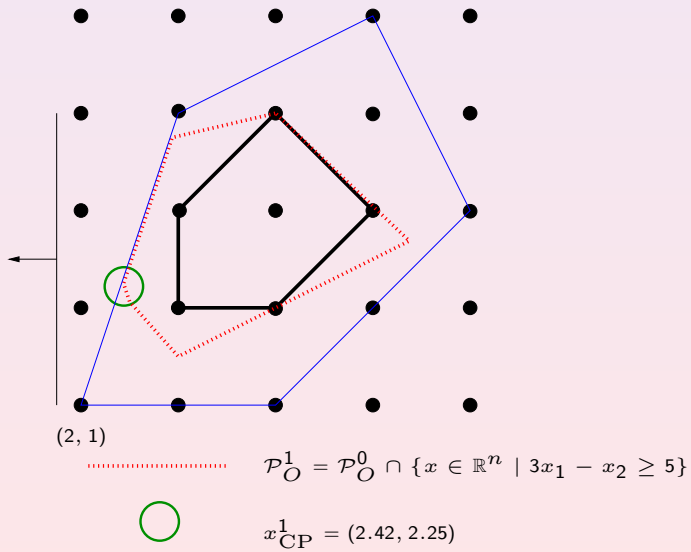
CPM combines an *outer* approximation of $\mathcal{P}'$ with an explicit description of $\mathcal{Q}''$

- **Master**: $z_{\text{CP}} = \min_{x \in \mathbb{R}^n} \left\{ c^\top x \mid Dx \geq d, A''x \geq b'' \right\}$
- **Subproblem**: $\text{SEP}(\mathcal{P}', x_{\text{CP}})$

$$\mathcal{P}' = \{x \in \mathbb{R}^n \mid Dx \geq d\}$$

Exponential number of constraints





Dantzig-Wolfe Method (DW)

DW combines an *inner* approximation of $\mathcal{P}'$ with an explicit description of $\mathcal{Q}''$

- **Master**: $z_{\text{DW}} = \min_{\lambda \in \mathbb{R}^{\mathcal{E}}_+} \left\{ c^\top \left(\sum_{s \in \mathcal{E}} s\lambda_s \right) \mid A'' \left(\sum_{s \in \mathcal{E}} s\lambda_s \right) \geq b'', \sum_{s \in \mathcal{E}} \lambda_s = 1 \right\}$
- **Subproblem**: $\text{OPT}\left(\mathcal{P}', c^\top - u_{\text{DW}}^\top A''\right)$

$$\mathcal{P}' = \left\{ x \in \mathbb{R}^n \;\middle|\; x = \sum_{s \in \mathcal{E}} s\lambda_s, \sum_{s \in \mathcal{E}} \lambda_s = 1, \lambda_s \geq 0 \; \forall s \in \mathcal{E} \right\}$$

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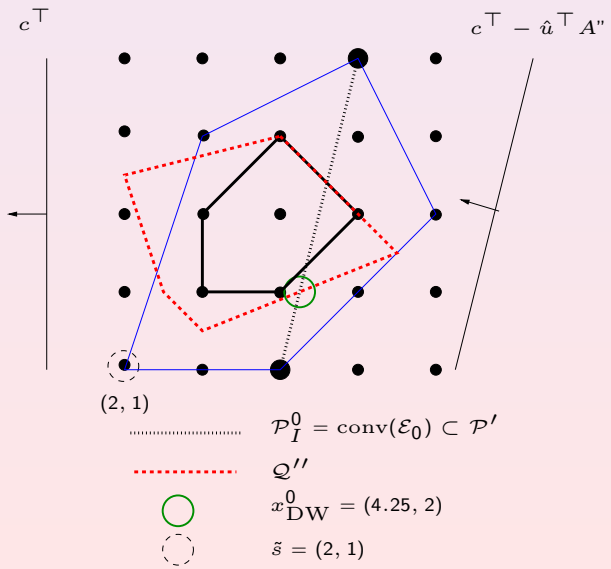

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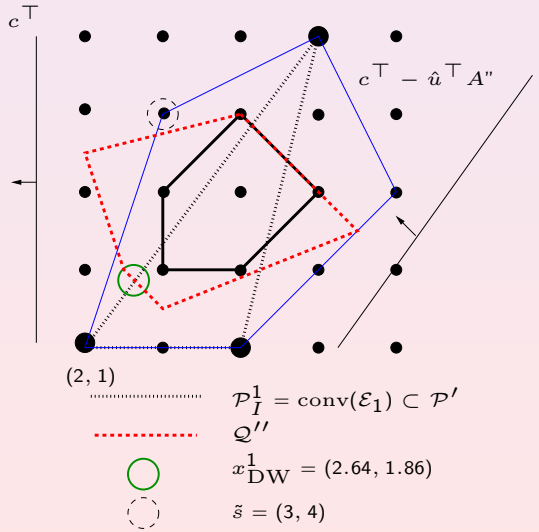

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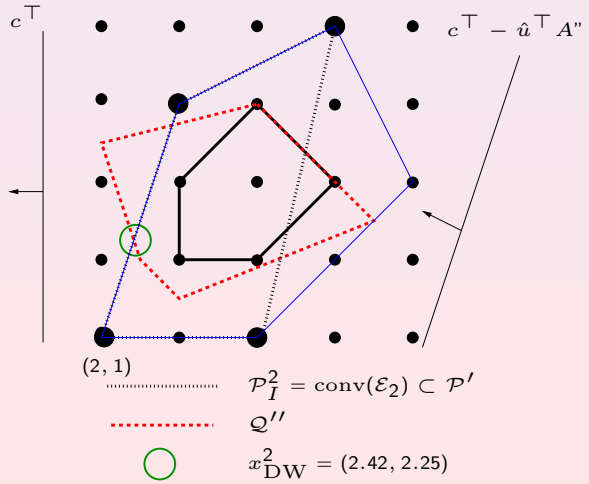

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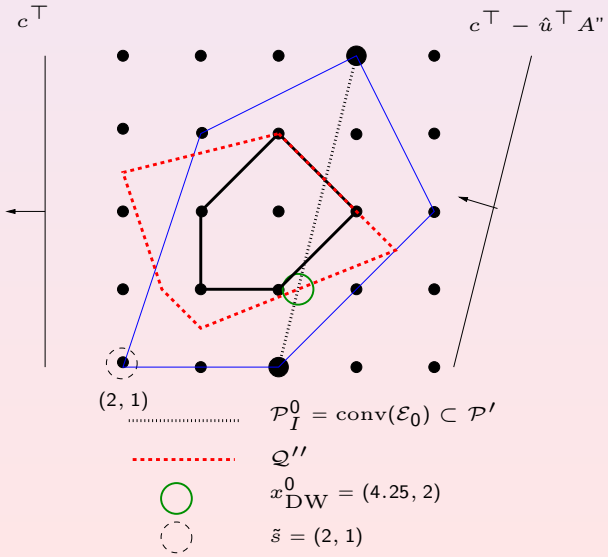

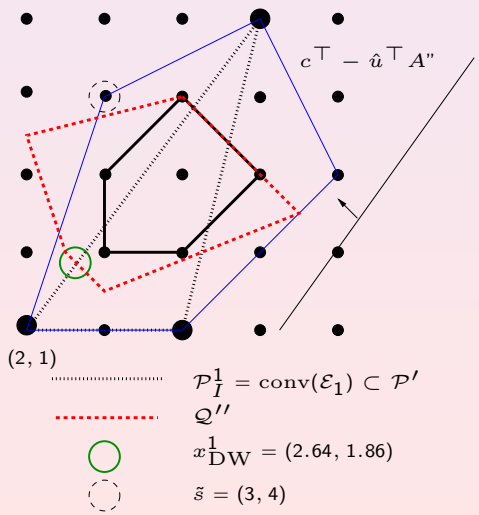

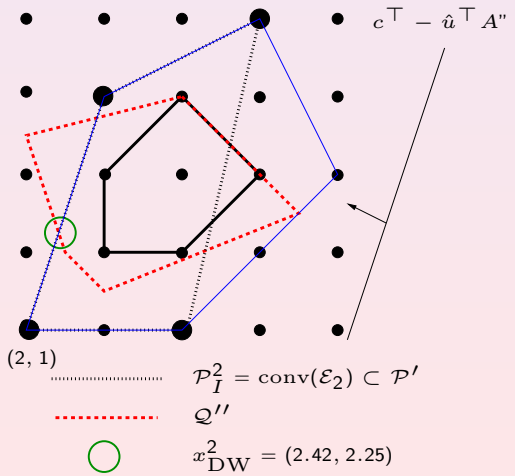

Lagrangian Method (LD)

LD iteratively produces single extreme points of $\mathcal{P}'$ and uses their violation of constraints of $\mathcal{Q}''$ to converge to the same optimal face of $\mathcal{P}'$ as CPM and DW.

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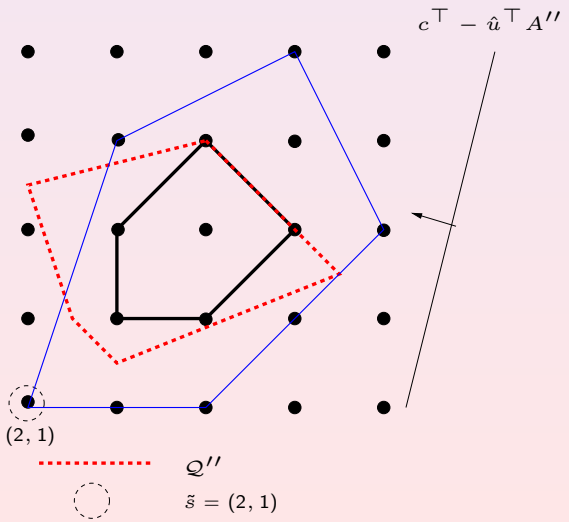

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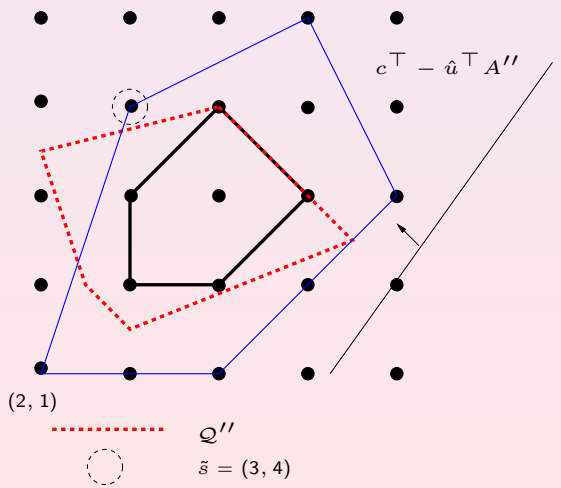

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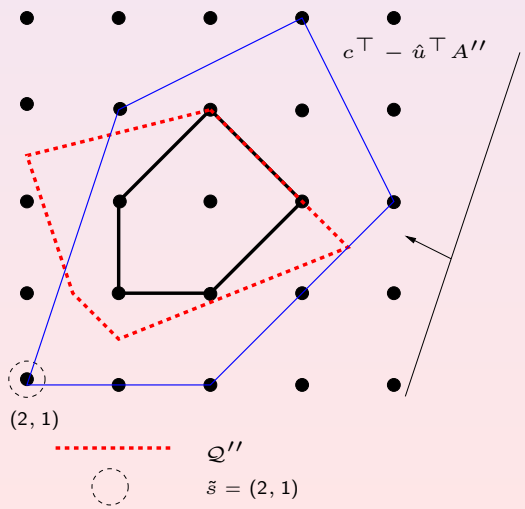

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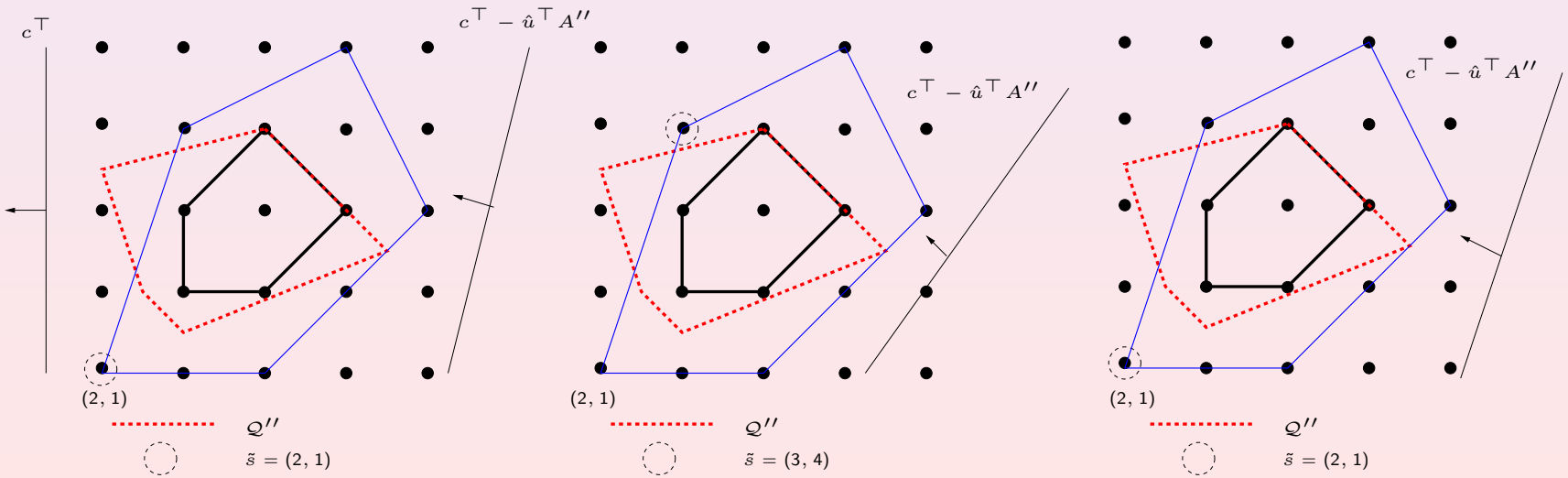

Common Threads

- The LP bound is obtained by optimizing over the intersection of two explicitly defined polyhedra.

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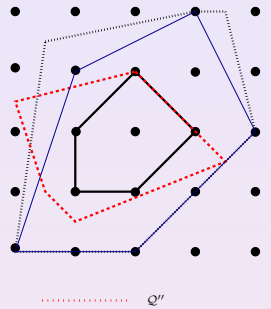

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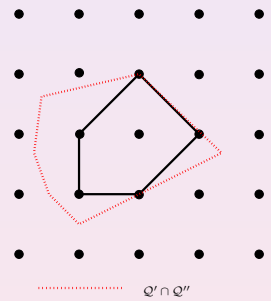

- Traditional decomp-based bounding methods contain two primary steps
  - **Master Problem:** Update the primal/dual solution information
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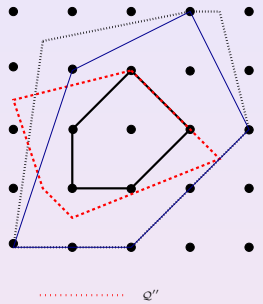

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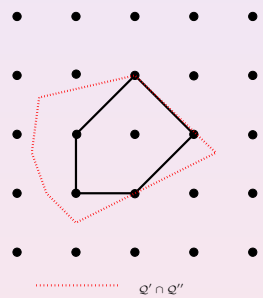

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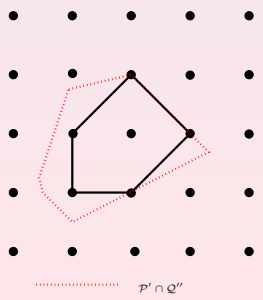

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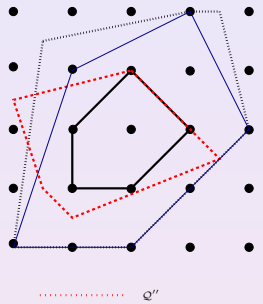

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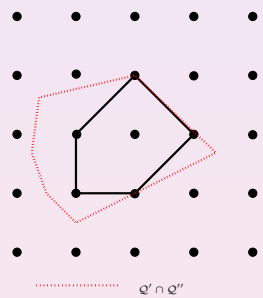

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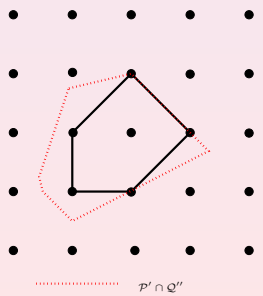

Decompose-and-Cut (DC)

Decompose-and-Cut: Each iteration of CPM, decompose into convex combo of e.p.'s of $\mathcal{P}'$

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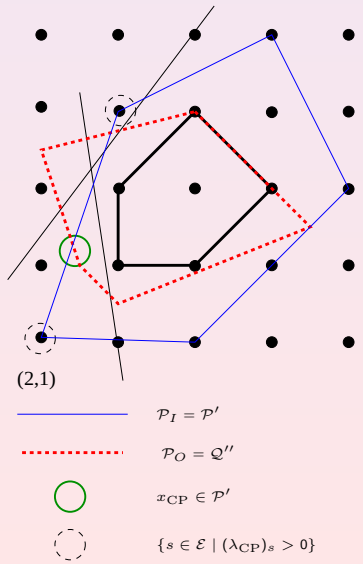
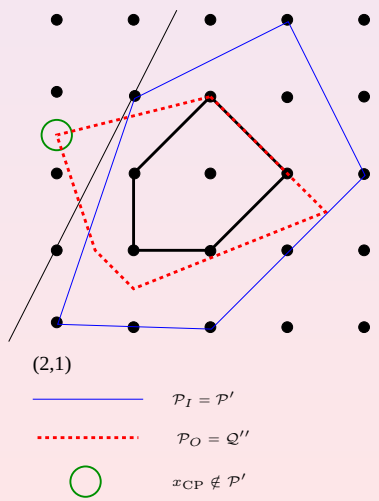
- If $\hat{x}_{\text{CP}}$ lies outside $\mathcal{P}'$ the decomposition will fail
- By the *Farkas Lemma* the proof of infeasibility provides a valid and violated inequality

Decomposition Cuts

$$\begin{array}{lcl} u^t_{\text{DC}} s + \alpha^t_{\text{DC}} & \leq & 0 \; \forall s \in \mathcal{P}' \quad \text{and} \\ u^t_{\text{DC}} \hat{x}_{\text{CP}} + \alpha^t_{\text{DC}} & > & 0 \end{array}$$





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- Originally proposed as a method to solve the VRP with TSP as relaxation.
- Essentially, we are transforming an optimization algorithm into a separation algorithm.
- The machinery for solving this already exists (=column generation)
- Much easier than DW problem because it's a *feasibility* problem and
  - $\hat{x}_i = 0 \Rightarrow s_i = 0$, can remove constraints not in support, and
  - $\hat{x}_i = 1$ and $s_i \in \{0, 1\} \Rightarrow$ constraint is redundant with convexity constraint
  - Often gets *lucky* and produces incumbent solutions to original IP

Branching

- By default, we branch on variables in the compact space.
- In PC, this is done by mapping back to the compact space $\hat{x} = \sum_{s \in \mathcal{E}} s\hat{\lambda}_s$.
- Variable branching in the compact space is constraint branching in the extended space
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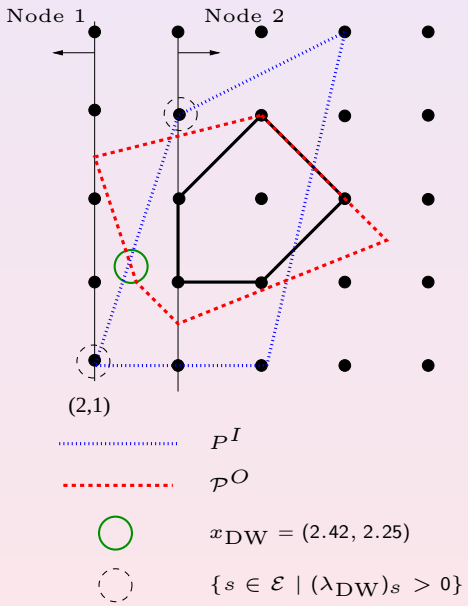

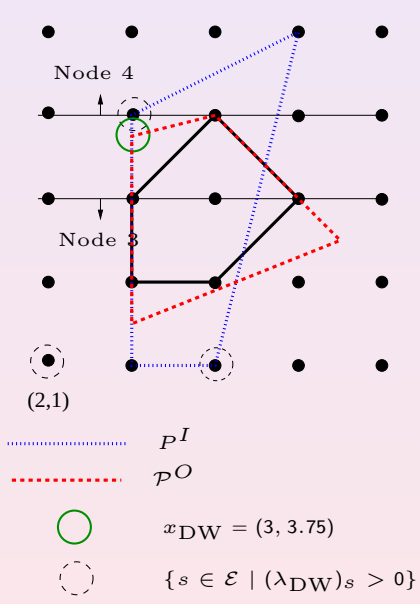

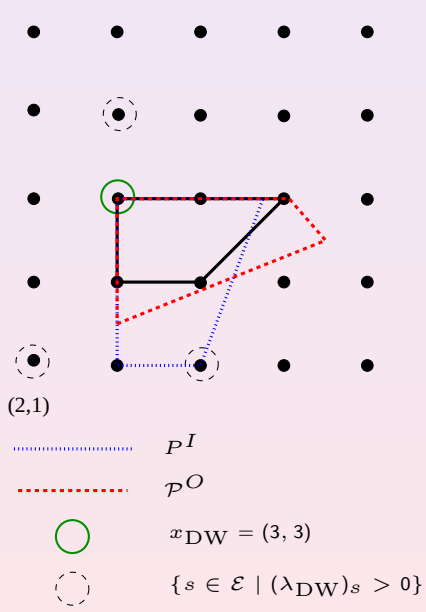

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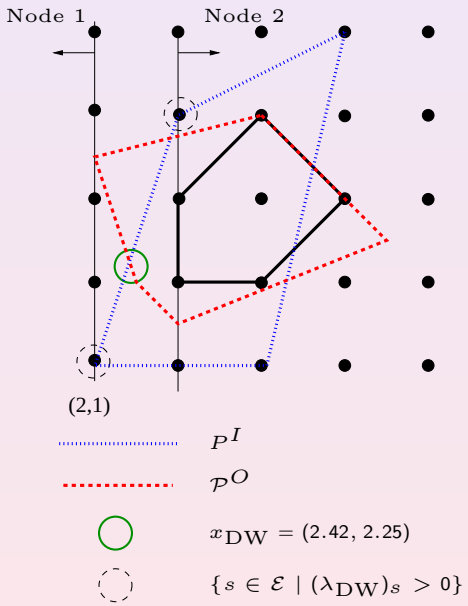

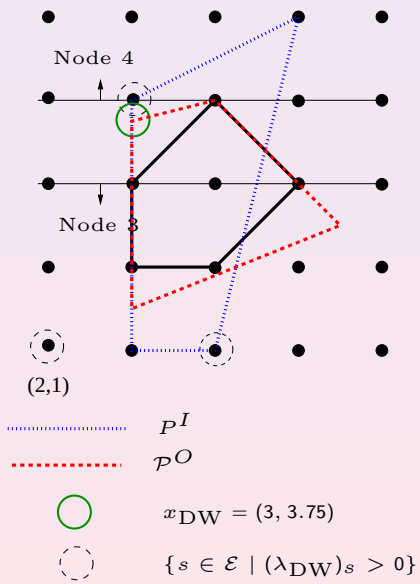

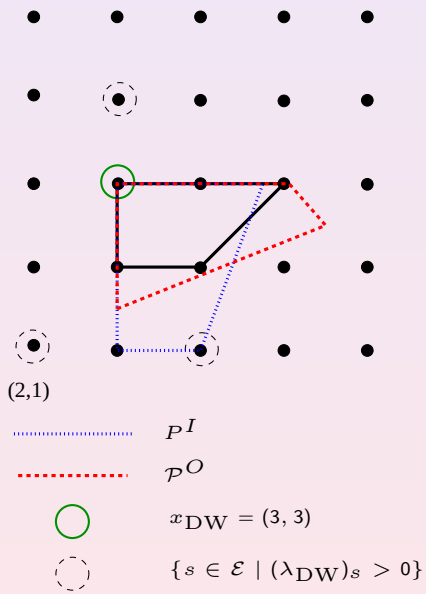

$$\begin{aligned} \text{Node 1:} \quad & 4\lambda_{(4,1)} + 5\lambda_{(5,5)} + 2\lambda_{(2,1)} + 3\lambda_{(3,4)} \leq 2 \\ \text{Node 2:} \quad & 4\lambda_{(4,1)} + 5\lambda_{(5,5)} + 2\lambda_{(2,1)} + 3\lambda_{(3,4)} \geq 3 \end{aligned}$$

Branching for Lagrangian Method

- In general, Lagrangian methods do *not* provide a primal solution λ
- Let $\mathcal{B}$ define the extreme points found in solving subproblems for z_{LD}
- Build an inner approximation using this set, then proceed as in PC

$$\mathcal{P}_I = \left\{ x \in \mathbb{R}^n \;\middle|\; x = \sum_{s \in \mathcal{B}} s\lambda_s, \sum_{s \in \mathcal{B}} \lambda_s = 1, \lambda_s \geq 0 \; \forall s \in \mathcal{B} \right\}$$

$$\min_{\lambda \in \mathbb{R}^{\mathcal{B}}_+} \left\{ c^\top \left(\sum_{s \in \mathcal{B}} s\lambda_s \right) \;\middle|\; A'' \left(\sum_{s \in \mathcal{B}} s\lambda_s \right) \geq b'', \sum_{s \in \mathcal{B}} \lambda_s = 1 \right\}$$

- Closely related to *volume* algorithm and *bundle* methods

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Algorithmic Details and Extensions

- **Separable subproblems** (Important!)
  - Identical subproblems (symmetry)
  - Parallel solution of subproblems
  - Automatic detection
- **Use of generic MILP solution technology**
  - Using the mapping $\hat{x} = \sum_{s \in \mathcal{E}} s\hat{\lambda}_s$ we can use generic MILP generation in RC/PC context
  - Use generic MILP solver to solve subproblems.
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  - For block-angular case, at end of each node, solve with $\lambda \in \mathbb{Z}$
  - Used in *root node* by Barahona and Jensen ('98), we extend to tree

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  - Dual simplex after adding rows or adjusting bounds (warm-start dual feasible)
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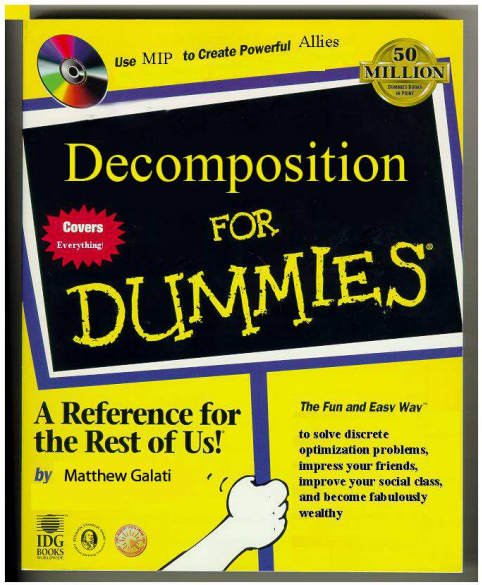
Outline

DIP and CHiPPS

- The application of decomposition methods in practice is hindered by a number of serious drawbacks.
  - *Implementation is difficult*, usually requiring development of sophisticated customized codes.
  - Choosing an algorithmic strategy *requires in-depth knowledge* of theory and strategies are *difficult to compare empirically*.
  - The powerful techniques modern solvers use to solve integer programs are *difficult to integrate* with decomposition-based approaches.
- DIP and CHiPPS are two frameworks that together allow for easier implementation of decomposition approaches.
  - CHiPPS (COIN High Performance Parallel Search Software) is a flexible library hierarchy for implementing parallel search algorithms.
  - DIP (Decomposition for Integer Programs) is a framework for implementing decomposition-based bounding methods.
  - DIP with CHiPPS is a full-blown branch-and-cut-and-price framework in which details of the implementation are hidden from the user.
- DIP can be accessed through a modeling language or by providing a model with notated structure.

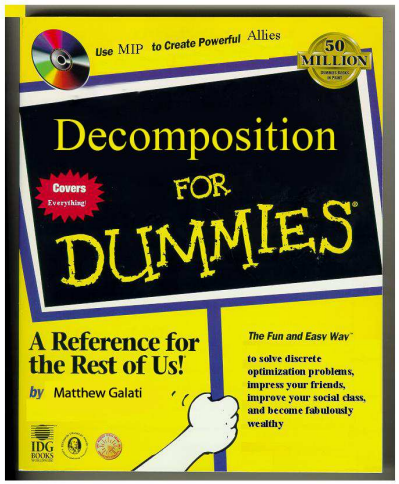


DIP Framework

DIP Framework

DIP (**D**ecomposition for **I**nteger **P**rogramming) is an open-source software framework that provides an implementation of various decomposition methods with minimal user responsibility

- Allows direct comparison CPM/DW/LD/PC/RC/DC in one framework
- DIP abstracts the common, generic elements of these methods
- **Key:** The user defines application-specific components in the space of the compact formulation - greatly simplifying the API
  - Define $[A'', b'']$ and/or $[A', b']$
  - Provide methods for $\mathrm{OPT}(\mathcal{P}', c)$ and/or $\mathrm{SEP}(\mathcal{P}', x)$
- Framework handles all of the algorithm-specific reformulation



DIP Framework: Implementation

COmputational INfrastructure for Operations Research
Have some DIP with your CHiPPS?

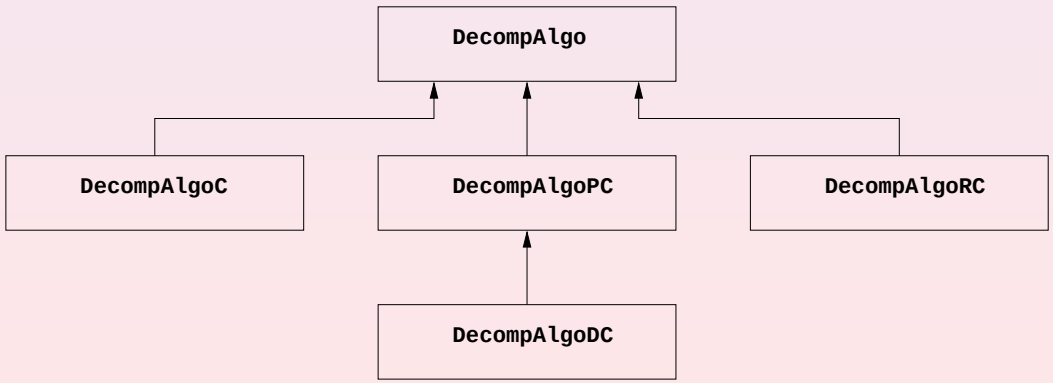
- **DIP** was built around data structures and interfaces provided by COIN-OR
- The **DIP** framework, written in C++, is accessed through two user interfaces:
  - Applications Interface: `DecompApp`
  - Algorithms Interface: `DecompAlgo`
- **DIP** provides the bounding method for branch and bound
- **ALPS** (Abstract Library for Parallel Search) provides the framework for tree search
  - `AlpsDecompModel : public AlpsModel`
    - a wrapper class that calls (data access) methods from `DecompApp`
  - `AlpsDecompTreeNode : public AlpsTreeNode`
    - a wrapper class that calls (algorithmic) methods from `DecompAlgo`

DIP Framework: Creating an Application (C++ API)

- The base class `DecompApp` provides an interface for user to define the application-specific components of their algorithm
- Define the model(s)
  - `setModelObjective(double * c)`: define c
  - `setModelCore(DecompConstraintSet * model)`: define $\mathcal{Q}''$
  - `setModelRelaxed(DecompConstraintSet * model, int block)`: define $\mathcal{Q}'$ [optional]
- `solveRelaxed()`: define a method for $\mathrm{OPT}(\mathcal{P}', c)$ [optional, if $\mathcal{Q}'$, **CBC** is built-in]
- `generateCuts()`: define a method for $\mathrm{SEP}(\mathcal{P}', x)$ [optional, **CGL** is built-in]
- `isUserFeasible()`: is $\hat{x} \in \mathcal{P}$? [optional, if $\mathcal{P} = \mathrm{conv}(\mathcal{P}' \cap \mathcal{Q}'' \cap \mathbb{Z})$]
- All other methods have appropriate defaults but are `virtual` and may be overridden

DIP Framework: Algorithms

- The base class `DecompAlgo` provides the shell (init / master / subproblem / update).
- Each of the methods described has derived default implementations `DecompAlgoX : public DecompAlgo` which are accessible by any application class, allowing full flexibility.
- New, hybrid or extended methods can be easily derived by overriding the various subroutines, which are called from the base class. For example,
  - Alternative methods for solving the master LP in DW, such as **interior point methods**
  - Add stabilization to the dual updates in LD (stability centers)
  - For LD, replace subgradient with **volume** providing an approximate primal solution
  - Hybrid init methods like using LD or DC to initialize the columns of the DW master
  - During PC, adding cuts to either master and/or subproblem.
  - ...



DIP Framework: Example Applications

Application	Description	$\mathcal{P}'$	$\mathbf{OPT}(c)$	$\mathbf{SEP}(x)$	Input
AP3	3-index assignment	AP	Jonker	user	user
ATM	cash management (SAS COE)	MILP(s)	CBC	CGL	user
GAP	generalized assignment	KP(s)	Pisinger	CGL	user
MAD	matrix decomposition	MaxClique	Cliquer	CGL	user
MILP	random partition into A', A''	MILP	CBC	CGL	mps
MILPBlock	user-defined blocks for A'	MILP(s)	CBC	CGL	mps, block
MMKP	multi-dim/choice knapsack	MCKP	Pisinger	CGL	user
		MDKP	CBC	CGL	user
SILP	intro example, tiny IP	MILP	CBC	CGL	user
TSP	traveling salesman problem	1-Tree	Boost	Concorde	user
		2-Match	CBC	Concorde	user
VRP	vehicle routing problem	k-TSP	Concorde	CVRPSEP	user
		b-Match	CBC	CVRPSEP	user

Outline

DIPPY

- DIPPY provides an interface to DIP through the modeling language PuLP.
- PuLP is a modeling language that provides functionality similar to other modeling languages.
- It is built on top of Python so you get the full power of that language for free.
- PuLP and DIPPY are being developed by Stuart Mitchell and Mike O'Sullivan in Auckland and are part of COIN.
- Through DIPPY, a user can
  - Specify the model and the relaxation, including the block structure.
  - Implement methods (coded in Python) for solving the relaxation, generating cuts, custom branching, etc.
- With DIP and DIPPY, it is possible to code a customized column-generation method from scratch in a few hours.
- This would have taken months with previously available tools.

Example: Facility Location Problem

- We are given n facility locations and m customers to be serviced from those locations.
- There is a fixed cost c_j and a capacity W_j associated with facility j.
- There is a cost d_{ij} and demand w_{ij} associated with serving customer i from facility j.
- We have two sets of binary variables.
  - y_j is 1 if facility j is opened, 0 otherwise.
  - x_{ij} is 1 if customer i is served by facility j, 0 otherwise.

Capacitated Facility Location Problem

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{n} c_j y_j + \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} x_{ij} & \\
\text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} = 1 & \forall i \\
& \sum_{i=1}^{m} w_{ij} x_{ij} \leq W_j & \forall j \\
& x_{ij} \leq y_j & \forall i, j \\
& x_{ij}, y_j \in \{0, 1\} & \forall i, j
\end{aligned}
$$

DIPPY Code for Facility Location

DIPPY

```
from facility_data import REQUIREMENT, PRODUCTS, LOCATIONS, CAPACITY

prob = dippy.DipProblem("Facility_Location")

assign = LpVariable.dicts("Assignment", [(i, j) for i in LOCATIONS for
                          j in PRODUCTS], 0, 1, LpBinary)
open   = LpVariable.dicts("FixedCharge", LOCATIONS, 0, 1, LpBinary)

# objective: minimise waste
prob += lpSum(excess[i] for i in LOCATIONS), "min"

# assignment constraints
for j in PRODUCTS:
    prob += lpSum(assign[(i, j)] for i in LOCATIONS) == 1

# Aggregate capacity constraints
for i in LOCATIONS:
    prob.relaxation[i] += lpSum(assign[(i, j)]*REQUIREMENT[j] for j in
                                PRODUCTS) + excess[i] == CAPACITY * open[i]

# Disaggregated capacity constraints
for i in LOCATIONS:
    for j in PRODUCTS:
        prob.relaxation[i] += assign[(i, j)] <= open[i]

# Ordering constraints
for index, location in enumerate(LOCATIONS):
    if index > 0:
        prob += use[LOCATIONS[index-1]] >= open[location]
```

DIPPY Auxiliary Methods for Facility Location

DIPPY

```
def solve_subproblem(prob, index, redCosts, convexDual):
    ...
    z, solution = knapsack01(obj, weights, CAPACITY)
    ...
    return []

prob.relaxed_solver = solve_subproblem

def knapsack01(obj, weights, capacity):
    ...
    return c[n-1][capacity], solution

def first_fit(prob):
    ...
    return bvs

def one_each(prob):
    ...
    return bvs

prob.init_vars = first_fit

dippy.Solve(prob, {
    'TolZero': '%s' % tol,
    'doPriceCut': '1',
    'generateInitVars': '1', })
```

MILPBlock: Decomposition-based MILP Solver

- Many difficult MILPs have a block structure, but this structure is not part of the input (MPS) or is not exploitable by the solver.
- In practice, it is common to have models composed of independent subsystems coupled by global constraints.
- The result may be models that are highly symmetric and difficult to solve using traditional methods, but would be easy to solve if the structure were known.

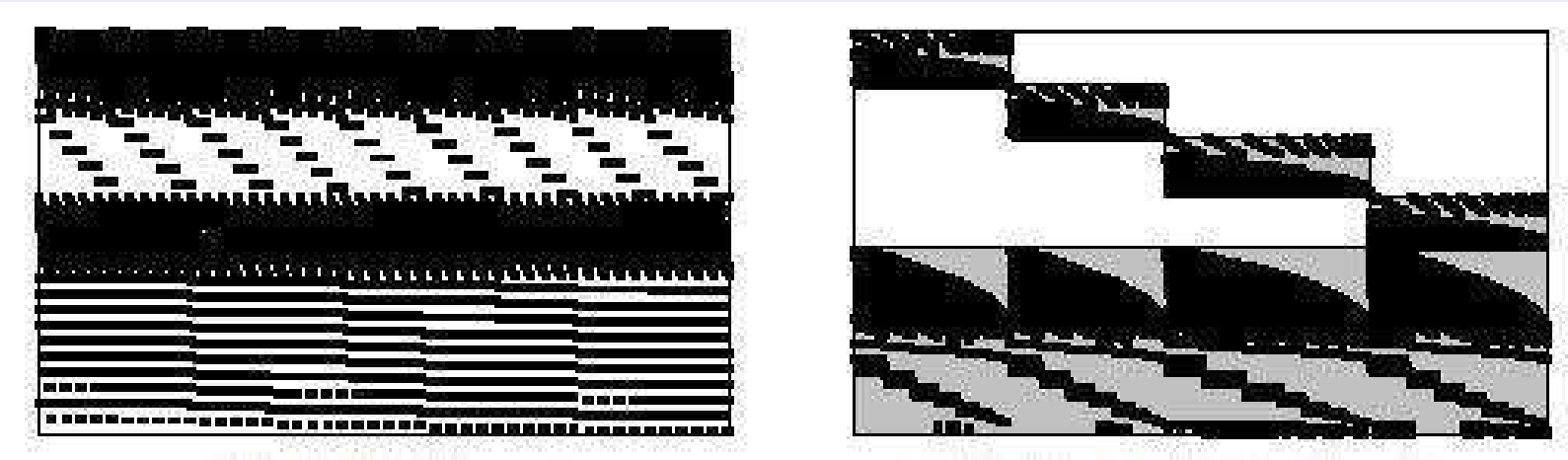
$$\begin{pmatrix} A''_1 & A''_2 & \cdots & A''_\kappa \\ A'_1 & & & \\ & A'_2 & & \\ & & \ddots & \\ & & & A'_\kappa \end{pmatrix}$$

- MILPBlock provides a black-box solver for applying integrated methods to generic MILP
- Input is an MPS/LP and a *block file* specifying structure.
- Optionally, the block file can be automatically generated using the hypergraph partitioning algorithm of HMetis.
- This is the engine underlying DIPPY.

Identifying Block Structure

- The problem of identifying the block structure of a matrix is difficult.
- At the moment, we identify block structure heuristically using a package for hypergraph partitioning called *HMetis*.
  - The columns of the matrix are identified with nodes in a hypergraph.
  - The edges of the hypergraph are the sets of columns corresponding to nonzeros in each row.
  - We partition the nodes in order to minimize the number of hyperedges in the resulting cut.
  - The hyperedges represent the linking rows.
- So far, this seems pretty effective, but this research is in its infancy.

Hidden Block Structure



[1]

Detected block structure for *10teams* instance

[1] Picture from "Generic Dantzig-Wolfe Reformulation of Mixed Integer Programs", M. Bergner et.al., in *Proceeding of the* 15^{th} *International Conference on Integer Programming and Combinatorial Optimization*, pp. 39–51.

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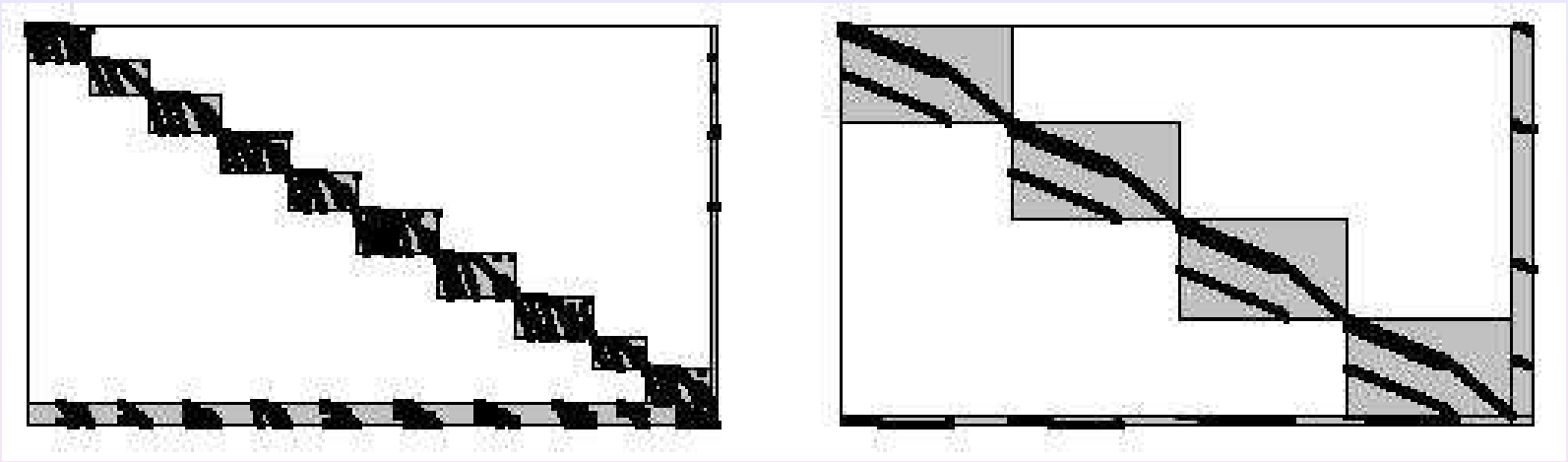

[2]

Detected block structure for *aflow30a* and *set1ch* instances

[2] Picture from "Generic Dantzig-Wolfe Reformulation of Mixed Integer Programs", M. Bergner et.al., in *Proceeding of the* 15^{th} *International Conference on Integer Programming and Combinatorial Optimization*, pp. 39–51.

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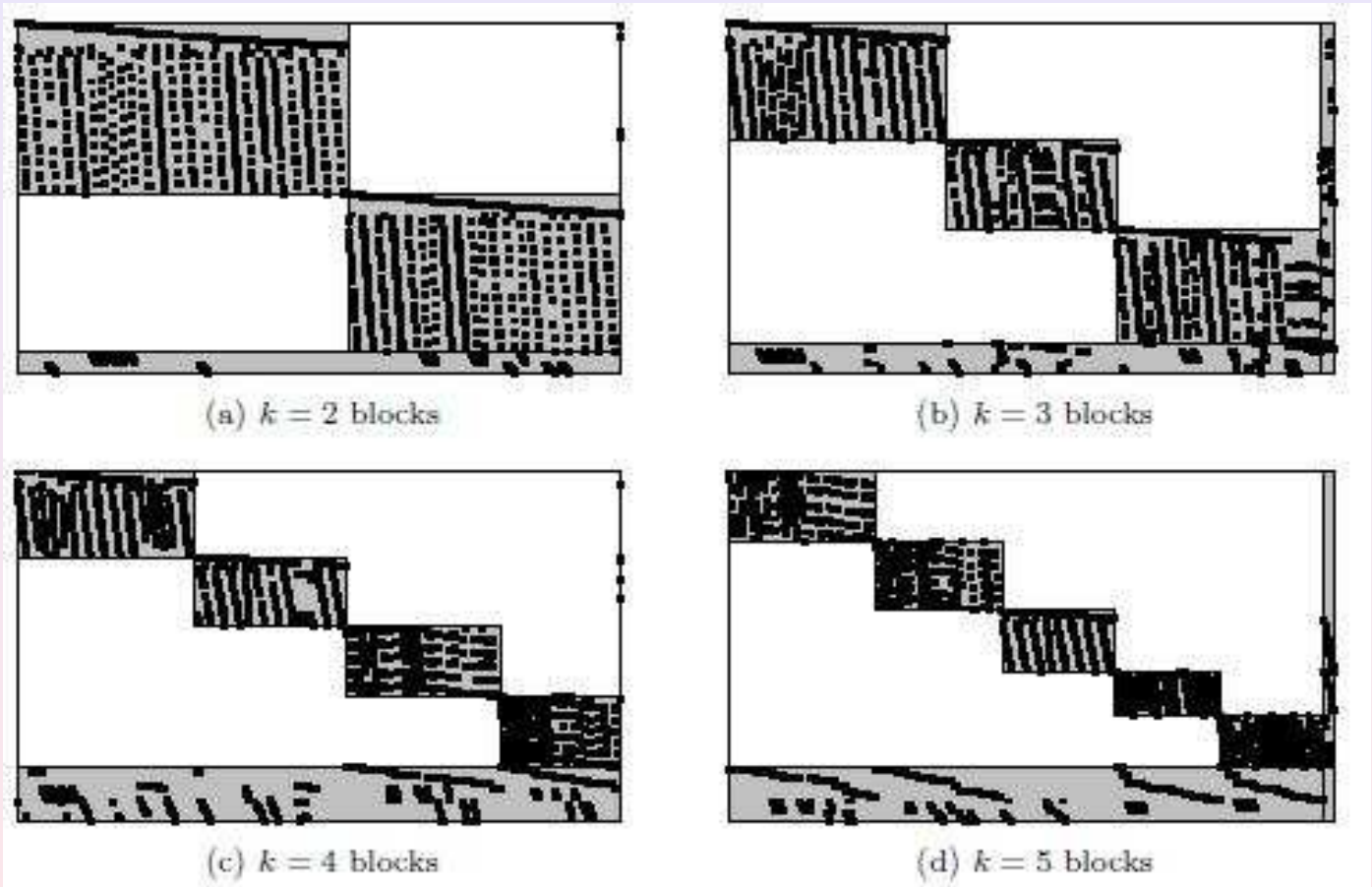
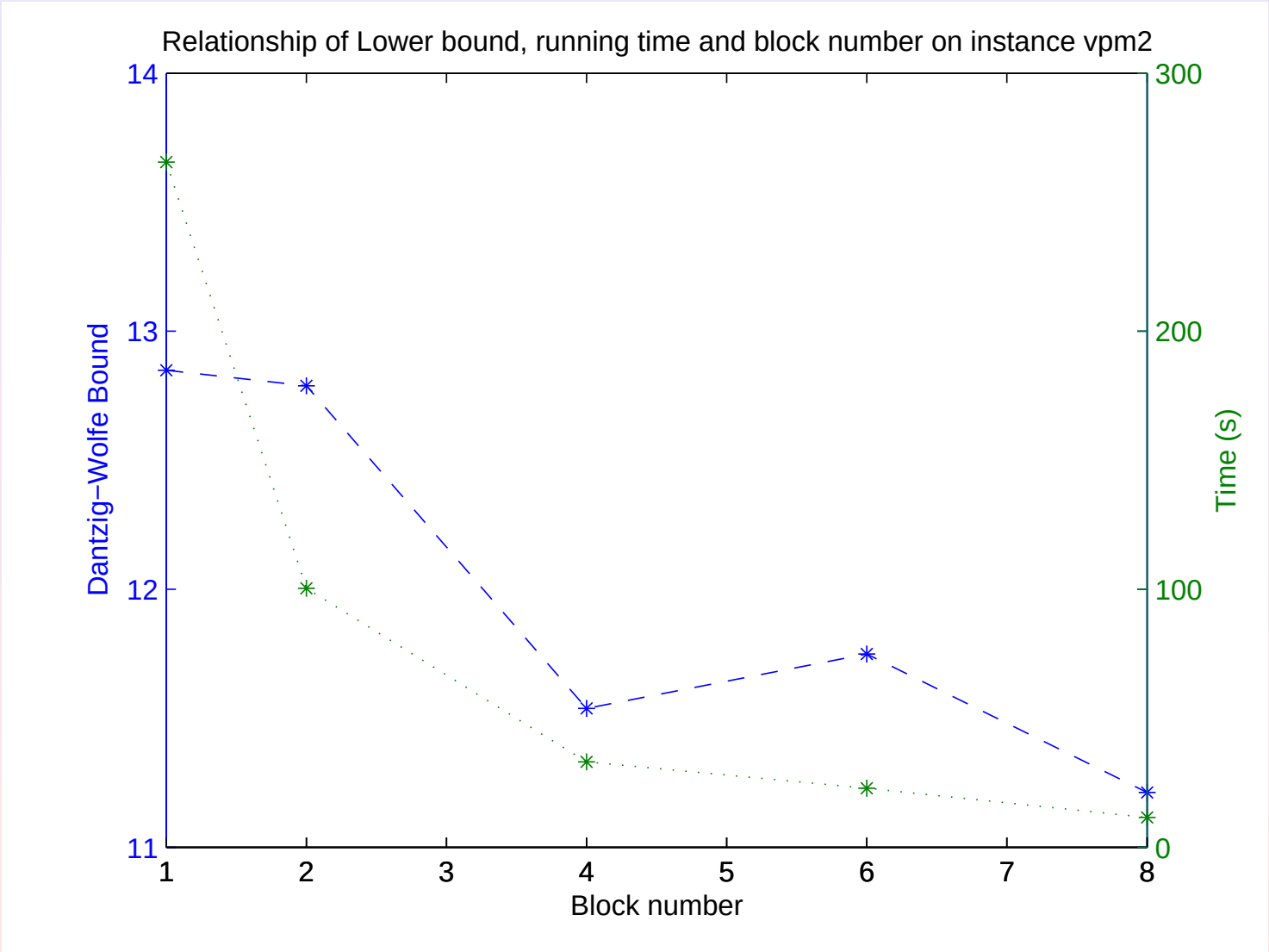


[3]

Structure with different numbers of blocks for *fiber* instance

[3] Picture from "Generic Dantzig-Wolfe Reformulation of Mixed Integer Programs", M. Bergner et.al., in *Proceeding of the* 15^{th} *International Conference on Integer Programming and Combinatorial Optimization*, pp. 39–51.

Exploiting Block Structure



Bound Improvement

insta	cols	rows	opt	k	DWR bound	CBC root
10teams	2025	230	924	3	918.1	917
noswot	128	182	563.8	3	-41.2	-43
p2756	2756	755	3124	3	3115.5	2688.7
timtab1	397	171	764772	3	350885	28694
timtab2	675	294	1096560	3	431963	83592
vpm2	378	234	13.7	3	12.2	9.8
pg5_34	2600	125	-14339.4	3	-15179.2	-16646.5
pg	2700	125	-8674.34	3	-15179.2	-16646.5
k16x240	480	256	10674	3	3303.6	2769.8

Application - Block-Angular MILP (applied to Retail Optimization)

SAS Retail Optimization Solution

- *Multi-tiered supply chain distribution problem* where each block represents a store
- Prototype model developed in SAS/OR's OPTMODEL (algebraic modeling language)

	CPX11			DIP-PC		
Instance	**Time**	**Gap**	**Nodes**	**Time**	**Gap**	**Nodes**
retail27	T	2.30%	2674921	3.18	OPT	1
retail31	T	0.49%	1434931	767.36	OPT	41
retail3	529.77	OPT	2632157	0.54	OPT	1
retail4	T	1.61%	1606911	116.55	OPT	1
retail6	1.12	OPT	803	264.59	OPT	303

Outline

MILPBlock: Recently Added Features

Interfaces for Pricing Algorithms (for IBM Project)

- User can provide an initial dual vector
- User can manipulate duals used at each pass (and specify per block)
- User can select which block to process next (alternative to *all* or *round-robin*)

New Options

- Branching can be auto enforced in subproblem or master (when oracle is MILP)
- Ability to stop subproblem calculation on gap/time and calculate LB (can branch early)
- For oracles that provide it, allow multiple columns for each subproblem call
- Management of compression of columns - once master gap is tight

Performance

- Detection and removal of columns that are close to parallel
- Added basic dual stabilization (Wentges smoothing)
- Redesign (and simplification) of treatment of master-only variables.

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Related Projects Currently using DIP

- **OSDip** – Optimization Services (**OS**) wraps DIP (in CoinBazaar)
  - University of Chicago – Kipp Martin
- **Dippy** – Python interface for **DIP** through **PuLP**
  - University of Auckland – Michael O'Sullivan
- **SAS** – surface MILPBlock-like solver for `PROC OPTMODEL`
  - SAS Institute – Matthew Galati
- **National Workforce Management, Cross-Training and Scheduling Project**
  - IBM Business Process Re-engineering – Alper Uygur
- **Transmission Switching Problem for Electricity Networks**
  - University of Denmark – Jonas Villumsem
  - University of Auckland – Andy Philipott

DIP@SAS in PROC OPTMODEL

- Prototype **PC** algorithm embedded in PROC OPTMODEL (based on MILPBlock)
- Minor API change - one new suffix on rows *or* cols (`.block`)

Preliminary Results (Recent Clients):

Client Problem	**IP-GAP**		**Real-Time**	
	DIP@SAS	**CPX12.1**	**DIP@SAS**	**CPX12.1**
ATM Cash Management and Predictive Model (India)	OPT	∞	103	2000 (T)
ATM Cash Management (Singapore)	OPT	OPT	86	831
	OPT	OPT	90	783
Retail Inventory Optimization (UK)	1.6%	9%	1200	1200 (T)
	4.7%	19%	1200	1200 (T)
	2.6%	∞	1200	1200 (T)

Future Research

- **Branch-and-Relax-and-Cut** - computational focus thus far has been on CPM/DC/PC
- Can we implement **Gomory cuts** in Price-and-Cut?
  - Similar to Interior Point crossover to Simplex, we can crossover from $\hat{x}$ to a feasible basis, load that into the solver and generate tableau cuts
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