DIP with CHiPPS: Decomposition Methods for Integer Linear Programming

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Outline

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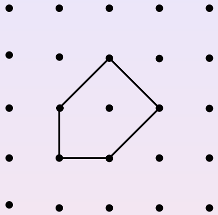
Basic Idea: By leveraging our ability to solve the optimization/separation problem for a relaxation, we can improve the bound yielded by the LP relaxation.

$$z_{\mathrm{IP}} = \min_{x \in \mathbb{Z}^n} \left\{ c^\top x \mid A'x \geq b', A''x \geq b'' \right\}$$

$$z_{\mathrm{LP}} = \min_{x \in \mathbb{R}^n} \left\{ c^\top x \mid A'x \geq b', A''x \geq b'' \right\}$$

$$z_{\mathrm{D}} = \min_{x \in \mathcal{P}'} \left\{ c^\top x \mid A''x \geq b'' \right\}$$

$$z_{\mathrm{IP}} \geq z_{\mathrm{D}} \geq z_{\mathrm{LP}}$$



$\mathcal{P} = \mathrm{conv}\{x \in \mathbb{Z}^n \mid A'x \geq b', A''x \geq b''\}$

Assumptions:

- $\mathrm{OPT}(\mathcal{P}, c)$ and $\mathrm{SEP}(\mathcal{P}, x)$ are *"hard"*
- $\mathrm{OPT}(\mathcal{P}', c)$ and $\mathrm{SEP}(\mathcal{P}', x)$ are *"easy"*
- $\mathcal{Q}''$ can be represented explicitly (description has polynomial size)
- $\mathcal{P}'$ must be represented implicitly (description has exponential size)

The Decomposition Principle in Integer Programming

Basic Idea: By leveraging our ability to solve the optimization/separation problem for a relaxation, we can improve the bound yielded by the LP relaxation.

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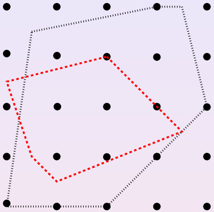

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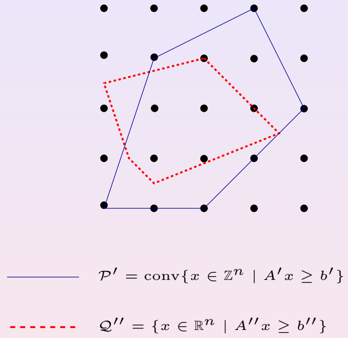

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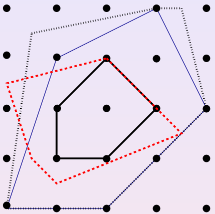

- (solid black line) $\mathcal{P} = \mathrm{conv}\{x \in \mathbb{Z}^n \mid A'x \geq b', A''x \geq b''\}$
- (thin blue line) $\mathcal{P}' = \mathrm{conv}\{x \in \mathbb{Z}^n \mid A'x \geq b'\}$
- (dotted line) $\mathcal{Q}' = \{x \in \mathbb{R}^n \mid A'x \geq b'\}$
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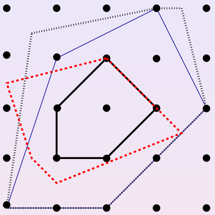

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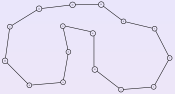
Traveling Salesman Problem Formulation

$$
\begin{array}{rcll}
x(\delta(\{u\})) & = & 2 & \forall u \in V \\
x(E(S)) & \leq & |S| - 1 & \forall S \subset V,\ 3 \leq |S| \leq |V| - 1 \\
x_e \in \{0,1\} & & & \forall e \in E
\end{array}
$$



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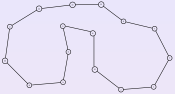

Two possible decompositions

Find a spanning subgraph with $|V|$ edges that satisfies the 2-degree constraints ($\mathcal{P}' =$ 1-Tree)

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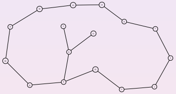

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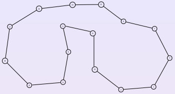

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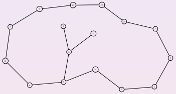

Find a 2-matching that satisfies the subtour constraints ($\mathcal{P}' =$ 2-Matching)

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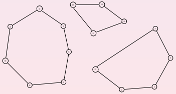
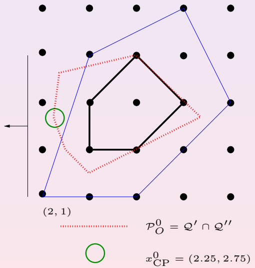


CPM combines an *outer* approximation of $\mathcal{P}'$ with an explicit description of $\mathcal{Q}''$

- **Master**: $z_{\mathrm{CP}} = \min_{x \in \mathbb{R}^n} \left\{ c^\top x \mid Dx \geq d, A''x \geq b'' \right\}$
- **Subproblem**: $\mathrm{SEP}(\mathcal{P}', x_{\mathrm{CP}})$

$$\mathcal{P}' = \{x \in \mathbb{R}^n \mid Dx \geq d\}$$

Exponential number of constraints



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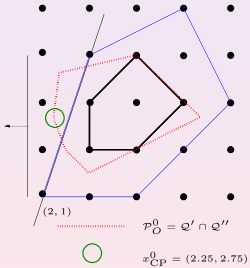

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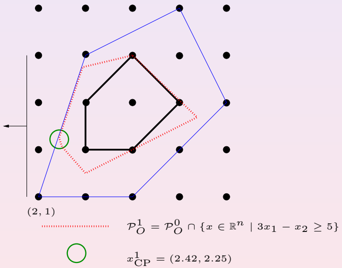

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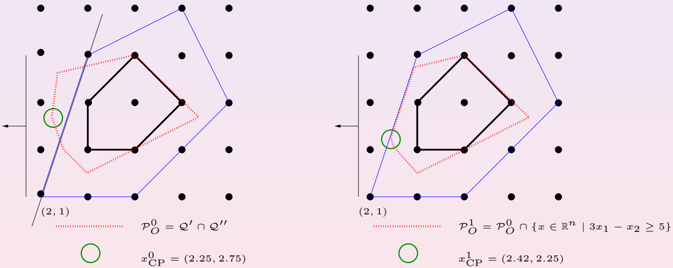

DW combines an *inner* approximation of $\mathcal{P}'$ with an explicit description of $\mathcal{Q}''$

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- **Subproblem**: $\mathrm{OPT}\left(\mathcal{P}', c^\top - u_{\mathrm{DW}}^\top A''\right)$

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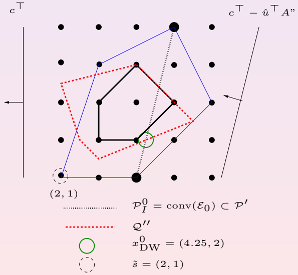
Exponential number of variables



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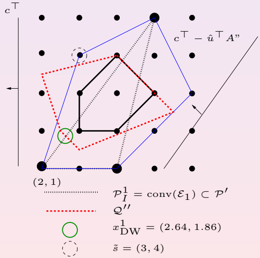

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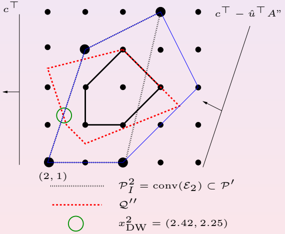

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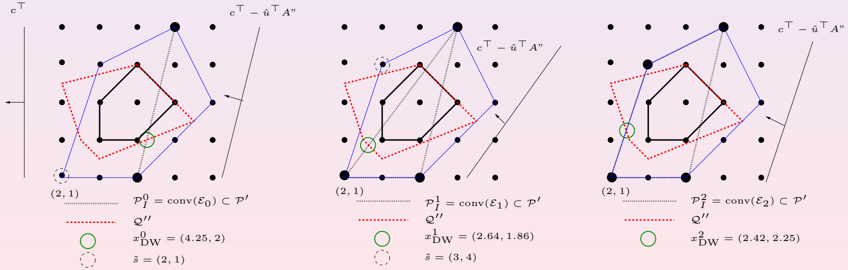

LD iteratively produces single extreme points of $\mathcal{P}'$ and uses their violation of constraints of $\mathcal{Q}''$ to converge to the same optimal face of $\mathcal{P}'$ as CPM and DW.

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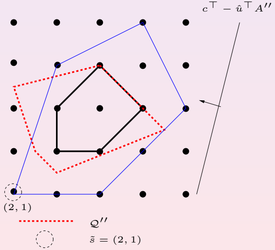

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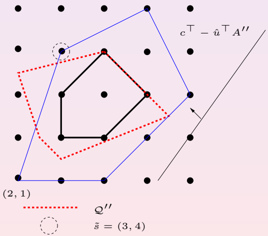

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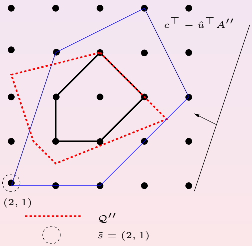

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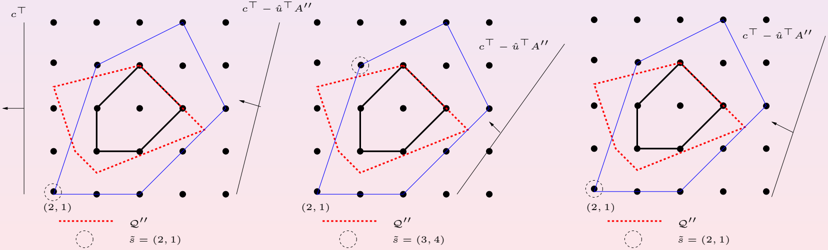


Common Threads

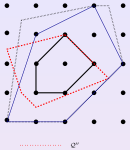
- The LP bound is obtained by optimizing over the intersection of two explicitly defined polyhedra.

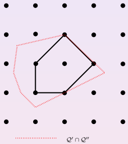
$$z_{\mathrm{LP}} = \min_{x \in \mathbb{R}^n} \{c^\top x \mid x \in \mathcal{Q}' \cap \mathcal{Q}''\}$$

- The decomposition bound is obtained by optimizing over the intersection of one explicitly defined polyhedron and one implicitly defined polyhedron.

$$z_{\mathrm{CP}} = z_{\mathrm{DW}} = z_{\mathrm{LD}} = z_{\mathrm{D}} = \min_{x \in \mathbb{R}^n} \{c^\top x \mid x \in \mathcal{P}' \cap \mathcal{Q}''\} \geq z_{\mathrm{LP}}$$

- Traditional decomp-based bounding methods contain two primary steps
  - **Master Problem:** Update the primal/dual solution information
  - **Subproblem:** Update the approximation of $\mathcal{P}'$: $\mathrm{SEP}(\mathcal{P}', x)$ or $\mathrm{OPT}(\mathcal{P}', c)$
- Integrated decomposition methods further improve the bound by considering two implicitly defined polyhedra whose descriptions are iteratively refined.
  - Price-and-Cut (PC)
  - Relax-and-Cut (RC)
  - Decompose-and-Cut (DC)





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- The LP bound is obtained by optimizing over the intersection of two explicitly defined polyhedra.

$$z_{\mathrm{LP}} = \min_{x \in \mathbb{R}^n} \{c^\top x \mid x \in \mathcal{Q}' \cap \mathcal{Q}''\}$$

- The decomposition bound is obtained by optimizing over the intersection of one explicitly defined polyhedron and one implicitly defined polyhedron.

$$z_{\mathrm{CP}} = z_{\mathrm{DW}} = z_{\mathrm{LD}} = z_{\mathrm{D}} = \min_{x \in \mathbb{R}^n} \{c^\top x \mid x \in \mathcal{P}' \cap \mathcal{Q}''\} \geq z_{\mathrm{LP}}$$

- Traditional decomp-based bounding methods contain two primary steps
  - **Master Problem:** Update the primal/dual solution information
  - **Subproblem:** Update the approximation of $\mathcal{P}'$: $\mathrm{SEP}(\mathcal{P}', x)$ or $\mathrm{OPT}(\mathcal{P}', c)$

- Integrated decomposition methods further improve the bound by considering two implicitly defined polyhedra whose descriptions are iteratively refined.
  - Price-and-Cut (PC)
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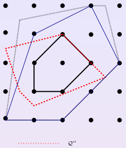

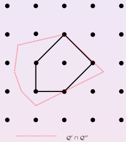

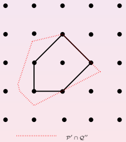

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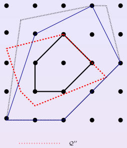
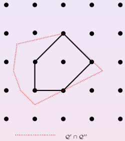
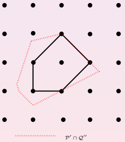


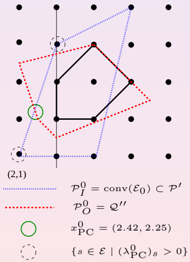




PC approximates $\mathcal{P}$ by building an *inner* approximation of $\mathcal{P}'$ (as in DW) intersected with an *outer* approximation of $\mathcal{P}$ (as in CPM)

- **Master**: $z_{\mathrm{PC}} = \min_{\lambda \in \mathbb{R}_+^{\mathcal{E}}} \left\{ c^\top \left(\sum_{s \in \mathcal{E}} s\lambda_s \right) \mid D \left(\sum_{s \in \mathcal{E}} s\lambda_s \right) \geq d, \sum_{s \in \mathcal{E}} \lambda_s = 1 \right\}$
- **Subproblem**: $\mathrm{OPT}\left(\mathcal{P}', c^\top - u_{\mathrm{PC}}^\top D\right)$ or $\mathrm{SEP}\left(\mathcal{P}, x_{\mathrm{PC}}\right)$

- As in CPM, separate $\hat{x}_{\mathrm{PC}} = \sum_{s \in \mathcal{E}} s\hat{\lambda}_s$ from $\mathcal{P}$ and add cuts to $[D, d]$.
- **Key Idea**: Cut generation takes place in the space of the compact formulation, maintaining the structure of the column generation subproblem.



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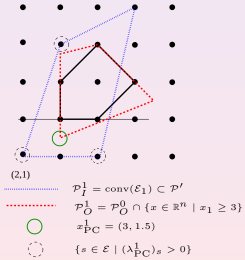

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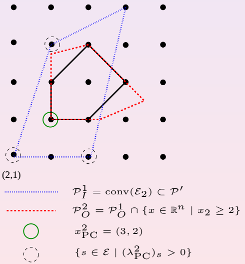

Price-and-Cut Method (PC)

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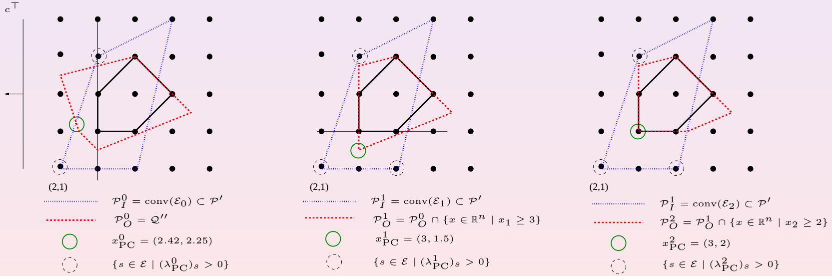
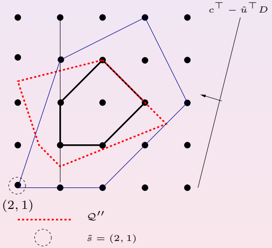


RC approximates $\mathcal{P}$ by tracing an *inner* approximation of $\mathcal{P}'$ (as in LD) penalizing points outside of a dynamically generated *outer* approximation of $\mathcal{P}$ (as in CPM)

- **Master**: $z_{\mathrm{LD}} = \max_{u \in \mathbb{R}_+^{m''}} \left\{ \min_{s \in \mathcal{E}} \left\{ c^\top s + u^\top (d - Ds) \right\} \right\}$
- **Subproblem**: $\mathrm{OPT}\left(\mathcal{P}', c^\top - u_{\mathrm{LD}}^\top D\right)$ or $\mathrm{SEP}\left(\mathcal{P}, s\right)$

- In each iteration, separate $\hat{s} \in \mathcal{E}$, a solution to the Lagrangian relaxation.
- **Advantage**: Often easier to separate $s \in \mathcal{E}$ from $\mathcal{P}$ than $\hat{x} \in \mathbb{R}^n$.



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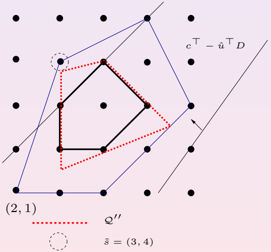

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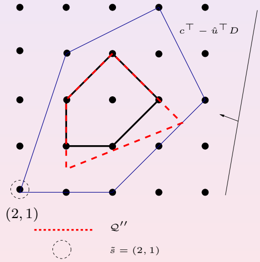

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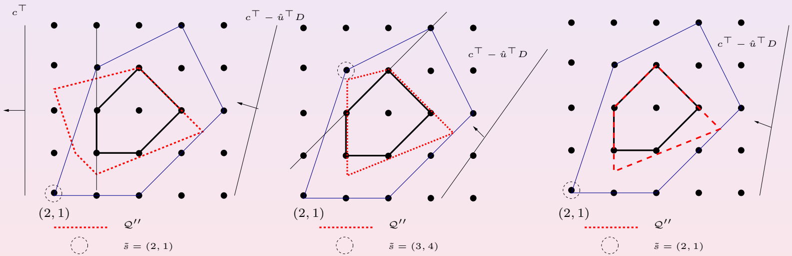

Structured Separation

- In general, $\mathrm{OPT}(X, c)$ and $\mathrm{SEP}(X, x)$ are polynomially equivalent.
- **Observation:** Restrictions on input or output can change their complexity.
- Template Paradigm, restricts the *output* of $\mathrm{SEP}(X, x)$ to valid inequalities that conform to a certain structure. This class of inequalities forms a polyhedron $\mathcal{C} \supset X$ (the *closure*).
- For example, let $\mathcal{P}$ be the convex hull of solutions to the TSP.
  - $\mathrm{SEP}(\mathcal{P}, x)$ is $\mathcal{NP}$-Complete.
  - $\mathrm{SEP}(\mathcal{C}, x)$ is polynomially solvable, for $\mathcal{C} \supset \mathcal{P}$
    - $\mathcal{P}^{\mathrm{Subtour}}$, the Subtour Polytope (separation using Min-Cut), or
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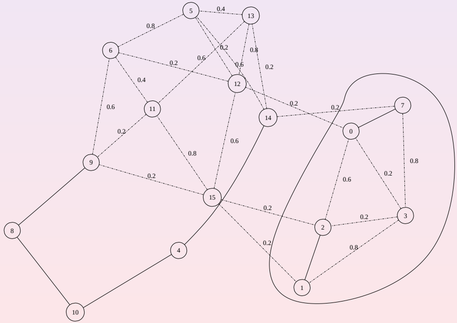
- Separation of Subtour Inequalities:

$$x(E(S)) \leq |S| - 1$$

- $\text{SEP}(\mathcal{P}^{\text{Subtour}}, x)$ for $x \in \mathbb{R}^n$ can be solved in $O\left(|E||V| + |V|^2 \log |V|\right)$ (Min-Cut)
- $\text{SEP}(\mathcal{P}^{\text{Subtour}}, s)$ for s a 2-matching, can be solved in $O(|V|)$
  - Simply determine the connected components C_i, and set $S = C_i$ for each component (each gives a violation of 1).



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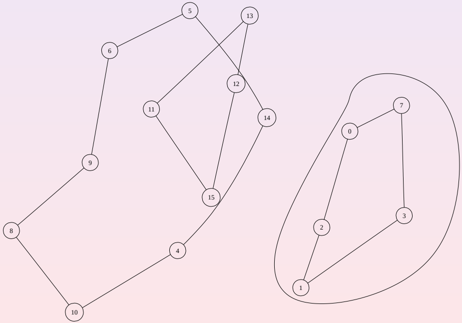

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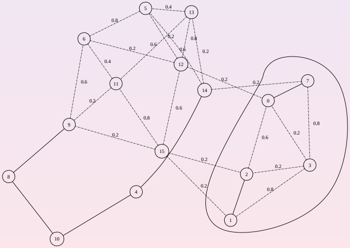

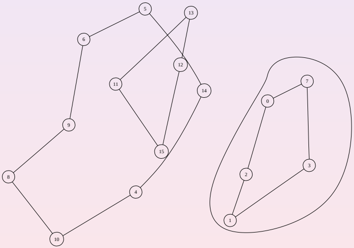

Example - TSP

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$$x(E(H)) + \sum_{i=1}^{k} x(E(T_i)) \leq |H| + \sum_{i=1}^{k} (|T_i| - 1) - \lceil k/2 \rceil$$

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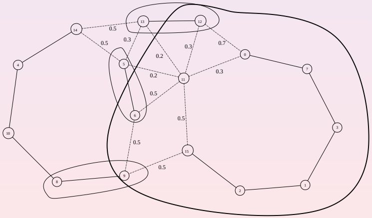
  - This can also be used as a quick heuristic to separate 1-trees for more general comb structures, for which there is no known polynomial algorithm for separation of arbitrary vectors.



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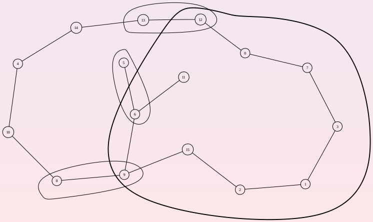

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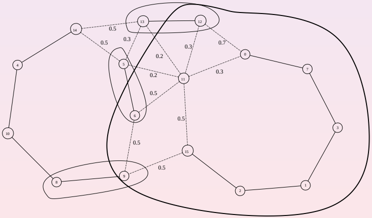

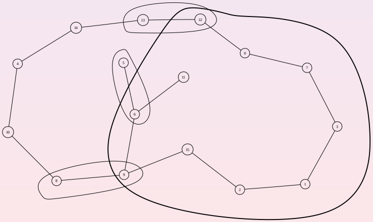

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- **Question:** Can we take advantage of this in other contexts?
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- In Relax-and-Cut, solutions to the Lagrangian subproblem $s \in \mathcal{E}$ typically have some *nice* combinatorial structure.
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- Consider the following set

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- $\mathcal{S}(u^t_{\mathrm{PC}}, \alpha^t_{\mathrm{PC}})$ is the set of extreme points with $rc(s) = 0$ in the DW-LP master or the set of alternative optimal solutions to the Lagrangian subproblem.

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1. $F^t \subseteq conv(\mathcal{S}(u^t_{\mathrm{PC}}, \alpha^t_{\mathrm{PC}}))$
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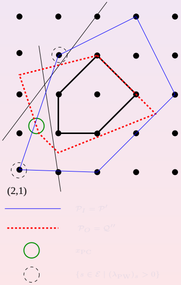

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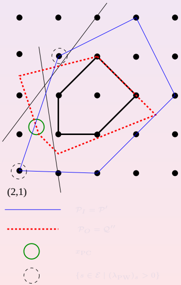

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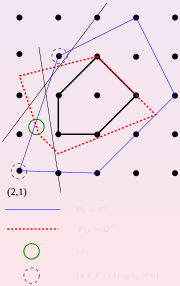

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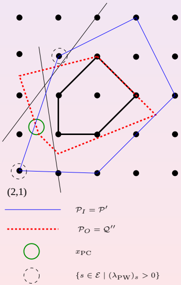

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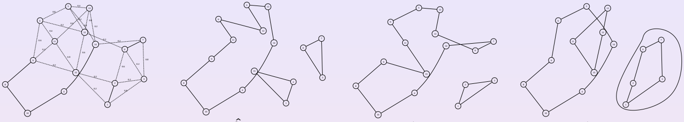

$\hat{x}$ $\hat{\lambda}_0 = 0.2$ $\hat{\lambda}_1 = 0.2$ $\hat{\lambda}_2 = 0.6$

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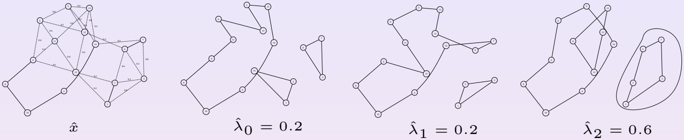

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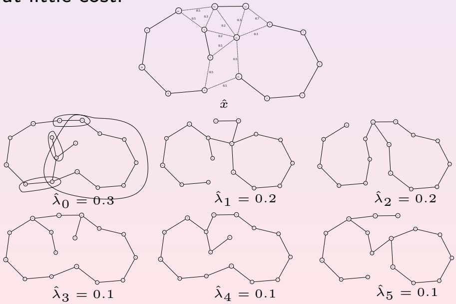

Decompose-and-Cut: Each iteration of CPM, decompose into convex combo of e.p.'s of $\mathcal{P}'$

$$\min_{\lambda \in \mathbb{R}_+^{\mathcal{E}}, (x^+, x^-) \in \mathbb{R}_+^n} \left\{ x^+ + x^- \;\middle|\; \sum_{s \in \mathcal{E}} s\lambda_s + x^+ - x^- = \hat{x}_{\text{CP}}, \sum_{s \in \mathcal{E}} \lambda_s = 1 \right\}$$

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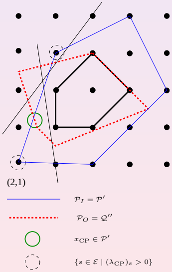
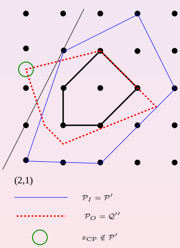
- If $\hat{x}_{\text{CP}}$ lies outside $\mathcal{P}'$ the decomposition will fail
- By the *Farkas Lemma* the proof of infeasibility provides a valid and violated inequality

Decomposition Cuts

$$\begin{aligned} u^t_{\text{DC}} s + \alpha^t_{\text{DC}} &\leq 0 \;\forall s \in \mathcal{P}' \quad \text{and} \\ u^t_{\text{DC}} \hat{x}_{\text{CP}} + \alpha^t_{\text{DC}} &> 0 \end{aligned}$$





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- The machinery for solving this already exists (=column generation)
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Branching for Inner Methods (PC)

- Add column bounds to $[A'', b'']$ and map back to the compact space $\hat{x} = \sum_{s \in \mathcal{E}} s\hat{\lambda}_s$
- Variable branching in the compact space is constraint branching in the extended space
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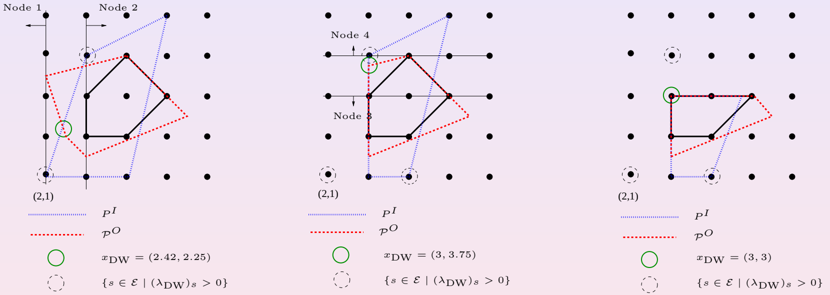

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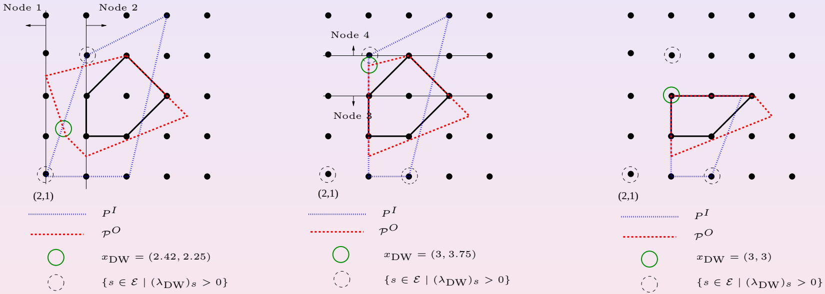

$$\begin{array}{lrcl} \text{Node 1:} & 4\lambda_{(4,1)} + 5\lambda_{(5,5)} + 2\lambda_{(2,1)} + 3\lambda_{(3,4)} & \leq & 2 \\ \text{Node 2:} & 4\lambda_{(4,1)} + 5\lambda_{(5,5)} + 2\lambda_{(2,1)} + 3\lambda_{(3,4)} & \geq & 3 \end{array}$$

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- In general, Lagrangian methods do *not* provide a primal solution λ
- Let $\mathcal{B}$ define the extreme points found in solving subproblems for z_{LD}
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  - Identical subproblems (symmetry)
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Algorithmic Details and Extensions

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Recent Added Features

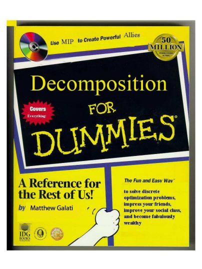
- User API for selection of which block to process next (can help alot!)
- Support for enforcing branching in subproblem.
- Sparse solution of subproblems for block decomposition.
- Option to detect and remove columns that are close to parallel.
- Dual stabilization (Wegntes).
- Allow to stop subproblem calculation on gap/time and calculate LB.
- For MILP oracle, now have option to allow multiple columns for each subproblem call.
- Better support for "master-only variables."
- Option to use PC solution as warm-start to CPLEX direct solve—try and finish it off.
- API to provide an initial dual vector.
- Option to NOT compress columns until master gap is tight.

Outline

DIP Framework

DIP (**D**ecomposition for **I**nteger **P**rogramming) is an open-source software framework that provides an implementation of various decomposition methods with minimal user responsibility

- Allows direct comparison CPM/DW/LD/PC/RC/DC in one framework
- DIP abstracts the common, generic elements of these methods
- Key: The user defines application-specific components in the space of the compact formulation - greatly simplifying the API
    - Define $[A'', b'']$ and/or $[A', b']$
    - Provide methods for $\mathrm{OPT}(\mathcal{P}', c)$ and/or $\mathrm{SEP}(\mathcal{P}', x)$
- Framework handles all of the algorithm-specific reformulation



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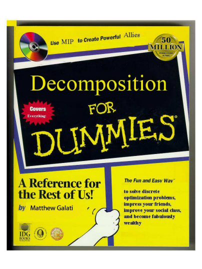

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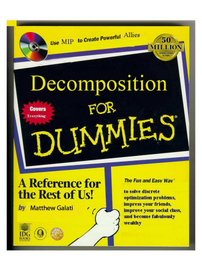

COmputational INfrastructure for Operations Research
Have some DIP with your CHiPPs?

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  - Applications Interface: `DecompApp`
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- Define the model(s)
  - setModelObjective(double * c): define c
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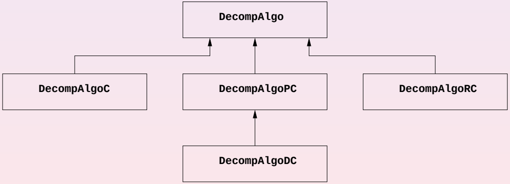
```
int main(int argc, char ** argv){
   //create the utility class for parsing parameters
   UtilParameters utilParam(argc, argv);
   bool doCut          = utilParam.GetSetting("doCut",       true);
   bool doPriceCut     = utilParam.GetSetting("doPriceCut",  false);
   bool doRelaxCut     = utilParam.GetSetting("doRelaxCut",  false);

   //create the user application (a DecompApp)
   SILP_DecompApp sip(utilParam);

   //create the CPM/PC/RC algorithm objects (a DecompAlgo)
   DecompAlgo   * algo = NULL;
   if(doCut)      algo = new DecompAlgoC (&sip, &utilParam);
   if(doPriceCut) algo = new DecompAlgoPC(&sip, &utilParam);
   if(doRelaxCut) algo = new DecompAlgoRC(&sip, &utilParam);

   //create the driver AlpsDecomp model
   AlpsDecompModel alpsModel(utilParam, algo);

   //solve
   alpsModel.solve();
}
```

DIP - Algorithms

- The base class DecompAlgo provides the shell (init / master / subproblem / update).
- Each of the methods described has derived default implementations DecompAlgoX : public DecompAlgo which are accessible by any application class, allowing full flexibility.
- New, hybrid or extended methods can be easily derived by overriding the various subroutines, which are called from the base class. For example,
  - Alternative methods for solving the master LP in DW, such as interior point methods
  - Add stabilization to the dual updates in LD (stability centers)
  - For LD, replace subgradient with volume providing an approximate primal solution
  - Hybrid init methods like using LD or DC to initialize the columns of the DW master
  - During PC, adding cuts to either master and/or subproblem.
  - ...



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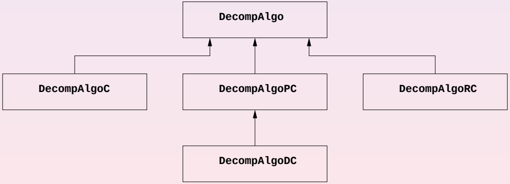

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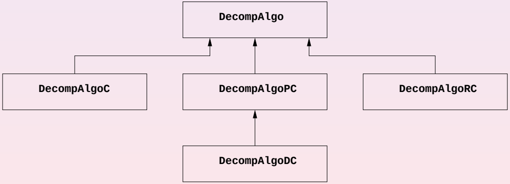


Application	Description	$\mathcal{P}'$	$\mathbf{OPT}(c)$	$\mathbf{SEP}(x)$	Input
AP3	3-index assignment	AP	Jonker	user	user
ATM	cash management (SAS COE)	MILP(s)	CBC	CGL	user
GAP	generalized assignment	KP(s)	Pisinger	CGL	user
MAD	matrix decomposition	MaxClique	Cliquer	CGL	user
MILP	random partition into A', A''	MILP	CBC	CGL	mps
MILPBlock	user-defined blocks for A'	MILP(s)	CBC	CGL	mps, block
MMKP	multi-dim/choice knapsack	MCKP	Pisinger	CGL	user
		MDKP	CBC	CGL	user
SILP	intro example, tiny IP	MILP	CBC	CGL	user
TSP	traveling salesman problem	1-Tree	Boost	Concorde	user
		2-Match	CBC	Concorde	user
VRP	vehicle routing problem	k-TSP	Concorde	CVRPSEP	user
		b-Match	CBC	CVRPSEP	user

Outline

Quick Introduction to CHiPPS

- CHiPPS stands for COIN-OR High Performance Parallel Search.
- CHiPPS is a set of C++ class libraries for implementing tree search algorithms for both sequential and parallel environments.

CHiPPS Components (Current)

ALPS (Abstract Library for Parallel Search)
- is the search-handling layer (parallel and sequential).
- provides various search strategies based on node priorities.

BiCePS (Branch, Constrain, and Price Software)
- is the data-handling layer for relaxation-based optimization.
- adds notion of variables and constraints.
- assumes iterative bounding process.

BLIS (BiCePS Linear Integer Solver)
- is a concretization of BiCePS.
- specific to models with linear constraints and objective function.

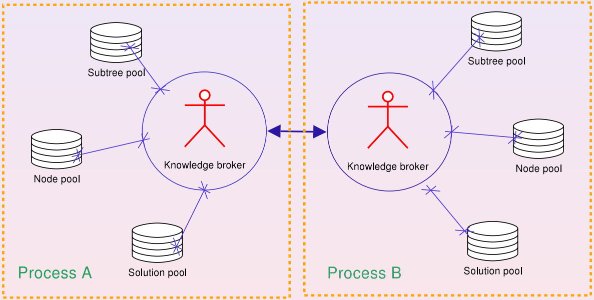
ALPS: Design Goals

- Intuitive object-oriented class structure.
  - `AlpsModel`
  - `AlpsTreeNode`
  - `AlpsNodeDesc`
  - `AlpsSolution`
  - `AlpsParameterSet`
- Minimal algorithmic assumptions in the base class.
  - Support for a wide range of problem classes and algorithms.
  - Support for constraint programming.
- Easy for user to develop a custom solver.
- Design for *parallel scalability*, but operate effective in a sequential environment.
- Explicit support for *memory compression* techniques (packing/differencing) important for implementing optimization algorithms.

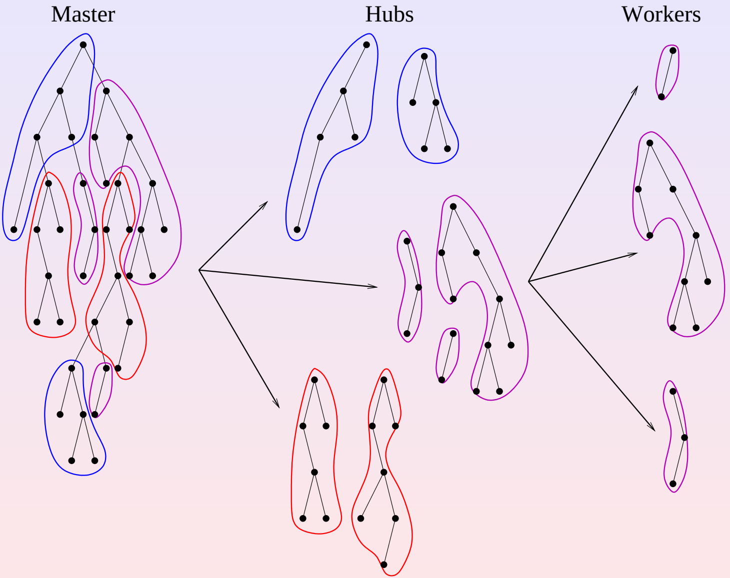
ALPS: Overview of Features

- The design is based on a very general concept of *knowledge*.
- Knowledge is shared asynchronously through *pools* and *brokers*.
- Management overhead is reduced with the *master-hub-worker* paradigm.
- Overhead is decreased using dynamic task granularity.
- Two static load balancing techniques are used.
- Three dynamic load balancing techniques are employed.
- Uses asynchronous messaging to the highest extent possible.
- A scheduler on each process manages tasks like
  - node processing,
  - load balaning,
  - update search states, and
  - termination checking, etc.

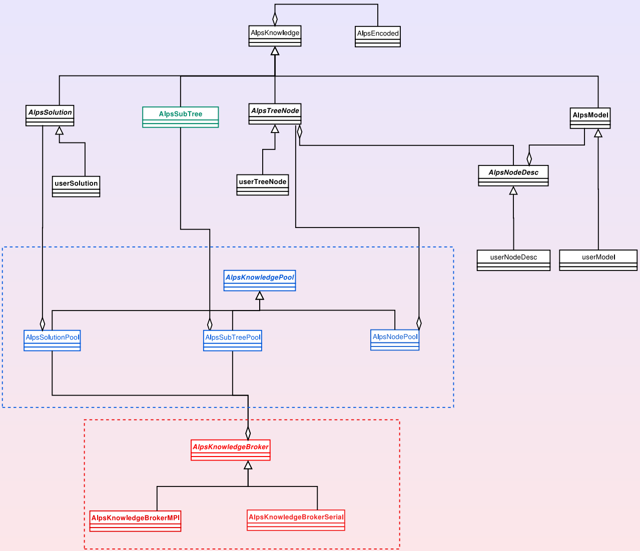
- All knowledge to be shared is derived from a single base class and has an associated *encoded form*.
- Encoded form is used for identification, storage, and communication.
- Knowledge is maintained by one or more *knowledge pools*.
- The knowledge pools communicate through *knowledge brokers*.



Master-Hub-Worker Paradigm



Alps Class Hierarchy



The formulation of the binary knapsack problem is

$$\max\{\sum_{i=1}^{m} p_i x_i : \sum_{i=1}^{m} s_i x_i \leq c, x_i \in \{0,1\}, i = 1, 2, \ldots, m\}, \tag{1}$$

We derive the following classes:

- `KnapModel` (from `AlpsModel`) : Stores the data used to describe the knapsack problem and implements `readInstance()`
- `KnapTreeNode` (from `AlpsTreeNode`) : Implements `process()` (bound) and `branch()`
- `KnapNodeDesc` (from `AlpsNodeDesc`) : Stores information about which variables/items have been fixed by branching and which are still free.
- `KnapSolution` (from `AlpsSolution`) Stores a solution (which items are in the knapsack).

Using ALPS: A Knapack Solver

Then, supply the main function.

```
int main(int argc, char* argv[])
{
    KnapModel model;

#if defined(SERIAL)
    AlpsKnowledgeBrokerSerial broker(argc, argv, model);
#elif defined(PARALLEL_MPI)
    AlpsKnowledgeBrokerMPI broker(argc, argv, model);
#endif

    broker.search();
    broker.printResult();
    return 0;
}
```

Outline

Multi-Choice Multi-Dimensional Knapsack Problem (MMKP)

- **SAS Marketing Optimization** - improve ROI for marketing campaign offers by targeting higher response rates, improving channel effectiveness, and reduce spending.

$$
\begin{array}{llll}
\max & \sum_{i \in N} \sum_{j \in L_i} v_{ij} x_{ij} & & \\
& \sum_{i \in N} \sum_{j \in L_i} r_{kij} x_{ij} \leq & b_k & \forall k \in M \\
& \sum_{j \in L_i} x_{ij} = & 1 & \forall i \in N \\
& x_{ij} \in & \{0,1\} & \forall i \in N, j \in L_i
\end{array}
$$

- Relaxation - Multi-Choice Knapsack Problem (MCKP)
  - solver *mcknap* by Pisinger a DP-based branch-and-bound

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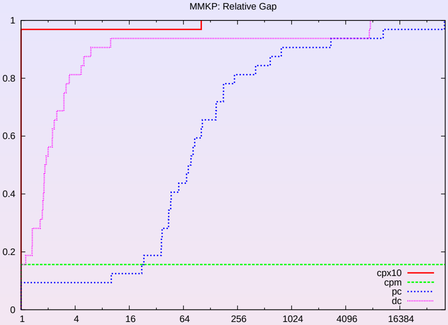
	CPX10.2		DIP-CPM		DIP-PC		DIP-DC	
Instance	**Time**	**Gap**	**Time**	**Gap**	**Time**	**Gap**	**Time**	**Gap**
I1	0.00	OPT	0.02	OPT	0.04	OPT	0.14	OPT
I10	T	0.05%	T	∞	T	11.86%	T	0.15%
I11	T	0.03%	T	∞	T	12.25%	T	0.14%
I12	T	0.01%	T	∞	T	7.93%	T	0.10%
I13	T	0.02%	T	∞	T	11.89%	T	0.12%
I2	0.01	OPT	0.01	OPT	0.05	OPT	0.05	OPT
I3	1.17	OPT	23.23	OPT	T	1.07%	T	0.75%
I4	15.71	OPT	T	∞	T	5.14%	T	0.77%
I5	0.01	0.01%	0.01	OPT	0.13	OPT	0.05	OPT
I6	0.14	OPT	0.07	OPT	T	0.28%	0.63	OPT
I7	T	0.08%	T	∞	T	14.32%	T	0.09%
I8	T	0.09%	T	∞	T	13.36%	T	0.20%
I9	T	0.06%	T	∞	T	10.71%	T	0.19%
INST01	T	0.43%	T	∞	T	9.99%	T	0.70%
INST02	T	0.09%	T	∞	T	7.39%	T	0.45%
INST03	T	0.38%	T	∞	T	3.83%	T	0.85%
INST04	T	0.34%	T	∞	T	7.48%	T	0.45%
INST05	T	0.18%	T	∞	T	10.23%	T	0.62%
INST06	T	0.21%	T	∞	T	9.82%	T	0.38%
INST07	T	0.36%	T	∞	T	15.75%	T	0.62%
INST08	T	0.25%	T	∞	T	11.55%	T	0.46%
INST09	T	0.21%	T	∞	T	15.24%	T	0.40%
INST11	T	0.22%	T	∞	T	7.96%	T	0.39%
INST12	T	0.18%	T	∞	T	7.90%	T	0.42%
INST13	T	0.08%	T	∞	T	2.97%	T	0.14%
INST14	T	0.05%	T	∞	T	3.89%	T	0.09%
INST15	T	0.04%	T	∞	T	3.43%	T	0.10%
INST16	T	0.06%	T	∞	T	2.19%	T	0.06%
INST17	T	0.03%	T	∞	T	2.09%	T	0.09%
INST18	T	0.03%	T	∞	T	4.43%	T	0.06%
INST19	T	0.03%	T	∞	T	3.13%	T	0.04%
INST20	T	0.03%	T	∞	T	3.05%	T	0.04%

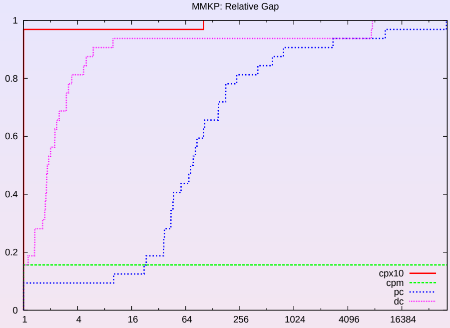


	CPX10.2	DIP-CPM	DIP-PC	DIP-DC
Optimal	5	5	3	4
$\leq$ 1% Gap	32	5	4	32
$\leq$ 10% Gap	32	5	22	32

CGL: missing *Gub Covers*

MMKP: CPX10.2 vs CPM/PC/DC

	CPX10.2		DIP-CPM		DIP-PC		DIP-DC	
Instance	**Time**	**Gap**	**Time**	**Gap**	**Time**	**Gap**	**Time**	**Gap**
I1	0.00	OPT	0.02	OPT	0.04	OPT	0.14	OPT
I10	T	0.05%	T	∞	T	11.86%	T	0.15%
I11	T	0.03%	T	∞	T	12.25%	T	0.14%
I12	T	0.01%	T	∞	T	7.93%	T	0.10%
I13	T	0.02%	T	∞	T	11.89%	T	0.12%
I2	0.01	OPT	0.01	OPT	0.05	OPT	0.05	OPT
I3	1.17	OPT	23.23	OPT	T	1.07%	T	0.75%
I4	15.71	OPT	T	∞	T	5.14%	T	0.77%
I5	0.01	0.01%	0.01	OPT	0.13	OPT	0.05	OPT
I6	0.14	OPT	0.07	OPT	T	0.28%	0.63	OPT
I7	T	0.08%	T	∞	T	14.32%	T	0.09%
I8	T	0.09%	T	∞	T	13.36%	T	0.20%
I9	T	0.06%	T	∞	T	10.71%	T	0.19%
INST01	T	0.43%	T	∞	T	9.99%	T	0.70%
INST02	T	0.09%	T	∞	T	7.39%	T	0.45%
INST03	T	0.38%	T	∞	T	3.83%	T	0.85%
INST04	T	0.34%	T	∞	T	7.48%	T	0.45%
INST05	T	0.18%	T	∞	T	10.23%	T	0.62%
INST06	T	0.21%	T	∞	T	9.82%	T	0.38%
INST07	T	0.36%	T	∞	T	15.75%	T	0.62%
INST08	T	0.25%	T	∞	T	11.55%	T	0.46%
INST09	T	0.21%	T	∞	T	15.24%	T	0.40%
INST11	T	0.22%	T	∞	T	7.96%	T	0.39%
INST12	T	0.18%	T	∞	T	7.90%	T	0.42%
INST13	T	0.08%	T	∞	T	2.97%	T	0.14%
INST14	T	0.05%	T	∞	T	3.89%	T	0.09%
INST15	T	0.04%	T	∞	T	3.43%	T	0.10%
INST16	T	0.06%	T	∞	T	2.19%	T	0.06%
INST17	T	0.03%	T	∞	T	2.09%	T	0.09%
INST18	T	0.03%	T	∞	T	4.43%	T	0.06%
INST19	T	0.03%	T	∞	T	3.13%	T	0.04%
INST20	T	0.03%	T	∞	T	3.05%	T	0.04%



	CPX10.2	DIP-CPM	DIP-PC	DIP-DC
Optimal	5	5	3	4
$\leq$ 1% Gap	32	5	4	32
$\leq$ 10% Gap	32	5	22	32

CGL: missing *Gub Covers*

MMKP: Nested Pricing

- Nested Relaxations:
  - Multi-Choice 2-D Knapsack Problem (MC2KP): $\mathcal{P}_p^{\text{MC2KP}} \subset \mathcal{P}^{\text{MCKP}} \ \forall p \in M \setminus \{m\}$

$$\begin{array}{lcll}
\sum_{i \in N} \sum_{j \in L_i} r_{pij} x_{ij} & \leq & b_p & \\
\sum_{i \in N} \sum_{j \in L_i} r_{mij} x_{ij} & \leq & b_m & \\
\sum_{j \in L_i} x_{ij} & = & 1 & \forall i \in N \\
x_{ij} & \in & \{0,1\} & \forall i \in N, j \in L_i
\end{array}$$

  - Multi-Choice Multi-Dimensional Knapsack Problem (MMKP): $\mathcal{P} \subset \mathcal{P}^{\text{MCKP}}$

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\begin{array}{lcll}
\sum_{i \in N} \sum_{j \in L_i} r_{pij} x_{ij} & \leq & b_p & \\
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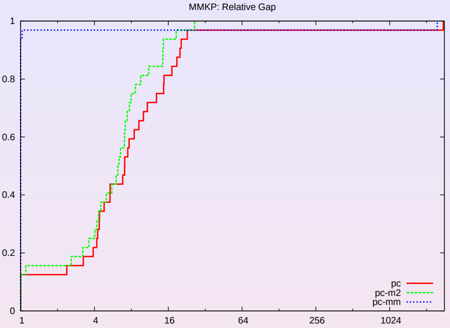
x_{ij} & \in & \{0,1\} & \forall i \in N, j \in L_i
\end{array}
$$

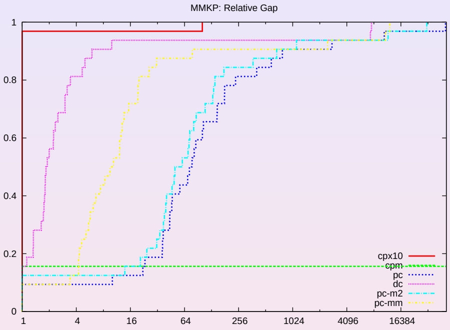
  - Multi-Choice Multi-Dimensional Knapsack Problem (MMKP): $\mathcal{P} \subset \mathcal{P}^{\text{MCKP}}$

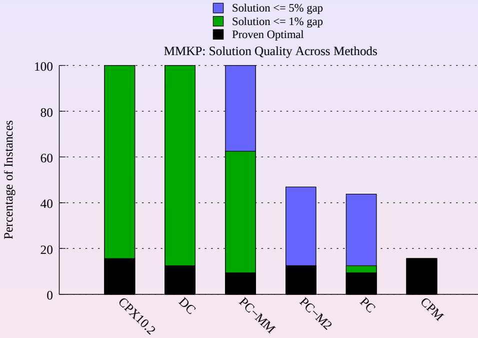
MMKP: PC vs PC Nested with MC2KP and MMKP

	DIP-PC		DIP-PC-M2		DIP-PC-MM	
Instance	**Time**	**Gap**	**Time**	**Gap**	**Time**	**Gap**
I1	0.04	OPT	0.16	OPT	0.08	OPT
I10	T	11.86%	T	6.99%	T	0.63%
I11	T	12.25%	T	11.15%	T	0.60%
I12	T	7.93%	T	11.41%	T	0.79%
I13	T	11.89%	T	13.65%	T	0.52%
I2	0.05	OPT	0.45	OPT	0.14	OPT
I3	T	1.07%	T	1.18%	T	1.10%
I4	T	5.14%	T	3.18%	T	1.23%
I5	0.13	OPT	0.14	OPT	0.07	OPT
I6	T	0.28%	483.53	OPT	T	0.25%
I7	T	14.32%	T	4.85%	T	0.97%
I8	T	13.36%	T	9.79%	T	0.67%
I9	T	10.71%	T	10.57%	T	0.73%
INST01	T	9.99%	T	5.97%	T	1.86%
INST02	T	7.39%	T	7.29%	T	1.74%
INST03	T	3.83%	T	11.93%	T	1.61%
INST04	T	7.48%	T	7.04%	T	1.56%
INST05	T	10.23%	T	8.84%	T	1.11%
INST06	T	9.82%	T	9.77%	T	1.39%
INST07	T	15.75%	T	8.78%	T	1.23%
INST08	T	11.55%	T	8.50%	T	1.37%
INST09	T	15.24%	T	8.48%	T	0.89%
INST11	T	7.96%	T	8.72%	T	1.13%
INST12	T	7.90%	T	6.72%	T	1.03%
INST13	T	2.97%	T	3.06%	T	0.76%
INST14	T	3.89%	T	3.67%	T	0.52%
INST15	T	3.43%	T	2.81%	T	0.78%
INST16	T	2.19%	T	3.01%	T	0.50%
INST17	T	2.09%	T	2.16%	T	0.39%
INST18	T	4.43%	T	2.60%	T	0.41%
INST19	T	3.13%	T	3.97%	T	0.46%
INST20	T	3.05%	T	4.06%	T	0.94%



	DIP-PC	DIP-PC-M2	DIP-PC-MM
Optimal	3	4	3
≤ 1% Gap	4	4	20
≤ 10% Gap	22	27	32


MMKP: Relative Gap
1
0.8
0.6
0.4
0.2
0
1
4
16
64
256
1024
4096
16384
cpx10
cpm
pc
dc
pc-m2
pc-mm


Solution <= 5% gap
Solution <= 1% gap
Proven Optimal
MMKP: Solution Quality Across Methods
Percentage of Instances
100
80
60
40
20
0
CPX10.2
DC
PC-MM
PC-M2
PC
CPM

SAS Center of Excellence in Operations Research Applications (OR COE)

- Determine schedule for allocation of cash inventory at branch banks to service ATMs
- Define a polynomial fit for predicted cash flow need per day/ATM
- Predictive model factors include:
  - days of the week
  - weeks of the month
  - holidays
  - salary disbursement days
  - location of the branches
- Cash allocation plans finalized at beginning of month - deviations from plan are costly
- **Goal:** Determine multipliers for fit to minimize mismatch based on predicted withdrawals
- **Constraints:**
  - Regulatory agencies enforce a minimum cash reserve ratio at branch banks (per day)
  - For each ATM, limit on number of days *cash-out* based on predictive model (customer satisfaction)

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- Simple *looking* nonconvex quadratic integer NLP.
- Linearize the absolute value, add binaries for count constraints.
- So far, no MINLP solvers seem to be able to solve this (several die with numerical failures).

$$
\begin{aligned}
\min \sum_{a \in A} \sum_{d \in D} |f_{ad}| & & & \\
\text{s.t. } \; c^x_{ad} x_a + c^y_{ad} y_a + c^{xy}_{ad} x_a y_a + c^u_{ad} u_a + c_{ad} - w_{ad} & = f_{ad} & & \forall a \in A, d \in D \\
\sum_{a \in A} (f_{ad} + w_{ad}) & \leq B_d & & \forall d \in D \\
|\{d \in D \mid f_{ad} < 0\}| & \leq K_a & & \forall a \in A \\
x_a, y_a & \in [0,1] & & \forall a \in A \\
u_a & \geq 0 & & \forall a \in A \\
f_{ad} & \geq -w_{ad} & & \forall a \in A, d \in D
\end{aligned}
$$

- Discretization of x domain $\{0, 0.1, 0.2, ..., 1.0\}$.
- Linearization of product of binary and continuous, and absolute value.

$$
\begin{aligned}
\min \sum_{a \in A} \sum_{d \in D} & \left(f^+_{ad} + f^-_{ad}\right) & & \\
\text{s.t. } c^x_{ad} \sum_{t \in T} c_t x_{at} + c^y_{ad} y_a + c^{xy}_{ad} \sum_{t \in T} c_t z_{at} + c^u_{ad} u_a - w_{ad} & = f^+_{ad} - f^-_{ad} & & \forall a \in A, d \in D \\
\sum_{t \in T} x_{at} & \leq 1 & & \forall a \in A \\
z_{at} & \leq x_{at} & & \forall a \in A, t \in T \\
z_{at} & \leq y_a & & \forall a \in A, t \in T \\
z_{at} & \geq x_{at} + y_a - 1 & & \forall a \in A, t \in T \\
f^-_{ad} & \leq w_{ad} v_{ad} & & \forall a \in A, d \in D \\
\sum_{a \in A} (f^+_{ad} - f^-_{ad} + w_{ad}) & \leq B_d & & \forall d \in D \\
\sum_{d \in D} v_{ad} & \leq K_a & & \forall a \in A
\end{aligned}
$$

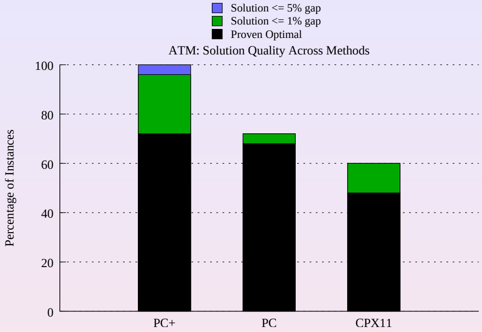
$$
\begin{aligned}
& x_{at} && \in \{0,1\} && \forall a \in A, t \in T \\
& z_{at} && \geq 0 && \forall a \in A, t \in T \\
& v_{ad} && \in \{0,1\} && \forall a \in A, d \in D \\
& y_{a} && \in [0,1] && \forall a \in A \\
& u_{a} && \geq 0 && \forall a \in A \\
& f^{+}_{ad}, f^{-}_{ad} && \in [0, w_{ad}] && \forall a \in A, d \in D
\end{aligned}
$$

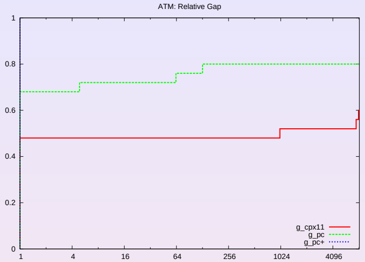
- The MILP formulation has a natural block-angular structure.
  - Master constraints are just the budget constraint.
  - Subproblem constraints (*the rest*) - one block for each ATM.

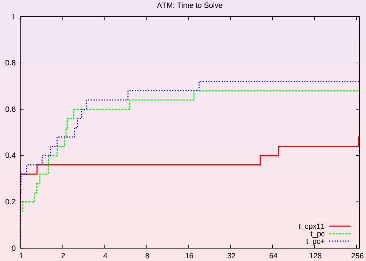
ATM: CPX11 vs PC/PC+

			CPX11			DIP-PC			DIP-PC+		
$\|A\|$	$\|D\|$	s	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes
5	25	1	0.76	OPT	467	1.62	OPT	6	1.96	OPT	6
5	25	2	1.41	OPT	804	1.95	OPT	9	1.57	OPT	7
5	25	3	0.42	OPT	147	7.38	OPT	32	8.03	OPT	32
5	25	4	1.49	OPT	714	2.74	OPT	14	2.45	OPT	13
5	25	5	0.16	OPT	32	0.98	OPT	7	0.95	OPT	6
5	50	1	T	0.10	1264574	162.74	OPT	127	164.46	OPT	131
5	50	2	87.96	OPT	38341	183.28	OPT	273	263.24	OPT	275
5	50	3	8.09	OPT	3576	17.58	OPT	36	22.28	OPT	35
5	50	4	4.13	OPT	1317	3.13	OPT	3	3.17	OPT	3
5	50	5	57.55	OPT	32443	91.30	OPT	145	141.29	OPT	147
10	50	1	T	0.76	998624	297.65	OPT	301	234.47	OPT	156
10	50	2	1507.84	OPT	351879	28.84	OPT	29	52.99	OPT	29
10	50	3	T	0.81	667371	64.72	OPT	64	49.20	OPT	47
10	50	4	1319.00	OPT	433155	7.97	OPT	1	5.00	OPT	1
10	50	5	365.51	OPT	181013	12.49	OPT	3	5.18	OPT	3
10	100	1	T	∞	128155	T	∞	20590	T	0.11	13190
10	100	2	T	∞	116522	T	∞	60554	2437.43	OPT	135
10	100	3	T	∞	118617	T	∞	52902	T	0.20	40793
10	100	4	T	∞	108899	T	∞	47931	T	1.51	59477
10	100	5	T	∞	167617	T	∞	40283	T	0.38	26490
20	100	1	T	∞	93519	379.75	OPT	9	544.49	OPT	9
20	100	2	T	∞	68863	T	16.44	14240	T	0.26	25756
20	100	3	T	∞	95981	T	15.37	41495	T	0.12	3834
20	100	4	T	∞	81836	T	0.39	7554	T	0.08	7918
20	100	5	T	∞	101917	635.59	OPT	21	608.68	OPT	19
Optimal				12			17			18	
$\leq$ 1% Gap				15			18			25	
$\leq$ 10% Gap				15			18			25	

ATM: CPX11 vs PC/PC+







MILPBlock - Block-Angular MILP (as a Generic Solver)

- Consulting work led to numerous MILPs that cannot be solved with generic (B&C) solvers
- Often consider a decomposition approach, since a common modeling paradigm is
  - independent departmental policies which are then coupled by some global constraints
- Development time was slow due to problem-specific implementations of methods

$$\begin{pmatrix} A''_1 & A''_2 & \cdots & A''_\kappa \\ \hline A'_1 & & & \\ & A'_2 & & \\ & & \ddots & \\ & & & A'_\kappa \end{pmatrix}$$

- MILPBlock provides a black-box solver for applying integrated methods to generic MILP
  - This is the *first* framework to do this (to my knowledge).
  - Similar efforts are being talked about by F. Vanderbeck BaPCod (no cuts)
- Currently, the only input needed is MPS/LP and a *block file*
- Future work will attempt to embed automatic recognition of the block-angular structure using packages from linear algebra like: MONET, hMETIS, Mondriaan

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SAS Retail Optimization Solution

- *Multi-tiered supply chain distribution problem* where each block represents a store
- Prototype model developed in SAS/OR's OPTMODEL (algebraic modeling language)

	CPX11			DIP-PC		
Instance	**Time**	**Gap**	**Nodes**	**Time**	**Gap**	**Nodes**
retail27	T	2.30%	2674921	3.18	OPT	1
retail31	T	0.49%	1434931	767.36	OPT	41
retail3	529.77	OPT	2632157	0.54	OPT	1
retail4	T	1.61%	1606911	116.55	OPT	1
retail6	1.12	OPT	803	264.59	OPT	303

Outline

Future Research

- **Branch-and-Relax-and-Cut** - computational focus thus far has been on CPM/DC/PC
- Can we implement **Gomory cuts** in Price-and-Cut?
  - Similar to Interior Point crossover to Simplex, we can crossover from $\hat{x}$ to a feasible basis, load that into the solver and generate tableau cuts
  - Will the design of OSI and CGL work like this? **YES.** J Forrest has added a `crossover` to OsiClp
- Other generic MILP techniques for **MILPBlock**: heuristics, branching strategies, presolve
- Better support for **identical subproblems** (using ideas of Vanderbeck)
- **Parallelization** of branch-and-bound
  - More work per node, communication overhead low - use ALPS
- **Parallelization** related to relaxed polyhedra (work-in-progress):
  - Pricing in block-angular case
  - Nested pricing - use idle cores to generate diverse set of columns simultaneously
  - Generation of decomposition cuts for various relaxed polyhedra - diversity of cuts

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