Duality for Mixed Integer Linear Programming

TED RALPHS
MENAL GÜZELSOY
ISE Department
COR@L Lab
Lehigh University
ted@lehigh.edu

University of Newcastle, Newcastle, Australia
13 May 2009

Thanks: Work supported in part by the National Science Foundation

Outline

1. Duality Theory
   - Introduction
   - IP Duality
2. The Subadditive Dual
   - Formulation
   - Properties
3. Constructing Dual Functions
   - The Value Function
   - Gomory's Procedure
   - Branch-and-Bound Method
   - Branch-and-Cut

What is Duality?

- It is difficult to give a general definition of mathematical duality, though mathematics is replete with various notions of it.
  - Set Theory and Logic (De Morgan Laws)
  - Geomety (Pascal's Theorem & Brianchon's Theorem)
  - Combinatorics (Graph Coloring)
- The duality we are interested in is a sort of *functional duality*.
- We define a generic optimization problem to be a mapping $f : X \to \mathbb{R}$, where X is the set of possible *inputs* and $f(x)$ is the *result*.
- Duality may then be defined as a method of transforming a given *primal problem* to an associated *dual problem* such that
  - the dual problem yields a bound on the primal problem, and
  - applying a related transformation to the dual produces the primal again.
- In many case, we would also like to require that the dual bound be "close" to the primal result for a specific input of interest.

Duality in Mathematical Programming

- In mathematical programming, the input is the problem data (e.g., the constraint matrix, right-hand side, and cost vector for a linear program).
- We view the primal and the dual as parametric problems, but some data is held constant.

Uses of the Dual in Mathematical Programing

- If the dual is easier to evaluate, we can use it to obtain a bound on the primal optimal value.
- We can also use the dual to perform sensitivity analysis on the parameterized primal input data.
- Finally, we can also use the dual to warm start solution procedure based on evaluation of the dual.

Duality in Integer Programming

- We will initially be interested in the mixed integer linear program (MILP) instance

$$z_{IP} = \min_{x \in \mathcal{S}} cx, \tag{P}$$

where, $c \in \mathbb{R}^n$, $S = \{x \in \mathbb{Z}^r_+ \times \mathbb{R}^{n-r}_+ \mid Ax = b\}$ with $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{R}^m$.

- We call this instance the *base primal instance*.
- To construct a dual, we need a parameterized version of this instance.
- For reasons that will become clear, the most relevant parameterization is of the right-hand side.
- The *value function* (or *primal function*) of the base primal instance (P) is

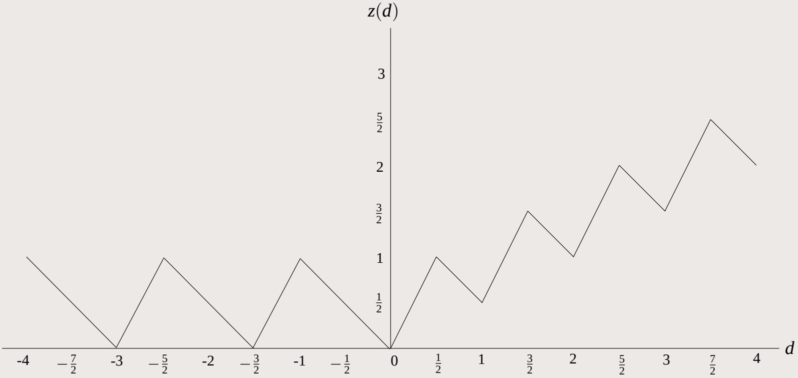
$$z(d) = \min_{x \in \mathcal{S}(d)} cx,$$

where for a given $d \in \mathbb{R}^m$, $\mathcal{S}(d) = \{x \in \mathbb{Z}^r_+ \times \mathbb{R}^{n-r}_+ \mid Ax = d\}$.

- We let $z(d) = \infty$ if $d \in \Omega = \{d \in \mathbb{R}^m \mid \mathcal{S}(d) = \emptyset\}$.

Example: Value Function

$$\begin{aligned} z_{IP} = \min \quad & \tfrac{1}{2}x_1 + 2x_3 + x_4 \\ \text{s.t} \quad & x_1 - \tfrac{3}{2}x_2 + x_3 - x_4 = b \quad \text{and} \\ & x_1, x_2 \in \mathbb{Z}_+, x_3, x_4 \in \mathbb{R}_+. \end{aligned}$$



Dual Functions

- A *dual function* $F : \mathbb{R}^m \to \mathbb{R}$ is one that satisfies $F(d) \leq z(d)$ for all $d \in \mathbb{R}^m$.
- How to select such a function?
- We choose may choose one that is easy to construct/evaluate and/or for which $F(b) \approx z(b)$.
- This results in the *base dual instance*

$$z_D = \max\,\{F(b) : F(d) \leq z(d),\ d \in \mathbb{R}^m, F \in \Upsilon^m\}$$

  where $\Upsilon^m \subseteq \{f \mid f : \mathbb{R}^m \to \mathbb{R}\}$
- We call F^* *strong* for this instance if F^* is a *feasible* dual function and $F^*(b) = z(b)$.
- This dual instance always has a solution F^* that is strong if the value function is bounded and $\Upsilon^m \equiv \{f \mid f : \mathbb{R}^m \to \mathbb{R}\}$. Why?

The LP Relaxation Dual Function

- It is easy to obtain a feasible dual function for any MILP.
- Consider the value function of the LP relaxation of the primal problem:

$$F_{LP}(d) = \max_{v \in \mathbb{R}^m} \{vd : vA \leq c\}.$$

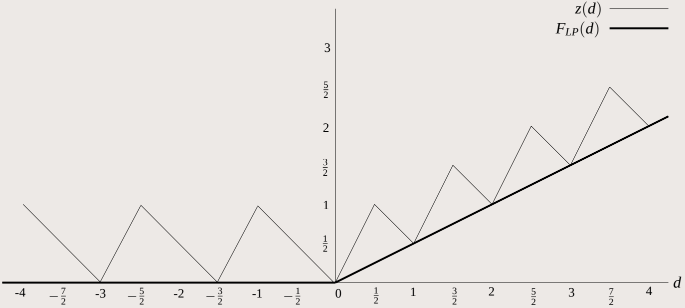
- By linear programming duality theory, we have $F_{LP}(d) \leq z(d)$ for all $d \in \mathbb{R}^m$.
- Of course, F_{LP} is not necessarily strong.

Example: LP Dual Function

$$F_{LP}(d) = \min \quad vd, \\ \text{s.t} \quad 0 \geq v \geq -\tfrac{1}{2}, \text{ and} \\ v \in \mathbb{R},$$

which can be written explicitly as

$$F_{LP}(d) = \begin{cases} 0, & d \leq 0 \\ -\frac{1}{2}d, & d > 0 \end{cases} .$$



The Subadditive Dual

By considering that

$$F(d) \leq z(d),\ d \in \mathbb{R}^m \quad \Longleftrightarrow \quad F(d) \leq cx\ ,\ x \in \mathcal{S}(d),\ d \in \mathbb{R}^m$$
$$\Longleftrightarrow \quad F(Ax) \leq cx\ ,\ x \in \mathbb{Z}^n_+,$$

the generalized dual problem can be rewritten as

$$z_D = \max\ \{F(b) : F(Ax) \leq cx,\ x \in \mathbb{Z}^r_+ \times \mathbb{R}^{n-r}_+,\ F \in \Upsilon^m\}.$$

Can we further restrict Υ^m and still guarantee a strong dual solution?

- The class of linear functions? NO!
- The class of convex functions? NO!
- The class of sudadditive functions? YES!

The Subadditive Dual

- Let a function F be defined over a domain V. Then F is subadditive if $F(v_1) + F(v_2) \geq F(v_1 + v_2) \forall v_1, v_2, v_1 + v_2 \in V$.
- Note that the value function z is subadditive over Ω. Why?
- If $\Upsilon^m \equiv \Gamma^m \equiv \{F \text{ is subadditive} \mid F : \mathbb{R}^m \rightarrow \mathbb{R}, F(0) = 0\}$, we can rewrite the dual problem above as the *subadditive dual*

$$\begin{aligned} z_D = \max \quad & F(b) \\ & F(a^j) \leq c_j \quad j = 1, ..., r, \\ & \bar{F}(a^j) \leq c_j \quad j = r+1, ..., n, \text{ and} \\ & F \in \Gamma^m, \end{aligned}$$

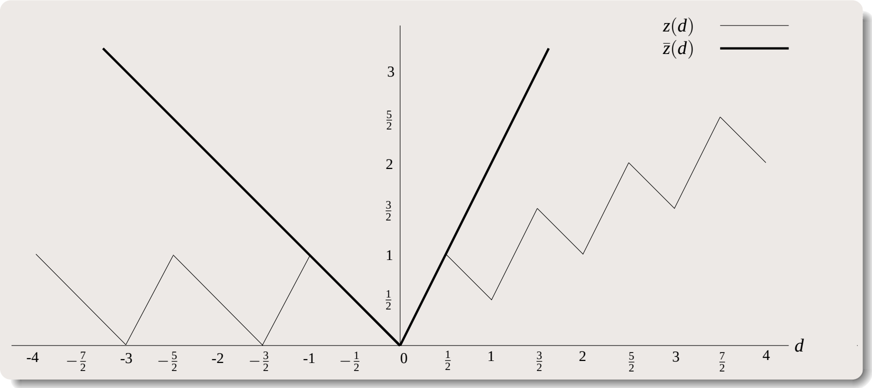
where the function $\bar{F}$ is defined by

$$\bar{F}(d) = \limsup_{\delta \rightarrow 0^+} \frac{F(\delta d)}{\delta} \quad \forall d \in \mathbb{R}^m.$$

- Here, $\bar{F}$ is the *upper d-directional derivative* of F at zero.

Example: Upper D-directional Derivative

- The upper d-directional derivative can be interpreted as the slope of the value function in direction d at 0.
- For the example, we have



Weak Duality

Weak Duality Theorem

Let x be a feasible solution to the primal problem and let F be a feasible solution to the subadditive dual. Then, $F(b) \leq cx$.

Proof.

Corollary

For the primal problem and its subadditive dual:

1. If the primal problem (resp., the dual) is unbounded then the dual problem (resp., the primal) is infeasible.
2. If the primal problem (resp., the dual) is infeasible, then the dual problem (resp., the primal) is infeasible or unbounded.

Strong Duality

Strong Duality Theorem

If the primal problem (resp., the dual) has a finite optimum, then so does the subadditive dual problem (resp., the primal) and they are equal.

Outline of the Proof. Show that the value function z or an extension to z is a feasible dual function.

- Note that z satisfies the dual constraints.
- $\Omega \equiv \mathbb{R}^m$: $z \in \Gamma^m$.
- $\Omega \subset \mathbb{R}^m$: $\exists\, z_e \in \Gamma^m$ with $z_e(d) = z(d)\ \forall\, d \in \Omega$ and $z_e(d) < \infty\ \forall d \in \mathbb{R}^m$.

Example: Subadditive Dual

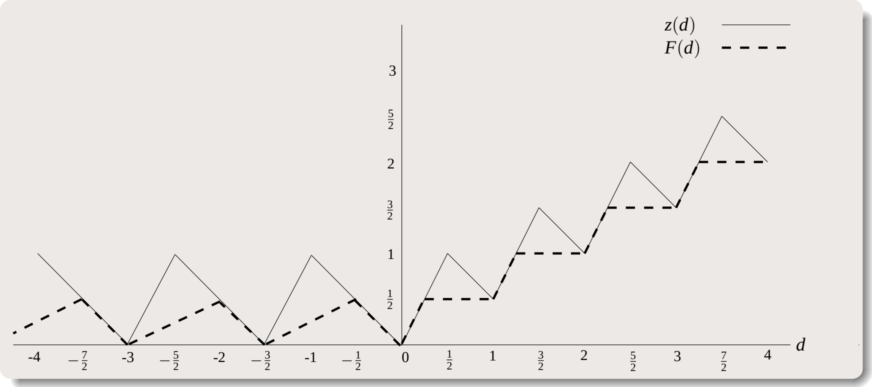
For our IP instance, the subadditive dual problem is

$$\begin{array}{lrl} \max & F(b) & \\ & F(1) & \leq \frac{1}{2} \\ & F(-\frac{3}{2}) & \leq 0 \\ & \bar{F}(1) & \leq 2 \\ & \bar{F}(-1) & \leq 1 \\ & F \in \Gamma^1. & \end{array} .$$

and we have the following feasible dual functions:

1. $F_1(d) = \frac{d}{2}$ is an optimal dual function for $b \in \{0, 1, 2, ...\}$.
2. $F_2(d) = 0$ is an optimal function for $b \in \{..., -3, -\frac{3}{2}, 0\}$.
3. $F_3(d) = \max\{\frac{1}{2}\lceil d - \frac{\lceil \lceil d \rceil - d \rceil}{4} \rceil, 2d - \frac{3}{2}\lceil d - \frac{\lceil \lceil d \rceil - d \rceil}{4} \rceil\}$ is an optimal function for $b \in \{[0, \frac{1}{4}] \cup [1, \frac{5}{4}] \cup [2, \frac{9}{4}] \cup ...\}$.
4. $F_4(d) = \max\{\frac{3}{2}\lceil \frac{2d}{3} - \frac{2\lceil \lceil \frac{2d}{3} \rceil - \frac{2d}{3} \rceil}{3} \rceil - d, -\frac{3}{4}\lceil \frac{2d}{3} - \frac{2\lceil \lceil \frac{2d}{3} \rceil - \frac{2d}{3} \rceil}{3} \rceil + \frac{d}{2}\}$ is an optimal function for $b \in \{... \cup [-\frac{7}{2}, -3] \cup [-2, -\frac{3}{2}] \cup [-\frac{1}{2}, 0]\}$

Example: Feasible Dual Functions



- Notice how different dual solutions are optimal for some right-hand sides and not for others.
- Only the value function is optimal for all right-hand sides.

Farkas' Lemma (Pure Integer)

For the primal problem, exactly one of the following holds:

1. $\mathcal{S} \neq \emptyset$
2. There is an $F \in \Gamma^m$ with $F(a^j) \geq 0, j = 1, ..., n$, and $F(b) < 0$.

Proof. Let $c = 0$ and apply strong duality theorem to subadditive dual.

Complementary Slackness (Pure Integer)

For a given right-hand side b, let x^* and F^* be feasible solutions to the primal and the subadditive dual problems, respectively. Then x^* and F^* are optimal solutions if and only if

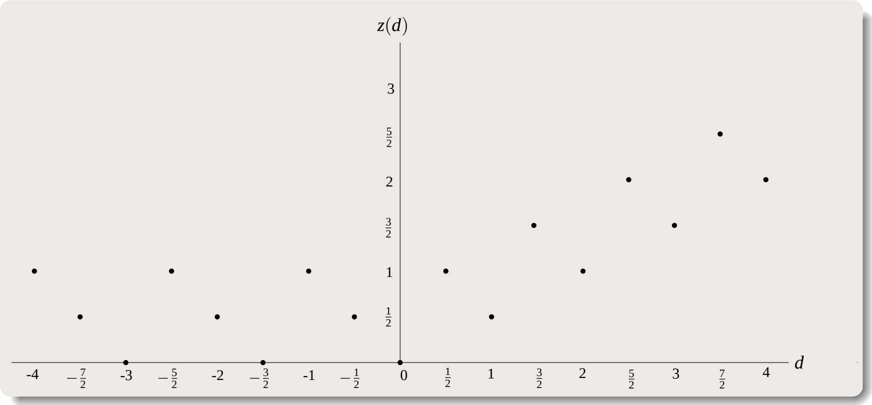
1. $x_j^*(c_j - F^*(a^j)) = 0, j = 1, \ldots, n$ and
2. $F^*(b) = \sum_{j=1}^n F^*(a^j)x_j^*$.

Proof. For an optimal pair we have

$$F^*(b) = F^*(Ax^*) = \sum_{j=1}^{n} F^*(a^j)x_j^* = cx^*.$$

Example: Pure Integer Case

If we require integrality of all variables in our previous example, then the value function becomes



Constructing Dual Functions

- Explicit construction
  - The Value Function
  - Generating Functions
- Relaxations
  - Lagrangian Relaxation
  - Quadratic Lagrangian Relaxation
  - Corrected Linear Dual Functions
- Primal Solution Algorithms
  - Cutting Plane Method
  - Branch-and-Bound Method
  - Branch-and-Cut Method

Properties of the Value Function

- It is subadditive over Ω.
- It is piecewise polyhedral.
- For an ILP, it can be obtained by a finite number of limited operations on elements of the RHS:

$$\left.\begin{array}{l}\left.\begin{array}{l}\text{(i) rational multiplication}\\ \text{(ii) nonnegative combination}\\ \text{(iii) rounding}\end{array}\right\}\textit{Chvátal fcns.}\\ \text{(iv) taking the minimum}\end{array}\right\}\textit{Gomory fcns.}$$

The Value Function for MILPs

- There is a one-to-one correspondence between ILP instances and Gomory functions.
- The *Jeroslow Formula* shows that the value function of a MILP can also be computed by taking the minimum of different values of a single Gomory function with a correction term to account for the continuous variables.
- The value function of the earlier example is

$$z(d) = \min \left\{ \begin{array}{l} \frac{3}{2} \max \left\{ \left\lceil \frac{\lfloor 2d \rfloor}{3} \right\rceil, \left\lceil \frac{\lfloor 2d \rfloor}{2} \right\rceil \right\} + \frac{3\lceil 2d \rceil}{2} + 2d, \\ \frac{3}{2} \max \left\{ \left\lceil \frac{\lceil 2d \rceil}{3} \right\rceil, \left\lceil \frac{\lceil 2d \rceil}{2} \right\rceil \right\} - d, \end{array} \right\}$$

Jeroslow Formula

Let the set $\mathscr{E}$ consist of the index sets of dual feasible bases of the linear program

$$\min\{\frac{1}{M}c_C x_C \ : \ \frac{1}{M}A_C x_C = b, x \geq 0\}$$

where $M \in \mathbb{Z}_+$ such that for any $E \in \mathscr{E}$, $MA_E^{-1}a^j \in \mathbb{Z}^m$ for all $j \in I$.

Theorem (Jeroslow Formula)

There is a $g \in \mathscr{G}^m$ such that

$$z(d) = \min_{E \in \mathscr{E}} g(\lfloor d \rfloor_E) + v_E(d - \lfloor d \rfloor_E) \ \ \forall d \in \mathbb{R}^m \ \ \textit{with} \ \ \mathcal{S}(d) \neq \emptyset,$$

where for $E \in \mathscr{E}$, $\lfloor d \rfloor_E = A_E \lfloor A_E^{-1} d \rfloor$ and v_E is the corresponding basic feasible solution.

The Single Constraint Case

- Let us now consider an MILP with a single constraint for the purposes of illustration:

$$\min_{x \in \mathcal{S}} cx, \tag{P1}$$

$c \in \mathbb{R}^n$, $S = \{x \in \mathbb{Z}^r_+ \times \mathbb{R}^{n-r}_+ \mid a'x = b\}$ with $a \in \mathbb{Q}^n$, $b \in \mathbb{R}$.

- The value function of (P) is

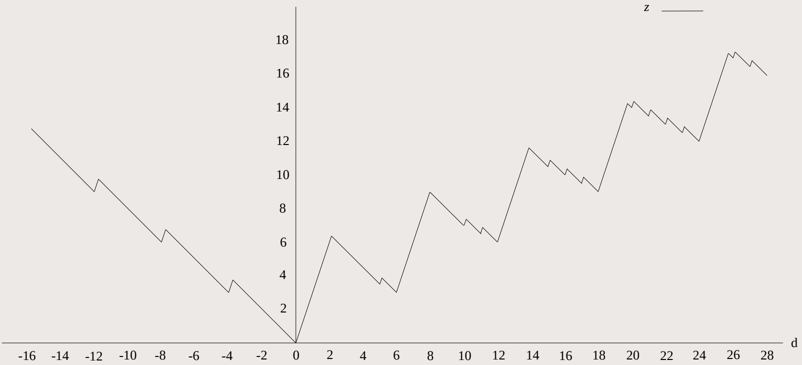
$$z(d) = \min_{x \in \mathcal{S}(d)} cx,$$

where for a given $d \in \mathbb{R}$, $\mathcal{S}(d) = \{x \in \mathbb{Z}^r_+ \times \mathbb{R}^{n-r}_+ \mid a'x = d\}$.

- Assumptions: Let $I = \{1, \ldots, r\}$, $C = \{r+1, \ldots, n\}$, $N = I \cup C$.
  - $z(0) = 0 \Longrightarrow z : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$,
  - $N^+ = \{i \in N \mid a_i > 0\} \neq \emptyset$ and $N^- = \{i \in N \mid a_i < 0\} \neq \emptyset$,
  - $r < n$, that is, $|C| \geq 1 \Longrightarrow z : \mathbb{R} \to \mathbb{R}$.

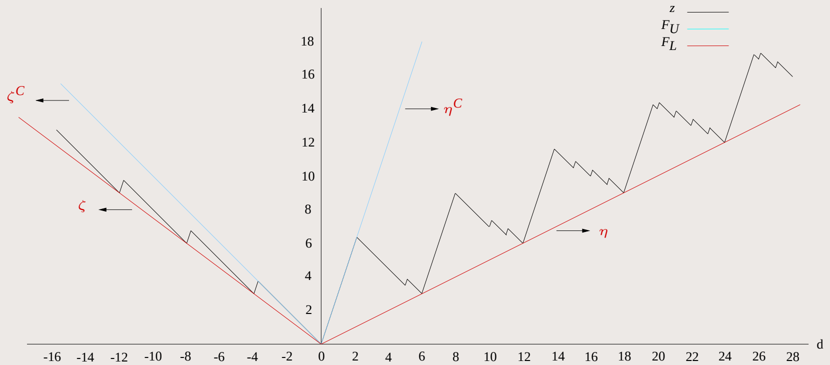
Example

$$\begin{array}{rl} \min & 3x_1 + \frac{7}{2}x_2 + 3x_3 + 6x_4 + 7x_5 + 5x_6 \\ \text{s.t} & 6x_1 + 5x_2 - 4x_3 + 2x_4 - 7x_5 + x_6 = b \quad \text{and} \\ & x_1, x_2, x_3 \in \mathbb{Z}_+, x_4, x_5, x_6 \in \mathbb{R}_+. \end{array} \qquad \text{(SP)}$$



Example (cont'd)

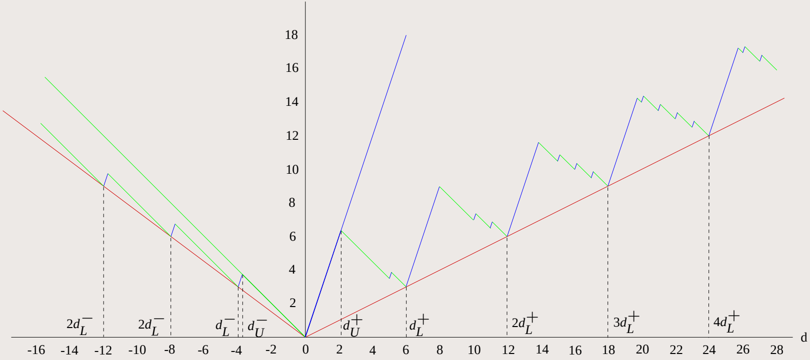
$\eta = \frac{1}{2}$, $\zeta = -\frac{3}{4}$, $\eta^C = 3$ and $\zeta^C = -1$:



- $\{\eta = \eta^C\} \Longleftrightarrow \{z(d) = F_U(d) = F_L(d)\ \forall d \in \mathbb{R}_+\}$
- $\{\zeta = \zeta^C\} \Longleftrightarrow \{z(d) = F_U(d) = F_L(d)\ \forall d \in \mathbb{R}_-\}$

Observations

Consider $d_U^+, d_U^-, d_L^+, d_L^-$:



The relation between F_U and the linear segments of z: $\{\eta^C, \zeta^C\}$

Redundant Variables

Let $T \subseteq C$ be such that

- $t^+ \in T$ if and only if $\eta^C < \infty$ and $\eta^C = \frac{c_{t^+}}{a_{t^+}}$ and similarly,
- $t^- \in T$ if and only if $\zeta^C > -\infty$ and $\zeta^C = \frac{c_{t^-}}{a_{t^-}}$.

and define

$$\begin{aligned} \nu(d) = \min \quad & c_I x_I + c_T x_T \\ s.t. \quad & a_I x_I + a_T x_T = d \\ & x_I \in \mathbb{Z}_+^I, \quad x_T \in \mathbb{R}_+^T \end{aligned}$$

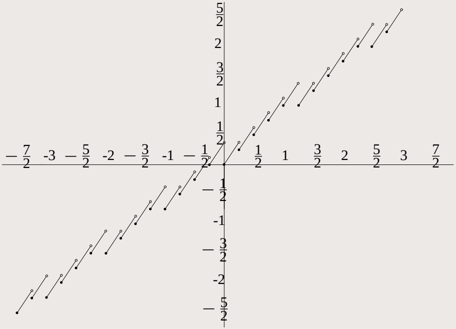
Then

- $\nu(d) = z(d)$ for all $d \in \mathbb{R}$.
- The variables in $C \backslash T$ are redundant.
- z can be represented with at most 2 continuous variables.

Example

$$\begin{array}{rl} \min & x_1 - 3/4x_2 + 3/4x_3 \\ \text{s.t} & 5/4x_1 - x_2 + 1/2x_3 = b, x_1, x_2 \in \mathbb{Z}_+, x_3 \in \mathbb{R}_+. \end{array}$$

$\eta^c = 3/2, \zeta^C = -\infty.$



For each discontinuous point d_i, we have $d_i - (5/4y_1^i - y_2^i) = 0$ and each linear segment has the slope of $\eta^C = 3/2$.

Jeroslow Formula

- Let $M \in \mathbb{Z}_+$ be such that for any $t \in T$, $\frac{Ma_j}{a_t} \in \mathbb{Z}$ for all $j \in I$.
- Then there is a Gomory function g such that

$$z(d) = \min_{t \in T}\{g(\lfloor d \rfloor_t) + \frac{c_t}{a_t}(d - \lfloor d \rfloor_t)\}, \ \ \lfloor d \rfloor_t = \frac{a_t}{M}\left\lfloor \frac{Md}{a_t} \right\rfloor, \ \ \forall d \in \mathbb{R}$$

- Such a Gomory function can be obtained from the value function of a related PILP.
- For $t \in T$, setting

$$\omega_t(d) = g(\lfloor d \rfloor_t) + \frac{c_t}{a_t}(d - \lfloor d \rfloor_t) \ \forall d \in \mathbb{R},$$

we can write

$$z(d) = \min_{t \in T} \omega_t(d) \ \forall d \in \mathbb{R}$$

Piecewise Linearity and Continuity

- For $t \in T$, ω_t is piecewise linear with finitely many linear segments on any closed interval and each of those linear segments has a slope of η^C if $t = t^+$ or ζ^C if $t = t^-$.
- ω_{t^+} is continuous from the right, ω_{t^-} is continuous from the left.
- ω_{t^+} and ω_{t^-} are both lower-semicontinuous.

Theorem

- *z is piecewise-linear with finitely many linear segments on any closed interval and each of those linear segments has a slope of η^C or ζ^C.*
- *(Meyer 1975) z is lower-semicontinuous.*
- *$\eta^C < \infty$ if and only if z is continuous from the right.*
- *$\zeta^C > -\infty$ if and only if z is continuous from the left.*
- *Both η^C and ζ^C are finite if and only if z is continuous everywhere.*

Maximal Subadditive Extension

- Let $f : [0, h] \rightarrow \mathbb{R}, h > 0$ be subadditive and $f(0) = 0$.
- The maximal subadditive extension of f from $[0, h]$ to $\mathbb{R}_+$ is

$$f_S(d) = \begin{cases} f(d) & \text{if } \ d \in [0, h] \\ \inf_{\mathcal{C} \in \mathcal{C}(d)} \sum_{\rho \in \mathcal{C}} f(\rho) & \text{if } \ d > h \end{cases},$$

  - $\mathcal{C}(d)$ is the set of all finite collections $\{\rho_1, ..., \rho_R\}$ such that $\rho_i \in [0, h], i = 1, ..., R$ and $\sum_{i=1}^{R} \rho_i = d$.
  - Each collection $\{\rho_1, ..., \rho_R\}$ is called an h-partition of d.
  - We can also extend a subadditive function $f : [h, 0] \rightarrow \mathbb{R}, h < 0$ to $\mathbb{R}_-$ similarly.
- (Bruckner 1960) f_S is subadditive and if g is any other subadditive extension of f from $[0, h]$ to $\mathbb{R}_+$, then $g \leq f_S$ (maximality).

Extending the Value Function

- Suppose we use z itself as the seed function.
- Observe that we can change the "inf" to "min":

Lemma

Let the function $f : [0, h] \to \mathbb{R}$ *be defined by* $f(d) = z(d) \; \forall d \in [0, h]$. *Then,*

$$f_S(d) = \begin{cases} z(d) & \text{if } d \in [0, h] \\ \min_{\mathcal{C} \in \mathcal{C}(d)} \sum_{\rho \in \mathcal{C}} z(\rho) & \text{if } d > h \end{cases} .$$

- For any $h > 0$, $z(d) \leq f_S(d) \; \forall d \in \mathbb{R}_+$.
- Observe that for $d \in \mathbb{R}_+$, $f_S(d) \to z(d)$ while $h \to \infty$.
- Is there an $h < \infty$ such that $f_S(d) = z(d) \; \forall d \in \mathbb{R}_+$?

Extending the Value Function (cont.)

Yes! For large enough h, maximal extension produces the value function itself.

Theorem

Let $d_r = \max\{a_i \mid i \in N\}$ and $d_l = \min\{a_i \mid i \in N\}$ and let the functions f_r and f_l be the maximal subadditive extensions of z from the intervals $[0, d_r]$ and $[d_l, 0]$ to $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively. Let

$$F(d) = \begin{cases} f_r(d) & d \in \mathbb{R}_+ \\ f_l(d) & d \in \mathbb{R}_- \end{cases}$$

then, $z = F$.

Outline of the Proof.

- $z \leq F$: By construction.
- $z \geq F$: Using MILP duality, F is dual feasible.

In other words, the value function is completely encoded by the breakpoints in $[d_l, d_r]$ and 2 slopes.

General Procedure

- We will construct the value function in two steps
  - Construct the value function on $[d_l, d_r]$.
  - Extend the value function to the entire real line from $[d_l, d_r]$.
- Assumptions
  - We assume $\eta^C < \infty$ and $\zeta^C < \infty$.
  - We construct the value function over $\mathbb{R}_+$ only.
  - These assmuptions are only needed to simplify the presentation.
- Constructing the Value Function on $[0, d_r]$
  - If both η^C and ζ^C are finite, the value function is continuous and the slopes of the linear segments alternate between η^C and ζ^C.
  - For $d_1, d_2 \in [0, d_r]$, if $z(d_1)$ and $z(d_2)$ are connected by a line with slope η^C or ζ^C, then z is linear over $[d_1, d_2]$ with the respective slope (subadditivity).
  - With these observations, we can formulate a finite algorithm to evaluate z in $[d_l, d_r]$.

Example (cont'd)

$d_r = 6$:

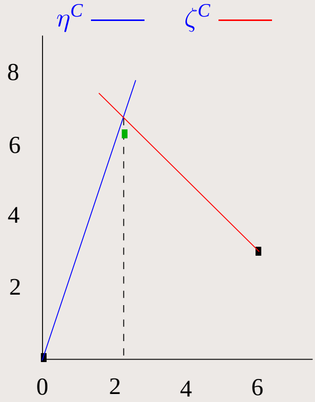


Figure: Evaluating z in $[0, 6]$

Example (cont'd)

$d_r = 6$:

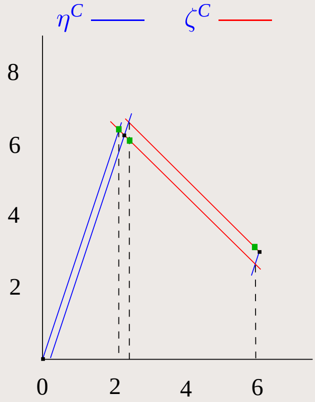


Figure: Evaluating z in $[0, 6]$

Example (cont'd)

$d_r = 6$:

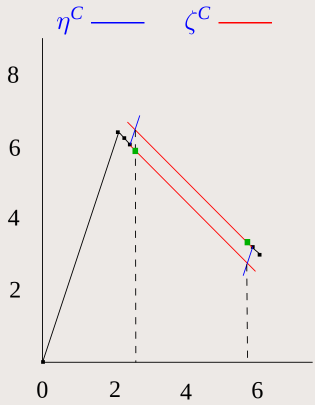


Figure: Evaluating z in $[0, 6]$

Example (cont'd)

$d_r = 6$:

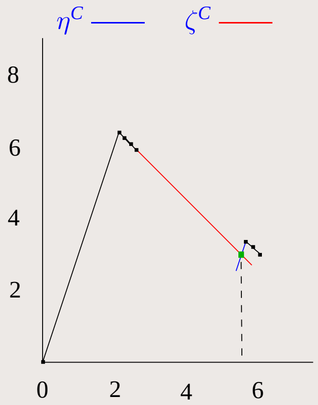


Figure: Evaluating z in $[0, 6]$

Example (cont'd)

$d_r = 6$:

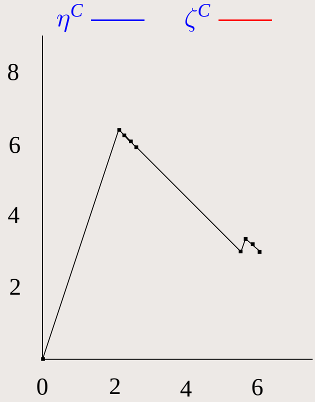
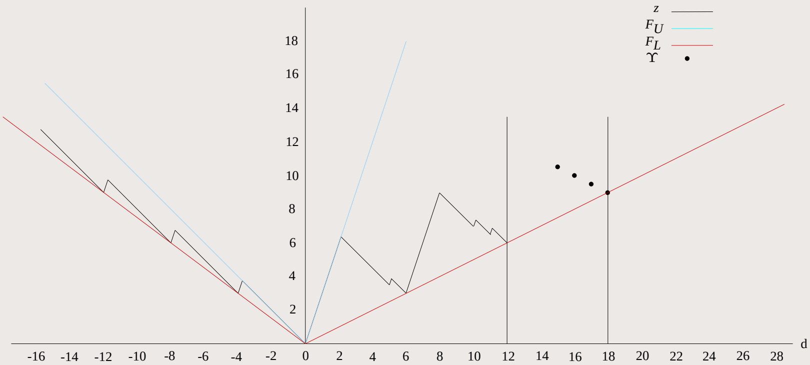


Figure: Evaluating z in $[0, 6]$

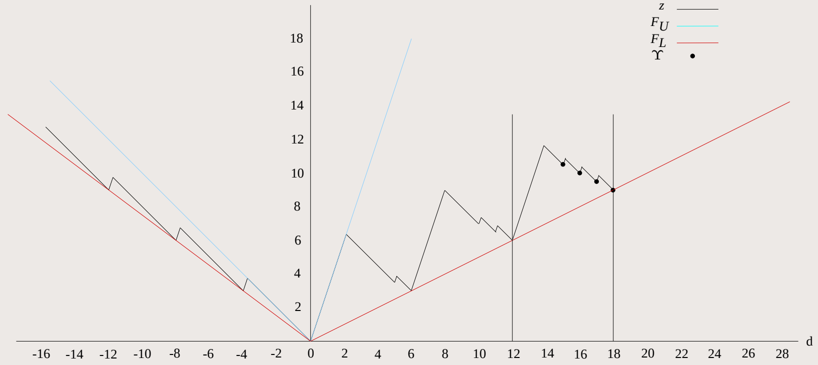
Example (cont'd)

Extending the value function of (SP) from $[0, 12]$ to $[0, 18]$



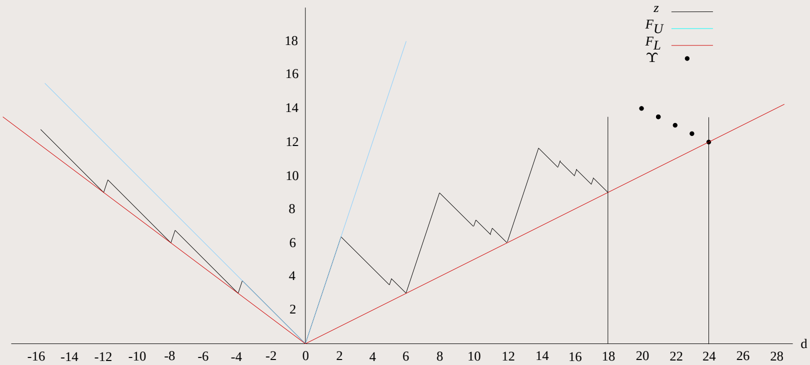
Example (cont'd)

Extending the value function of (SP) from $[0, 12]$ to $[0, 18]$



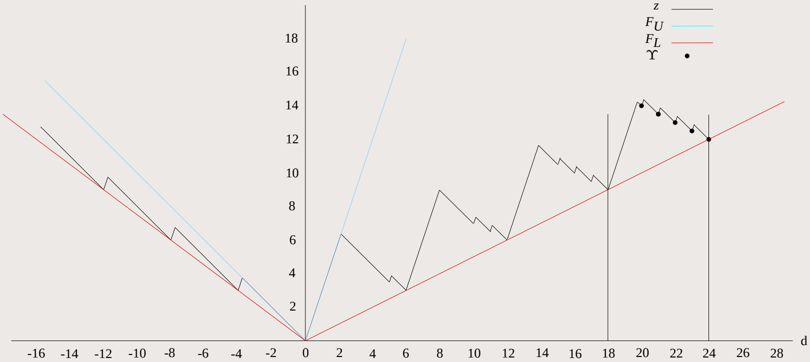
Example (cont'd)

Extending the value function of (SP) from $[0, 18]$ to $[0, 24]$



Example (cont'd)

Extending the value function of (SP) from $[0, 18]$ to $[0, 24]$



General Case

- For $E \in \mathscr{E}$, setting

$$\omega_E(d) = g(\lfloor d \rfloor_E) + \nu_E(d - \lfloor d \rfloor_E) \ \forall d \in \mathbb{R}^m \ \text{ with } \ \mathcal{S}(d) \neq \emptyset,$$

  we can write

$$z(d) = \min_{E \in \mathscr{E}} \omega_E(d) \ \forall d \in \mathbb{R}^m \ \text{ with } \ \mathcal{S}(d) \neq \emptyset.$$

- Many of our previous results can be extended to general case in the obvious way.
- Similarly, we can use maximal subadditive extensions to construct the value function.
- However, an obvious combinatorial explosion occurs.
- Therefore, we consider using single row relaxations to get a subadditive approximation.

Gomory's Procedure

- There is a Chvátal function that is optimal to the subadditive dual of an ILP with RHS $b \in \Omega_{IP}$ and $z_{IP}(b) > -\infty$.
- The procedure:
  In iteration k, we solve the following LP

$$\begin{aligned} z_{IP}(b)^{k-1} = \min \quad & cx \\ \text{s.t.} \quad & Ax = b \\ & \textstyle\sum_{j=1}^{n} f^i(a_j)x_j \geq f^i(b) \qquad i = 1, \ldots, k-1 \\ & x \geq 0 \end{aligned}$$

- The k^{th} cut, $k > 1$, is dependent on the RHS and written as:

$$f^k(d) = \left\lceil \sum_{i=1}^{m} \lambda_i^{k-1} d_i + \sum_{i=1}^{k-1} \lambda_{m+i}^{k-1} f^i(d) \right\rceil \quad \text{where} \;\; \lambda^{k-1} = (\lambda_1^{k-1}, \ldots, \lambda_{m+k-1}^{k-1}) \geq 0$$

Gomory's Procedure (cont.)

- Assume that $b \in \Omega_{IP}$, $z_{IP}(b) > -\infty$ and the algorithm terminates after $k+1$ iterations.
- If u^k is the optimal dual solution to the LP in the final iteration, then

$$F^k(d) = \sum_{i=1}^{m} u_i^k d_i + \sum_{i=1}^{k} u_{m+i}^k f^i(d),$$

is a Chvátal function with $F^k(b) = z_{IP}(b)$ and furthermore, it is optimal to the subadditive ILP dual problem.

Gomory's Procedure (cont.)

Example: Consider an integer program with two constraints. Suppose that the following is the first constraint:

$$2x_1 + -2x_2 + x_3 - x_4 = 3$$

At the first iteration, Gomory's procedure can be used to derive the valid inequality

$$\lceil 2/2 \rceil x_1 + \lceil -2/2 \rceil x_2 + \lceil 1/2 \rceil x_3 + \lceil -1/2 \rceil x_4 \geq \lceil 3/2 \rceil$$

by scaling the constraint with $\lambda = 1/2$. In other words, we obtain $x_1 - x_2 + x_3 \geq 2$. Then the corresponding function is:

$$F^1(d) = \lceil d_1/2 \rceil + 0d_2 = \lceil d_1/2 \rceil$$

By continuing Gomory's procedure, one can then build up an optimal dual function in this way.

Branch-and-Bound Method

- Assume that the primal problem is solved to optimality.
- Let T be the set of leaf nodes of the search tree.
- Thus, we've solved the LP relaxation of the following problem at node $t \in T$

$$
\begin{array}{rl} z^t(b) = \min & cx \\ \text{s.t} & x \in \mathcal{S}_t(b) \end{array},
$$

  where $\mathcal{S}_t(b) = \{Ax = b, x \geq l^t, -x \geq -u^t, x \in \mathbb{Z}^n\}$ and $u^t,\ l^t \in \mathbb{Z}^r$ are the branching bounds applied to the integer variables.
- Let $(v^t, \underline{v}^t, \overline{v}^t)$ be
  - the dual feasible solution used to prune node t, if t is feasibly pruned
  - a dual feasible solution (that can be obtained from it parent) to node t, if t is infeasibly pruned

Then,

$$
F_{BB}(d) = \min_{t \in T}\{v^t d + \underline{v}^t l^t - \overline{v}^t u^t\}
$$

is an optimal solution to the generalized dual problem.

Branch-and-Cut Method

- It is thus easy to get a strong dual function from branch-and-bound.
- Note, however, that it's not subadditive in general.
- To obtain a subaditive function, we can include the variable bounds explicitly as constraints, but then the function may not be strong.
- For branch-and-cut, we have to take care of the cuts.
  - Case 1: Do we know the subadditive representation of each cut?
  - Case 2: Do we know the RHS dependency of each cut?
  - Case 3: Otherwise, we can use some proximity results or the variable bounds.

Case 1

If we know the subadditive representation of each cut:
At a node t, we solve the LP relaxation of the following problem

$$\begin{array}{rrcl} z^t(b) = \min & cx & & \\ \text{s.t} & Ax & \geq & b \\ & x & \geq & l^t \\ & -x & \geq & -g^t \\ & H^t x & \geq & h^t \\ & x & \in & \mathbb{Z}_+^r \times \mathbb{R}_+^{n-r} \end{array}$$

where $g^t,\ l^t \in \mathbb{R}^r$ are the branching bounds applied to the integer variables and $H^t x \geq h^t$ is the set of added cuts in the form

$$\sum_{j \in I} F_k^t(a_j^k) x_j + \sum_{j \in N \setminus I} \bar{F}_k^t(a_j^k) x_j \ \geq\ F_k^t(\sigma_k(b)) \qquad k = 1, \ldots, \nu(t),$$

$\nu(t)$: the number of cuts generated so far,
$a_j^k, j = 1, \ldots, n$: the columns of the problem that the k^{th} cut is constructed from,
$\sigma_k(b)$: is the mapping of b to the RHS of the corresponding problem.

Case 1

Let T be the set of leaf nodes, $u^t, \underline{u}^t, \bar{u}^t$ and w^t be the dual feasible solution used to prune $t \in T$. Then,

$$F(d) = \min_{t \in T} \{u^t d + \underline{u}^t l^t - \bar{u}^t g^t + \sum_{k=1}^{\nu(t)} w^t{}_k F_k^t(\sigma_k(d))\}$$

is an optimal dual function, that is, $z(b) = F(b)$.

- Again, we obtain a subaddite function if the variables are bounded.
- However, we may not know the subadditive representation of each cut.

Case 2

If we know the RHS dependencies of each cut:
We know for

- Gomory fractional cuts.
- Knapsack cuts
- Mixed-integer Gomory cuts
- ?

Then, we do the same analysis as before.

Case 3

In the absence of subadditive representations or RHS dependencies:
For each node $t \in T$, let $\hat{h}^t$ be such that

$$\hat{h}^t_k = \sum_{j=1}^{n} h^t_{kj}\hat{x}_j \qquad \text{with} \qquad \hat{x}_j = \begin{cases} l^t_j & \text{if } h^t_{kj} \geq 0 \\ g^t_j & \text{otherwise} \end{cases}, \qquad k = 1, \ldots, \nu(t)$$

where h^t_{kj} is the k^{th} entry of column h^t_j. Furthermore, define

$$\tilde{h}^t = \sum_{j=1}^{n} h^t_j\tilde{x}_j \qquad \text{with} \qquad \tilde{x}_j = \begin{cases} l^t_j & \text{if } w^t h^t_j \geq 0 \\ g^t_j & \text{otherwise} \end{cases}.$$

Then the function

$$F(d) = \min_{t \in T}\{u^t d + \underline{u}^t l^t - \bar{u}^t g^t + \max\{w^t\tilde{h}^t, \ w^t\hat{h}^t\}\}$$

is a feasible dual solution and hence $F(b)$ yields a lower bound.

- This approach is the easiest way and can be used for bounded MILPs (binary programs), however it is unlikely to get an effective dual feasible solution for general MILPs.

Conclusions

- It is possible to generalize the duality concepts that are familiar to us from linear programming.
- However, it is extremely difficult to make any of it practical.
- There are some isolated cases where this theory has been applied in practice, but they are far and few between.
- We have a number of projects aimed in this direction, so stay tuned...