Decomposition Methods for Discrete Optimization

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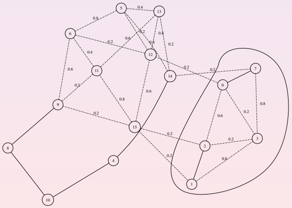
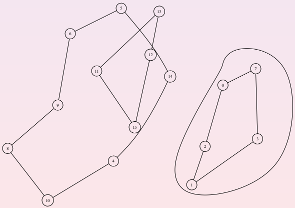
Outline

What is Decomposition?

- Many complex models are built up from simpler structures.
  - Subsystems linked by system-wide constraints or variables.
  - Complex combinatorial structures obtained by combining simpler ones.
- Decomposition is the process of taking a model and breaking it into smaller parts.
- The goal is either to
  - reformulate the model for easier solution;
  - reformulate the model to obtain an improved relaxation (bound); or
  - separate the model into stages or levels (possibly with separate objectives).





Block Structure

- "Classical" decomposition arises from *block structure* in the constraint matrix.
- By relaxing/fixing the linking variables/constraints, we then get a model that is separable.
- A separable model consists of multiple smaller submodels that are easier to solve.
- The separability lends itself nicely to parallel implementation.

$$\begin{pmatrix} A_{01} & A_{02} & \cdots & A_{0\kappa} \\ A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_{\kappa\kappa} \end{pmatrix} \quad \begin{pmatrix} A_{10} & A_{11} & & & \\ A_{20} & & A_{22} & & \\ \vdots & & & \ddots & \\ A_{\gamma 0} & & & & A_{\kappa\kappa} \end{pmatrix}$$

$$\begin{pmatrix} A_{00} & A_{01} & A_{02} & \cdots & A_{0\kappa} \\ A_{10} & A_{11} & & & \\ A_{20} & & A_{22} & & \\ \vdots & & & \ddots & \\ A_{\gamma 0} & & & & A_{\kappa\kappa} \end{pmatrix}$$

The Decomposition Principle

- Decomposition methods leverage our ability to solve either a relaxation or a restriction.
- Methodology is based on the ability to solve a given *subproblem* repeatedly with varying inputs.
- The goal of solving the subproblem repeatedly is to obtain information about its structure that can be incorporated into a *master problem*.
- An overarching theme in this tutorial will be that *most solution methods* for discrete optimization problems are, in a sense, based on the decomposition principle.

Constraint decomposition

- Relax a set of *linking constraints* to expose structure.
- Leverages ability to solve either the optimization or separation problem for a *relaxation* (with varying objectives and/or points to be separated).

Variable decomposition

- Fix the values of *linking variables* to to expose the structure.
- Leverages ability to solve a *restriction* (with varying right-hand sides).

Example: Block Structure (Linking Constraints)

Generalized Assignment Problem (GAP)

$$
\begin{array}{llll}
\min & \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} & & \\
& \sum_{j \in N} w_{ij} x_{ij} & \leq & b_i & \forall i \in M \\
& \sum_{i \in M} x_{ij} & = & 1 & \forall j \in N \\
& x_{ij} & \in & \{0,1\} & \forall i,j \in M \times N
\end{array}
$$

- The problem is to assign m tasks to n machines subject to capacity constraints.
- The variable x_{ij} is one if task i is assigned to machine j.
- The "profit" associated with assigning task i to machine j is c_{ij}.
- If we relax the requirement that each task be assigned to only one machine, the problem decomposes into n independent knapsack problems.

Example: Block Structure (Linking Variables)

Facility Location Problem

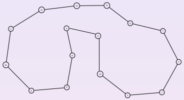
$$
\begin{array}{lll}
\min & \sum_{j=1}^{n} c_j y_j + \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} x_{ij} & \\
\text{s.t.} & \sum_{j=1}^{n} x_{ij} = 1 & \forall i \\
& x_{ij} \leq y_j & \forall i, j \\
& x_{ij}, y_j \in \{0, 1\} & \forall i, j
\end{array}
$$

- We are given n facility locations and m customers to be serviced from those locations.
- There is a fixed cost c_j associated with facility j.
- There is a cost d_{ij} associated with serving customer i from facility j.
- We have two sets of binary variables.
  - y_j is 1 if facility j is opened, 0 otherwise.
  - x_{ij} is 1 if customer i is served by facility j, 0 otherwise.
- If we fix the set of open facilities, then the problem becomes easy to solve.

Example: Underlying Combinatorial Structure

Traveling Salesman Problem Formulation

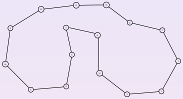
$$
\begin{array}{lcll}
x(\delta(\{u\})) & = & 2 & \forall u \in V \\
x(E(S)) & \leq & |S| - 1 & \forall S \subset V,\ 3 \leq |S| \leq |V| - 1 \\
x_e \in \{0, 1\} & & & \forall e \in E
\end{array}
$$



Example: Underlying Combinatorial Structure

Traveling Salesman Problem Formulation

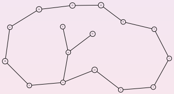
$$\begin{array}{lcll} x(\delta(\{u\})) & = & 2 & \forall u \in V \\ x(E(S)) & \leq & |S|-1 & \forall S \subset V,\ 3 \leq |S| \leq |V|-1 \\ x_e \in \{0,1\} & & & \forall e \in E \end{array}$$



Two relaxations

Find a spanning subgraph with $|V|$ edges ($\mathcal{P}' = $ 1-Tree)

$$\begin{array}{lcll} x(\delta(\{0\})) & = & 2 & \\ x(E(V)) & = & |V| & \\ x(E(S)) & \leq & |S|-1 & \forall S \subset V \setminus \{0\}, 3 \leq |S| \leq |V|-1 \\ x_e \in \{0,1\} & & & \forall e \in E \end{array}$$



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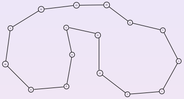

Two relaxations

Find a spanning subgraph with $|V|$ edges ($\mathcal{P}' =$ 1-Tree)

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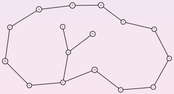
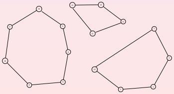


Find a 2-matching that satisfies the subtour constraints ($\mathcal{P}' =$ 2-Matching)

$$\begin{array}{lcll} x(\delta(\{u\})) & = & 2 & \forall u \in V \\ x_e \in \{0,1\} & & & \forall e \in E \end{array}$$



Example: Eliminating Symmetry

- In some cases, the identified blocks are *identical*.
- In such cases, the original formulation will often be highly symmetric.
- The decomposition eliminates the symmetry by collapsing the identical blocks.

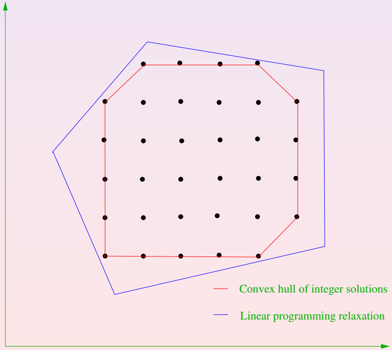
Vehicle Routing Problem (VRP)

$$
\begin{array}{llll}
\min & \sum_{k \in M} \sum_{(i,j) \in A} c_{ij} x_{ijk} & & \\
& \sum_{k \in M} \sum_{j \in N} x_{ijk} & = 1 & \forall i \in V \\
& \sum_{i \in V} \sum_{j \in N} d_i x_{ijk} & \leq C & \forall k \in M \\
& \sum_{j \in N} x_{0jk} & = 1 & \forall k \in M \\
& \sum_{i \in N} x_{ihk} - \sum_{j \in N} x_{hjk} & = 0 & \forall h \in V, k \in M \\
& \sum_{i \in N} x_{i,n+1,k} & = 1 & \forall k \in M \\
& x_{ijk} \in \{0,1\} & & \forall (i,j) \in A, k \in M
\end{array}
$$

Basic Setting

Integer Linear Program: Minimize/Maximize a linear *objective function* over a (discrete) set of *solutions* satisfying specified *linear constraints*.

$$z_{\mathrm{IP}} = \min_{x \in \mathbb{Z}^n} \left\{ c^\top x \mid Ax \geq b \right\}$$



Solving Integer Programs

- *Implicit enumeration* techniques try to enumerate the solution space in an intelligent way.
- The most common algorithm of this type is *branch and bound*.
- Suppose F is the set of feasible solutions for a given MILP. We wish to solve $\min_{x \in F} c^\top x$.

Divide and Conquer

Consider a partition of F into subsets $F_1, \ldots F_k$. Then

$$\min_{x \in F} c^\top x = \min_{1 \le i \le k} \{ \min_{x \in F_i} c^\top x \}.$$

We can then solve the resulting *subproblems* recursively.

- Dividing the original problem into subproblems is called *branching*.
- Taken to the extreme, this scheme is equivalent to complete enumeration.
- We avoid complete enumeration primarily by deriving bounds on the value of an optimal solution to each subproblem.

Branch and Bound

- A *relaxation* of an ILP is an auxiliary mathematical program for which
  - the feasible region contains the feasible region for the original ILP, and
  - the objective function value of each solution to the original ILP is not increased.
  - Relaxations can be used to efficiently get bounds on the value of the original integer program.
- Types of Relaxations
  - Continuous relaxation
  - Combinatorial relaxation
  - Lagrangian relaxations

Branch and Bound

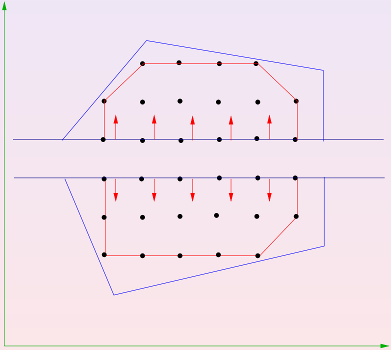
Initialize the queue with F. While there are subproblems in the queue, do

1. Remove a subproblem and solve its relaxation.
2. The relaxation is infeasible $\Rightarrow$ subproblem is infeasible and can be pruned.
3. Solution is feasible for the MILP $\Rightarrow$ subproblem solved (update upper bound).
4. Solution is not feasible for the MILP $\Rightarrow$ lower bound.
   - If the lower bound exceeds the global upper bound, we can *prune the node*.
   - Otherwise, we *branch* and add the resulting subproblems to the queue.

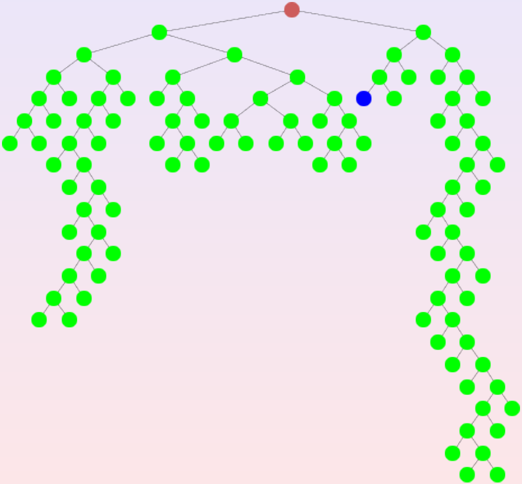
Branching

Branching involves partitioning the feasible region by imposing a *valid disjunction* such that:

- All optimal solutions are in one of the members of the partition.
- The solution to the current relaxation is not in any of the members of the partition.



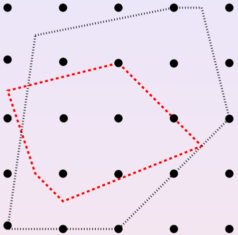
Branch and Bound Tree



Outline

Constraint Decomposition in Linear Programming

$$
\begin{aligned}
z_{\mathrm{LP}} &= \min_{x \in \mathbb{R}^n} \left\{ c^\top x \mid A'x \geq b', A''x \geq b'' \right\} \\
&= \min_{x \in \mathcal{P}'} \left\{ c^\top x \mid A''x \geq b'' \right\} \\
&= \min_{x \in \mathbb{R}^n} \left\{ c^\top x \,\middle|\, x = \sum_{s \in \mathcal{E}} s\lambda_s, \sum_{s \in \mathcal{E}} \lambda_s = 1, \right. \\
&\qquad\qquad \left. \lambda_s \geq 0, \forall s \in \mathcal{E} \right\} \\
&= \max_{u \in \mathbb{R}^{m''}_+} \left\{ \min_{s \in \mathcal{E}} \left\{ c^\top s + u^\top (b'' - A''s) \right\} \right\}
\end{aligned}
$$



$\mathcal{Q}' = \{x \in \mathbb{R}^n \mid A'x \geq b'\}$

$\mathcal{Q}'' = \{x \in \mathbb{R}^n \mid A''x \geq b''\}$

Basic Strategy:

- The original linear program is “hard” to solve because of its size or other properties.
- The matrix A' has structure that makes optimization “easy.”
  - Block structure
  - Network structure
- With decomposition, we can exploit the structure to obtain better solution methods.

Constraint Decomposition in Integer Programming

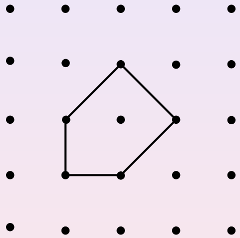
Basic Strategy: Leverage our ability to solve the optimization/separation problem for a relaxation to improve the bound yielded by the LP relaxation.

$$z_{\mathrm{IP}} = \min_{x \in \mathbb{Z}^n} \left\{ c^\top x \mid A'x \geq b', A''x \geq b'' \right\}$$

$$z_{\mathrm{LP}} = \min_{x \in \mathbb{R}^n} \left\{ c^\top x \mid A'x \geq b', A''x \geq b'' \right\}$$

$$z_{\mathrm{D}} = \min_{x \in \mathcal{P}'} \left\{ c^\top x \mid A''x \geq b'' \right\}$$

$$z_{\mathrm{IP}} \geq z_{\mathrm{D}} \geq z_{\mathrm{LP}}$$



$\mathcal{P} = \mathrm{conv}\{x \in \mathbb{Z}^n \mid A'x \geq b', A''x \geq b''\}$

Assumptions:

- $\mathrm{OPT}(\mathcal{P}, c)$ and $\mathrm{SEP}(\mathcal{P}, x)$ are *"hard"*
- $\mathrm{OPT}(\mathcal{P}', c)$ and $\mathrm{SEP}(\mathcal{P}', x)$ are *"easy"*
- $\mathcal{Q}''$ can be represented explicitly (description has polynomial size)
- $\mathcal{P}'$ may be represented implicitly (description has exponential size)

Constraint Decomposition in Integer Programming

Basic Strategy: Leverage our ability to solve the optimization/separation problem for a relaxation to improve the bound yielded by the LP relaxation.

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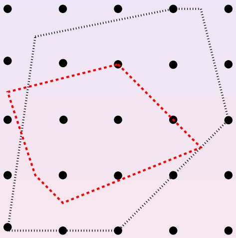

$\mathcal{Q}' = \{x \in \mathbb{R}^n \mid A'x \geq b'\}$

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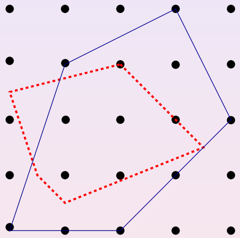

$\mathcal{P}' = \mathrm{conv}\{x \in \mathbb{Z}^n \mid A'x \geq b'\}$

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Constraint Decomposition in Integer Programming

Basic Strategy: Leverage our ability to solve the optimization/separation problem for a relaxation to improve the bound yielded by the LP relaxation.

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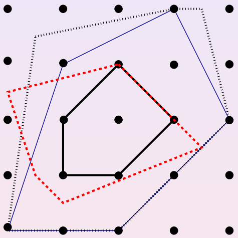

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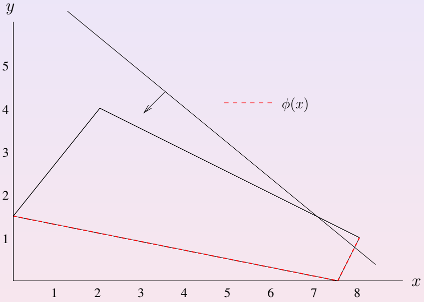
Variable Decomposition in Linear Programming

$$z_{\text{LP}} = \min_{(x,y)\in\mathbb{R}^n} \{c'x + c''y \mid A'x + A''y \geq b\}$$
$$= \min_{x\in\mathbb{R}^{n'}} \{c'x + \phi(b - A'x)\},$$

where

$$\phi(d) = \min c''y \quad \text{s.t. } A''y \geq d, \quad y \in \mathbb{R}^{n''}$$



Basic Strategy:

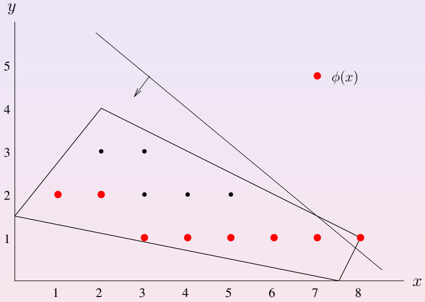
- The function ϕ is the *value function* of a linear program.
- The value function is piecewise linear and convex, but has a description of exponential size.
- We iteratively generate a lower approximation by evaluating ϕ for various values of it domain (Benders' Decomposition).
- The method is effective when we have an efficient method of evaluating ϕ.

Variable Decomposition in Integer Programming

$$z_{\mathrm{IP}} = \min_{(x,y)\in\mathbb{Z}^n} \{c'x + c''y \mid A'x + A''y \geq b\}$$
$$= \min_{x\in\mathbb{R}^{n'}} \{c'x + \phi(b - A'x)\},$$

where

$$\phi(d) = \min c''y$$
$$\text{s.t. } A''y \geq d$$
$$y \in \mathbb{Z}^{n''}$$



Basic Strategy:

- Here, ϕ is the value function of an *integer program*.
- In the general case, the function ϕ is piecewise linear but not convex.
- We can still iteratively generate a lower approximation by evaluating ϕ.

Connections Between Constraint and Variable Decomposition

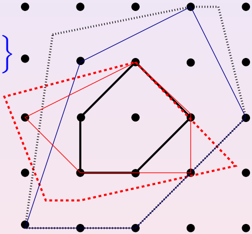
- Constraint and variable decompositions are related.
- Fixing all the variables in the linking constraints also yields a decomposition.
- In the facility location example, relaxing the constraints that require any assigned facility to be open yields a constraint decomposition.
- In the linear programming case, constraint decomposition is variables decomposition in the dual.
- In the discrete case, the situation is more complex and there is no simple relationship between constraint and variables decomposition.
- A technique known as *Lagrangian Decomposition* can be used to decompose linking variables using constraint decomposition.
  - We make a copy of each original variable in each block.
  - We impose a constraint that all copies must take the same value.
  - We relax the new constraint in a Lagrangian fashion.

Lagrangian Decomposition in Integer Programming

Basic Strategy: Leverage our ability to solve the optimization/separation problem for a relaxation to improve the bound yielded by the LP relaxation.

$$
\begin{aligned}
z_{\mathrm{IP}} &= \min_{x \in \mathbb{Z}^n} \left\{ c^\top x \mid A'x \geq b', A''x \geq b'' \right\} \\
&= \min_{x', x'' \in \mathbb{Z}^n} \left\{ c^\top x' \mid A'x' \geq b', A''x'' \geq b'', x' = x'' \right\} \\
z_{\mathrm{LP}} &= \min_{x \in \mathbb{R}^n} \left\{ c^\top x \mid A'x \geq b', A''x \geq b'' \right\} \\
z_{\mathrm{LD}} &= \min \left\{ c^\top x \mid x \in \mathcal{P}' \cap \mathcal{P}'' \right\}
\end{aligned}
$$

$$z_{\mathrm{IP}} \geq z_{\mathrm{LD}} \geq z_{\mathrm{LP}}$$



$\mathcal{P}' = \mathrm{conv}\{x \in \mathbb{Z}^n \mid A'x \geq b'\}$

$\mathcal{P}'' = \mathrm{conv}\{x \in \mathbb{Z}^n \mid A''x \geq b'\}$

$\mathcal{Q}' = \{x \in \mathbb{R}^n \mid A'x \geq b'\}$

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Assumptions:

- $\mathrm{OPT}(\mathcal{P}, c)$ and $\mathrm{SEP}(\mathcal{P}, x)$ are *"hard"*
- $\mathrm{OPT}(\mathcal{P}', c)$ and $\mathrm{OPT}(\mathcal{P}'', x)$ are *"easy"*

Outline

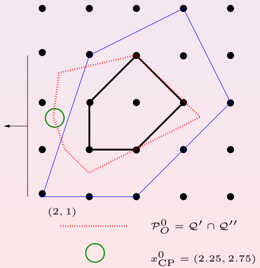
Cutting Plane Method (CPM)

CPM combines an *outer* approximation of $\mathcal{P}'$ with an explicit description of $\mathcal{Q}''$

- **Master**: $z_{\mathrm{CP}} = \min_{x \in \mathbb{R}^n} \left\{ c^\top x \mid Dx \geq d, A''x \geq b'' \right\}$
- **Subproblem**: $\mathrm{SEP}(\mathcal{P}', x_{\mathrm{CP}})$

$$\mathcal{P}' = \{x \in \mathbb{R}^n \mid Dx \geq d\}$$

Exponential number of constraints



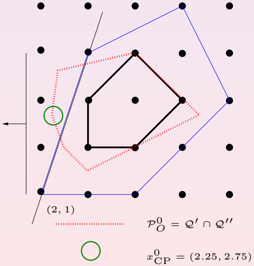
Cutting Plane Method (CPM)

CPM combines an *outer* approximation of $\mathcal{P}'$ with an explicit description of $\mathcal{Q}''$

- **Master**: $z_{\mathrm{CP}} = \min_{x \in \mathbb{R}^n} \left\{ c^\top x \mid Dx \geq d, A''x \geq b'' \right\}$
- **Subproblem**: $\mathrm{SEP}(\mathcal{P}', x_{\mathrm{CP}})$

$$\mathcal{P}' = \{x \in \mathbb{R}^n \mid Dx \geq d\}$$

Exponential number of constraints



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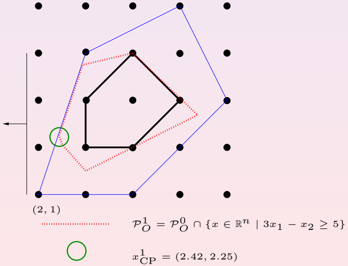

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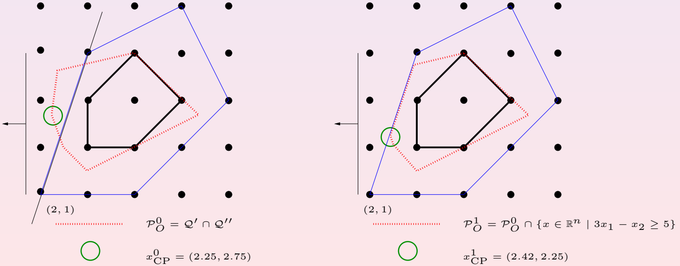

Dantzig-Wolfe Method (DW)

DW combines an *inner* approximation of $\mathcal{P}'$ with an explicit description of $\mathcal{Q}''$

- **Master**: $z_{\mathrm{DW}} = \min_{\lambda \in \mathbb{R}^{\mathcal{E}}_{+}} \left\{ c^\top \left(\sum_{s \in \mathcal{E}} s\lambda_s \right) \mid A'' \left(\sum_{s \in \mathcal{E}} s\lambda_s \right) \geq b'', \sum_{s \in \mathcal{E}} \lambda_s = 1 \right\}$
- **Subproblem**: $\mathrm{OPT}\left(\mathcal{P}', c^\top - u_{\mathrm{DW}}^\top A''\right)$

$$\mathcal{P}' = \left\{ x \in \mathbb{R}^n \;\middle|\; x = \sum_{s \in \mathcal{E}} s\lambda_s, \sum_{s \in \mathcal{E}} \lambda_s = 1, \lambda_s \geq 0 \; \forall s \in \mathcal{E} \right\}$$

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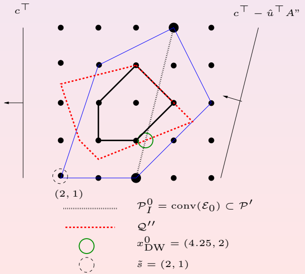

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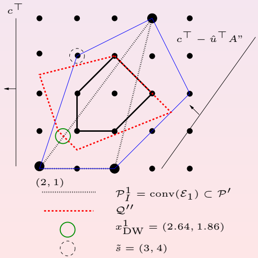

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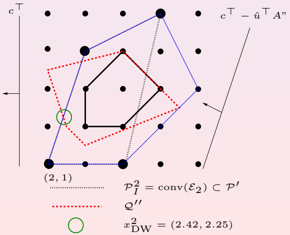

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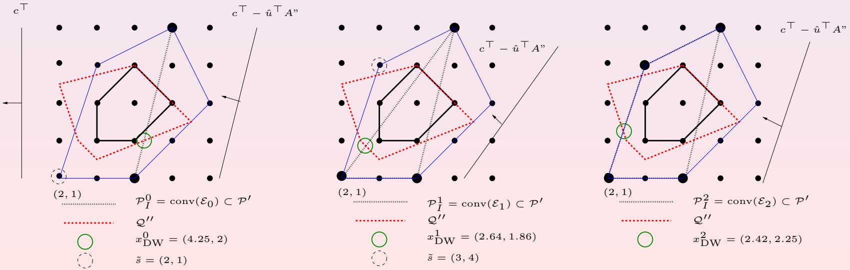
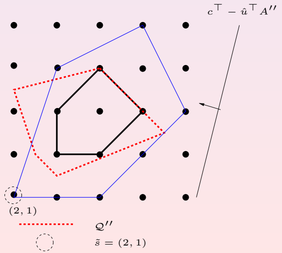


Lagrangian Method (LD)

LD iteratively produces single extreme points of $\mathcal{P}'$ and uses their violation of constraints of $\mathcal{Q}''$ to converge to the same optimal face of $\mathcal{P}'$ as CPM and DW.

- **Master**: $z_{\text{LD}} = \max_{u \in \mathbb{R}_+^{m''}} \left\{ \min_{s \in \mathcal{E}} \left\{ c^\top s + u^\top (b'' - A'' s) \right\} \right\}$
- **Subproblem**: $\text{OPT}\left(\mathcal{P}', c^\top - u_{\text{LD}}^\top A'' \right)$

$$z_{\text{LD}} = \max_{\alpha \in \mathbb{R}, u \in \mathbb{R}_+^{m''}} \left\{ \alpha + b''^\top u \;\middle|\; \left(c^\top - u^\top A'' \right) s - \alpha \geq 0 \; \forall s \in \mathcal{E} \right\} = z_{\text{DW}}$$



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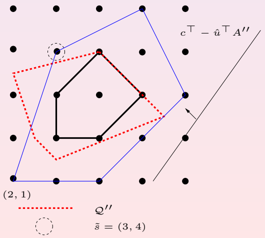

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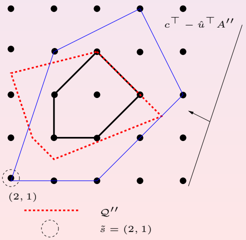

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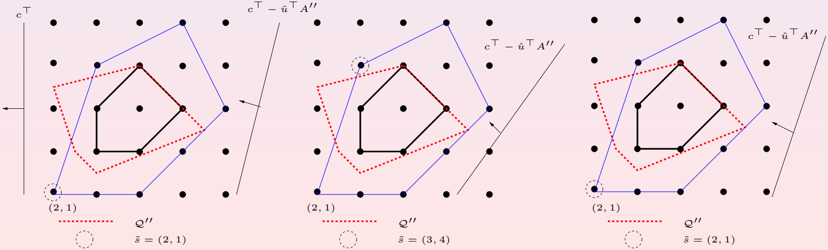
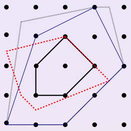


Common Threads

- The LP bound is obtained by optimizing over the intersection of two explicitly defined polyhedra.

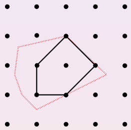
$$z_{\mathrm{LP}} = \min_{x \in \mathbb{R}^n} \left\{ c^\top x \mid x \in \mathcal{Q}' \cap \mathcal{Q}'' \right\}$$



$\mathcal{Q}''$

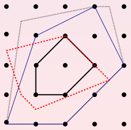
- The constraint decomposition bound is obtained by optimizing over the intersection of one explicitly defined polyhedron and one implicitly defined polyhedron.

$$z_{\mathrm{CP}} = z_{\mathrm{DW}} = z_{\mathrm{LD}} = z_{\mathrm{D}} = \min_{x \in \mathbb{R}^n} \left\{ c^\top x \mid x \in \mathcal{P}' \cap \mathcal{Q}'' \right\} \geq z_{\mathrm{LP}}$$



$\mathcal{Q}' \cap \mathcal{Q}''$

- Traditional constraint decomposition-based bounding methods contain two primary steps
  - **Master Problem:** Update the primal/dual **solution** information
  - **Subproblem:** Update the **approximation** of $\mathcal{P}'$: $\mathrm{SEP}(\mathcal{P}', x)$ or $\mathrm{OPT}(\mathcal{P}', c)$



$\mathcal{Q}''$

When to Apply Constraint Decomposition

Typical scenarios in which constraint decomposition is effective.

- The problem has block structure that makes solution of the subproblem very efficient and/or parallelizable.
- The subproblem has a substantial integrality gap, but we know an efficient algorithm for solving it.
- The original problem is highly symmetric (has identical blocks) and the decomposition eliminates the symmetry.

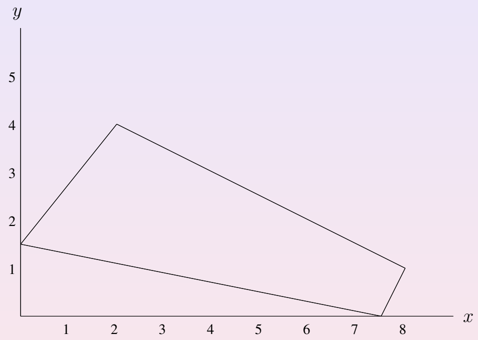
Choosing a particular algorithm raises additional issues.

- Cutting plane methods are hard to beat if strong cuts are known for the subproblem.
- Cutting plane methods also allow a wider variety of cuts to be generated (cuts from the tableau or from multiple relaxations).
- Among traditional decomposition methods, Dantzig-Wolfe is appropriate if cuts for the master are to be generated or when branching in the original space.
- Lagrangian methods offer fast solve times in the master and less overhead, but only approximate primal solution information.

Outline

Variable Decomposition in Linear Programming

$$
\begin{array}{rcll}
z_{LP} & = & \min & x + y \\
& & \text{s.t.} & 25x - 20y \geq -30 \\
& & & -x - 2y \geq -10 \\
& & & -2x + y \geq -15 \\
& & & 2x + 10y \geq 15 \\
& & & x, y \in \mathbb{R}
\end{array}
$$

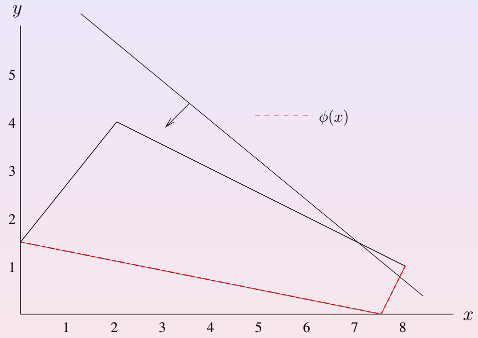


Value Function Reformulation

$$z_{LP} = \min_{x \in \mathbb{R}} x + \phi(x),$$

where

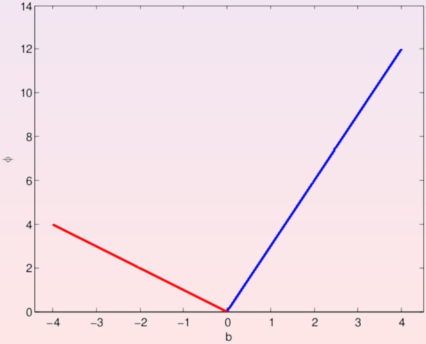
$$\begin{aligned} \phi(x) = \min \quad & y \\ \text{s.t.} \quad & -20y \geq -30 - 25x \\ & -2y \geq -10 + x \\ & y \geq -15 + 2x \\ & 10y \geq 15 - 2x \\ & y \in \mathbb{R} \end{aligned}$$



LP Value Function

Example

$$\begin{aligned} \phi(d) = \min\ & 6x_1 + 7x_2 + 5x_3 \\ \text{s.t. } & 2x_1 - 7x_2 + x_3 = d \\ & x_1, x_2, x_3 \in \mathbb{R}_+ \end{aligned}$$



LP Value Function Structure

LP Value Function

$$\begin{aligned} \phi(d) = \min\ & c^\top x \\ \text{s.t. } & Ax = b \\ & x \in \mathbb{R}^n_+ \end{aligned} \tag{LP}$$

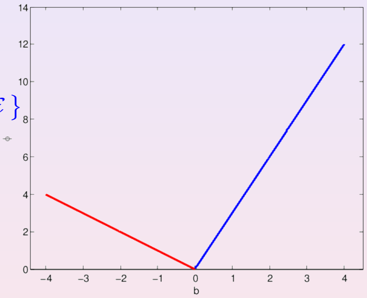
- Suppose the dual of (LP) is feasible and bounded.
- Then the epigraph of ϕ is the convex cone

$$\left\{ (b, z) \;\middle|\; z \geq \nu^\top b, \forall \nu \in \mathcal{E} \right\}$$

  where $\mathcal{E}$ is the set of extreme points of the dual of (LP).
- Thus, the value function is piecewise linear and convex with each piece corresponding to an extreme point of the dual.

Benders' Method for Linear Programs

$$\begin{aligned} z_{\mathrm{LP}} &= \min_{(x,y)\in\mathbb{R}^n} \left\{c'x + c''y \;\middle|\, A'x + A''y \geq b\right\} \\ &= \min_{x\in\mathbb{R}^{n'}} \left\{c'x + \phi(b - A'x)\right\} \\ &= \min_{x\in\mathbb{R}^{n'}} \left\{c'x + z \;\middle|\; z \geq \nu(b - A'x), \nu \in \mathcal{E}\right\} \end{aligned}$$

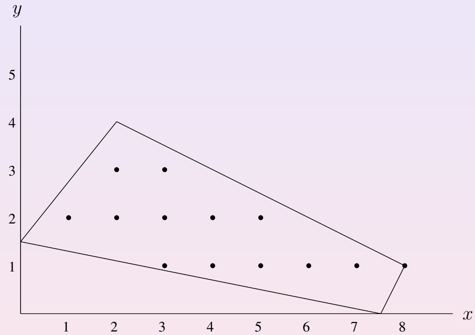


Basic Strategy:

- Solve the above linear program with a cutting plane method.
- We iteratively generate a lower approximation by evaluating ϕ for various values of x (Benders' Decomposition).

Variable Decomposition in Integer Programming

$$
\begin{array}{rrr}
z_{IP} = & \min & x + y \\
& \text{s.t.} & 25x - 20y \geq -30 \\
& & -x - 2y \geq -10 \\
& & -2x + y \geq -15 \\
& & 2x + 10y \geq 15 \\
& & x, y \in \mathbb{Z}
\end{array}
$$



Value Function Reformulation

$$z_{IP} = \min_{x \in \mathbb{Z}} x + \phi(x),$$

where

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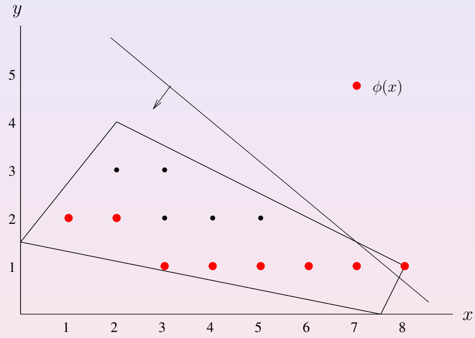
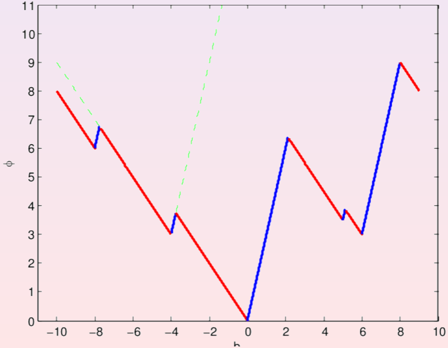


MILP Value Function

Example

$$
\begin{aligned}
\phi(d) = \min\ & 3x_1 + \frac{7}{2}x_2 + 3x_3 + 6x_4 + 7x_5 + 5x_6 \\
\text{s.t. } & 6x_1 + 5x_2 - 4x_3 + 2x_4 - 7x_5 + x_6 = d \\
& x_1, x_2, x_3 \in \mathbb{Z}_+,\ x_4, x_5, x_6 \in \mathbb{R}_+
\end{aligned}
$$



MILP Value Function Structure

MILP Value Function

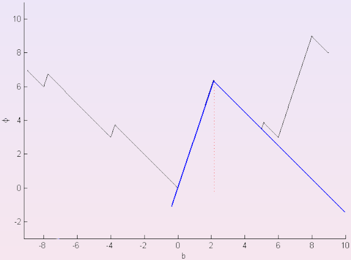
$$\begin{aligned} \phi(d) = \min\ & c^\top x \\ \text{s.t. } & Ax = b \\ & x \in \mathbb{R}^n_+ \end{aligned} \qquad \text{(MILP)}$$

- The epigraph of the MILP value function is the union of a countable collection of epigraphs of identical convex cones.
- These convex convex are translations of the value function of the *continuous restriction*.

Benders' Method for Integer Programs

$$
\begin{aligned}
z_{\mathrm{LP}} &= \min_{(x,y)\in\mathbb{R}^n} \left\{c'x + c''y \;\middle|\; A'x + A''y \geq b\right\} \\
&= \min_{x\in\mathbb{R}^{n'}} \left\{c'x + \phi(b - A'x)\right\} \\
&\geq \min_{x\in\mathbb{R}^{n'}} \left\{c'x + z \;\middle|\; z \geq \phi_D(b - A'x)\right\}
\end{aligned}
$$

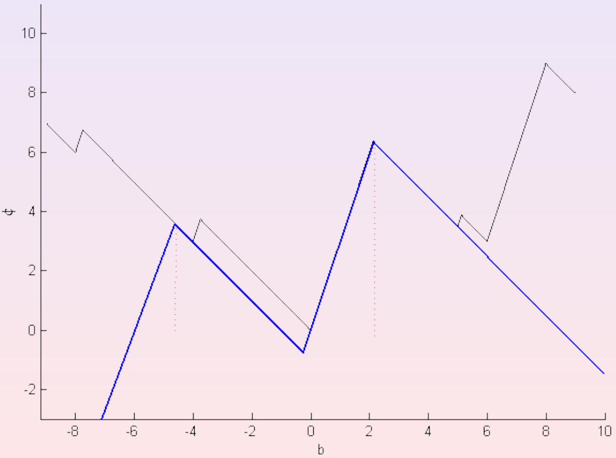
where ϕ_D is a function bounding ϕ from below.



Basic Strategy:

- Solve the above nonlinear program by iteratively constructing ϕ_D.
- The approximation can be updated each time we solve the MILP.
- The pieces of the approximation come from the branch-and-bound tree resulting from solution of the MILP for fixed d

Approximating the Value Function



Common Threads

- Just as in the case of constraint decomposition, variable decomposition methods contain two primary steps
  - **Master Problem:** Update the primal/dual **solution** information
  - **Subproblem:** Update the **approximation** of ϕ by evaluating $\phi(x)$.
- The motivation for applying variable decomposition methods is a bit different than for constraint decomposition methods.
- Generally, variable decomposition is appropriate when
  - we have an efficient method for evaluating ϕ (it has block structure) or
  - we have a multilevel problem with multiple objectives.
- In cases like stochastic programming, the blocks may only differ in their right-hand side, so there is only one (lower-dimensional) function needed to describe all blocks.
- It may also be possible to exploit symmetry in variable decomposition using a strategy similar to that used in constraint decomposition.

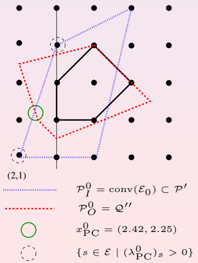
Outline

Price-and-Cut Method (PC)

PC approximates $\mathcal{P}$ by building an *inner* approximation of $\mathcal{P}'$ (as in DW) intersected with an *outer* approximation of $\mathcal{P}$ (as in CPM)

- **Master**: $z_{\mathrm{PC}} = \min_{\lambda \in \mathbb{R}_+^{\mathcal{E}}} \left\{ c^\top \left(\sum_{s \in \mathcal{E}} s \lambda_s \right) \mid D \left(\sum_{s \in \mathcal{E}} s \lambda_s \right) \geq d, \sum_{s \in \mathcal{E}} \lambda_s = 1 \right\}$
- **Subproblem**: $\mathrm{OPT}\left(\mathcal{P}', c^\top - u_{\mathrm{PC}}^\top D\right)$ or $\mathrm{SEP}\left(\mathcal{P}, x_{\mathrm{PC}}\right)$

- As in CPM, separate $\hat{x}_{\mathrm{PC}} = \sum_{s \in \mathcal{E}} s \hat{\lambda}_s$ from $\mathcal{P}$ and add cuts to $[D, d]$.
- **Key Idea**: Cut generation takes place in the space of the compact formulation, maintaining the structure of the column generation subproblem.



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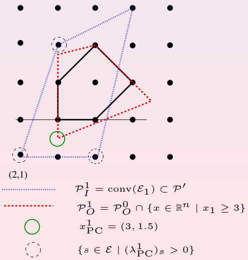

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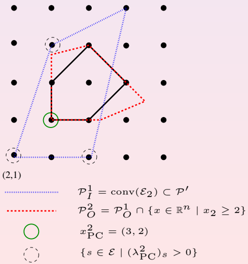
- **Master**: $z_{\mathrm{PC}} = \min_{\lambda \in \mathbb{R}_+^{\mathcal{E}}} \left\{ c^\top \left(\sum_{s \in \mathcal{E}} s \lambda_s \right) \mid D \left(\sum_{s \in \mathcal{E}} s \lambda_s \right) \geq d, \sum_{s \in \mathcal{E}} \lambda_s = 1 \right\}$
- **Subproblem**: $\mathrm{OPT}\left(\mathcal{P}', c^\top - u_{\mathrm{PC}}^\top D\right)$ or $\mathrm{SEP}\left(\mathcal{P}, x_{\mathrm{PC}}\right)$

- As in CPM, separate $\hat{x}_{\mathrm{PC}} = \sum_{s \in \mathcal{E}} s \hat{\lambda}_s$ from $\mathcal{P}$ and add cuts to $[D, d]$.
- **Key Idea**: Cut generation takes place in the space of the compact formulation, maintaining the structure of the column generation subproblem.



Price-and-Cut Method (PC)

PC approximates $\mathcal{P}$ by building an *inner* approximation of $\mathcal{P}'$ (as in DW) intersected with an *outer* approximation of $\mathcal{P}$ (as in CPM)

- **Master**: $z_{\mathrm{PC}} = \min_{\lambda \in \mathbb{R}_+^{\mathcal{E}}} \left\{ c^\top \left(\sum_{s\in\mathcal{E}} s\lambda_s\right) \mid D\left(\sum_{s\in\mathcal{E}} s\lambda_s\right) \geq d, \sum_{s\in\mathcal{E}} \lambda_s = 1 \right\}$
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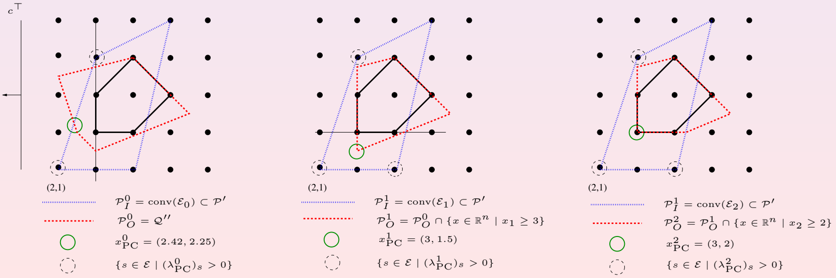
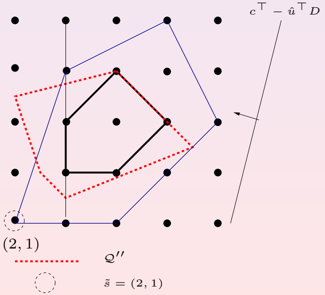


Relax-and-Cut Method (RC)

RC approximates $\mathcal{P}$ by tracing an *inner* approximation of $\mathcal{P}'$ (as in LD) penalizing points outside of a dynamically generated *outer* approximation of $\mathcal{P}$ (as in CPM)

- **Master**: $z_{\mathrm{LD}} = \max_{u \in \mathbb{R}_+^{m''}} \left\{ \min_{s \in \mathcal{E}} \left\{ c^\top s + u^\top (d - Ds) \right\} \right\}$
- **Subproblem**: $\mathrm{OPT}\left(\mathcal{P}', c^\top - u_{\mathrm{LD}}^\top D\right)$ or $\mathrm{SEP}\left(\mathcal{P}, s\right)$

- In each iteration, separate $\hat{s} \in \mathcal{E}$, a solution to the Lagrangian relaxation.
- **Advantage**: Often easier to separate $s \in \mathcal{E}$ from $\mathcal{P}$ than $\hat{x} \in \mathbb{R}^n$.



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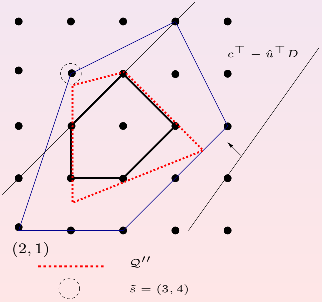

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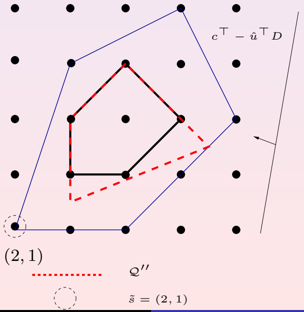

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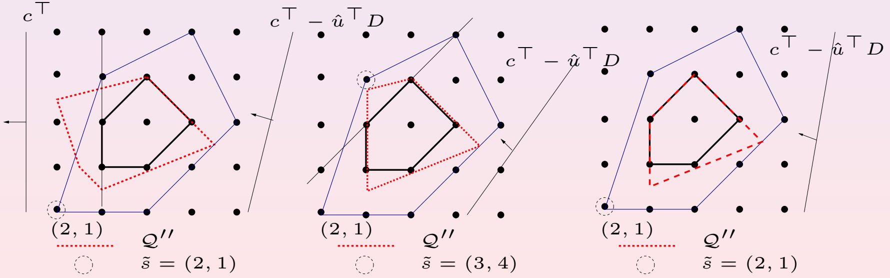


Outline

Structured Separation

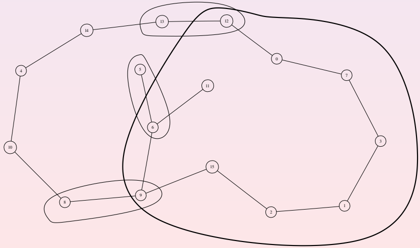
- In general, $\mathrm{OPT}(X, c)$ and $\mathrm{SEP}(X, x)$ are polynomially equivalent.
- **Observation:** Restrictions on input or output can change their complexity.
- The Template Paradigm, restricts the *output* of $\mathrm{SEP}(X, x)$ to valid inequalities that conform to a certain structure. This class of inequalities forms a polyhedron $\mathcal{C} \supset X$ (the *closure*).
- For example, let $\mathcal{P}$ be the convex hull of solutions to the TSP.
  - $\mathrm{SEP}(\mathcal{P}, x)$ is $\mathcal{NP}$-Complete.
  - $\mathrm{SEP}(\mathcal{C}, x)$ is polynomially solvable, for $\mathcal{C} \supset \mathcal{P}$
    - $\mathcal{P}^{\mathrm{Subtour}}$, the Subtour Polytope (separation using Min-Cut), or
    - $\mathcal{P}^{\mathrm{Blossom}}$, the Blossom Polytope (separation using Letchford, et al.).
- Structured Separation, restricts the *input* of $\mathrm{SEP}(X, x)$, such that x conforms to some structure. For example, if x is restricted to solutions to a combinatorial problem, then separation often becomes much easier.

Structured Separation: Example

- Separation of Comb Inequalities:

$$x(E(H)) + \sum_{i=1}^{k} x(E(T_i)) \leq |H| + \sum_{i=1}^{k} (|T_i| - 1) - \lceil k/2 \rceil$$

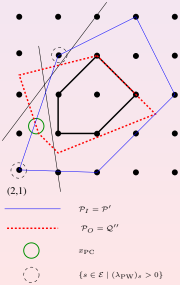
- $\mathrm{SEP}(\mathcal{P}^{\mathrm{Blossom}}, s)$, for s a 1-tree, can be solved in $O(|V|^2)$
  - Construct candidate handles H from BFS tree traversal and an odd (≥ 3) set of edges with one endpoint in H and one in $V \setminus H$ as candidate teeth (each gives a violation of $\lceil k/2 \rceil - 1$).
  - This can also be used as a quick heuristic to separate 1-trees for more general comb structures, for which there is no known polynomial algorithm for separation of arbitrary vectors.



Price-and-Cut (Revisited)

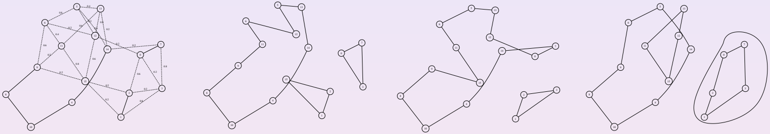
Price-and-Cut (Revisited): As normal, use **DW** as the bounding method, but use the decomposition obtained in each iteration to generate improving inequalities, as in **RC**.

- **Key Idea:** Rather than (or in addition to) separating $\hat{x}_{\mathrm{PC}}$, separate each member of D
- As with **RC**, often much easier to separate $s \in \mathcal{E}$ than $\hat{x}_{\mathrm{PC}} \in \mathbb{R}^n$
- **RC** only gives us one member of $\mathcal{E}$ to separate, while **PC** gives us a set, one of which must be violated by any inequality violated by $\hat{x}_{\mathrm{PC}}$
- Provides an alternative *necessary* (but not *sufficient*) condition to find an improving inequality which is very **easy to implement and understand**.



Price-and-Cut (Revisited)

- The violated subtour found by separating the 2-matching *also* violates the fractional point, but was found at little cost.



$\hat{x}$ $\quad$ $\hat{\lambda}_0 = 0.2$ $\quad$ $\hat{\lambda}_1 = 0.2$ $\quad$ $\hat{\lambda}_2 = 0.6$

- Similarly, the violated blossom found by separating the 1-tree *also* violates the fractional point, but was found at little cost.

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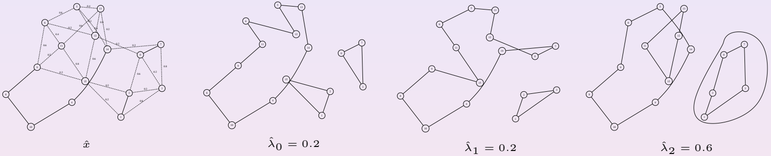

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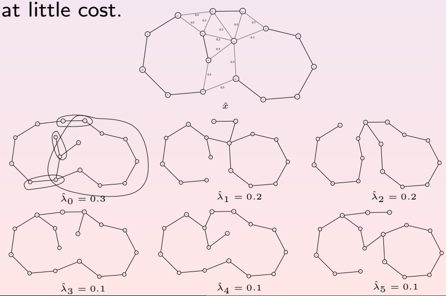

Outline

Decompose-and-Cut (DC)

Decompose-and-Cut: Each iteration of CPM, decompose into convex combo of e.p.'s of $\mathcal{P}'$

$$\min_{\lambda \in \mathbb{R}^{\mathcal{E}}_{+}, (x^+, x^-) \in \mathbb{R}^n_+} \left\{ x^+ + x^- \;\middle|\; \sum_{s \in \mathcal{E}} s\lambda_s + x^+ - x^- = \hat{x}_{\mathrm{CP}}, \sum_{s \in \mathcal{E}} \lambda_s = 1 \right\}$$

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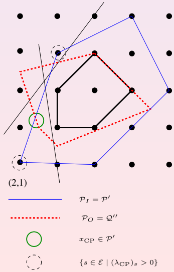
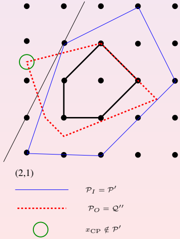
- If $\hat{x}_{\mathrm{CP}}$ lies outside $\mathcal{P}'$ the decomposition will fail
- By the *Farkas Lemma* the proof of infeasibility provides a valid and violated inequality

Decomposition Cuts

$$\begin{aligned} u_{\mathrm{DC}}^t s + \alpha_{\mathrm{DC}}^t &\leq 0 \; \forall s \in \mathcal{P}' \quad \text{and} \\ u_{\mathrm{DC}}^t \hat{x}_{\mathrm{CP}} + \alpha_{\mathrm{DC}}^t &> 0 \end{aligned}$$





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- Originally proposed as a method to solve the VRP with TSP as relaxation.
- Essentially, we are transforming an optimization algorithm into a separation algorithm.
- The machinery for solving this already exists (=column generation)
- Much easier than DW problem because it's a *feasibility* problem and
  - $\hat{x}_i = 0 \Rightarrow s_i = 0$, can remove constraints not in support, and
  - $\hat{x}_i = 1 \text{ and } s_i \in \{0, 1\} \Rightarrow$ constraint is redundant with convexity constraint
  - Often gets *lucky* and produces incumbent solutions to original IP

Outline

Generic Constraint Decomposition

- Traditionally, decomposition-based branch-and-bound methods have required extensive problem-specific customization.
  - identifying the decomposition (which constraints to relax);
  - formulating and solving the subproblem (either optimization or separation over $\mathcal{P}'$);
  - formulating and solving the master problem; and
  - performing the branching operation.
- However, it is possible to replace these components with generic alternatives.
  - The decomposition can be identified automatically by analyzing the matrix or through a modeling language.
  - The subproblem can be solved with a generic MILP solver.
  - The branching can be done in the original compact space.
- The remainder of this talk focuses on our recent efforts to develop a completely generic decomposition-based MILP solver.

Working in the Compact Space

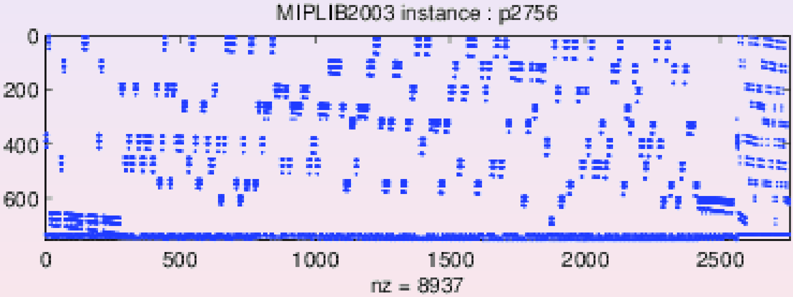
- The key to the implementation of this unified framework is that we always maintain a representation of the problem in the compact space.
- This allows us to employ most of the usual techniques used in LP-based branch and bound without modification, even in this more general setting.
- There are some challenges related to this approach that we are still working on.
  - Gomory cuts
  - Preprocessing
  - Identical subproblems
  - Strong branching
- Allowing the user to express all methods in the compact space is extremely powerful when it comes to modeling language support.
- It is important to note that DIP currently assumes the existence of a formulation in the compact space.
- We are working on relaxing this assumption, but this means the loss of the fully generic implementation of some techniques.

Automatic Structure Detection

- For unstructured problems, block structure may be detected automatically.
- This is done using hypergraph partitioning methods.
- We map each row of the original matrix to a hyperedge and the nonzero elements to nodes in a hypergraph.
- Hypergraph partitioning results in identification of the blocks in a singly-bordered block diagonal matrix.

$$
\begin{pmatrix}
* & * & & & & * \\
* & & & * & * & \\
& * & & & * & * \\
* & & * & & & \\
* & * & & * & * & \\
& * & * & & * & \\
& & * & * & * & \\
* & & & & &
\end{pmatrix}
$$

Hidden Block Structure



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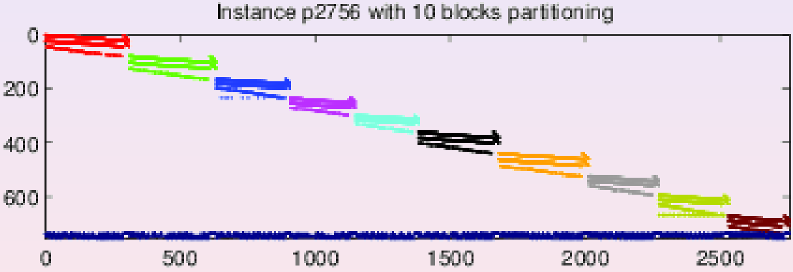
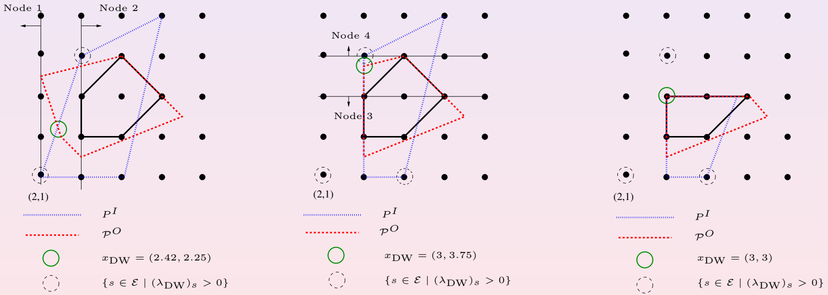


Generic Branching

- By default, we branch on variables in the compact space.
- In PC, this is done by mapping back to the compact space $\hat{x} = \sum_{s \in \mathcal{E}} s\hat{\lambda}_s$.
- Variable branching in the compact space is constraint branching in the extended space
- This idea makes it possible define generic branching procedures.



$$
\begin{array}{lccc}
\text{Node 1:} & 4\lambda_{(4,1)} + 5\lambda_{(5,5)} + 2\lambda_{(2,1)} + 3\lambda_{(3,4)} & \leq & 2 \\
\text{Node 2:} & 4\lambda_{(4,1)} + 5\lambda_{(5,5)} + 2\lambda_{(2,1)} + 3\lambda_{(3,4)} & \geq & 3
\end{array}
$$

Branching for Lagrangian Method

- In general, Lagrangian methods do *not* provide a primal solution λ
- Let $\mathcal{B}$ define the extreme points found in solving subproblems for z_{LD}
- Build an inner approximation using this set, then proceed as in PC

$$\mathcal{P}_I = \left\{ x \in \mathbb{R}^n \;\middle|\; x = \sum_{s \in \mathcal{B}} s\lambda_s, \sum_{s \in \mathcal{B}} \lambda_s = 1, \lambda_s \geq 0 \; \forall s \in \mathcal{B} \right\}$$

$$\min_{\lambda \in \mathbb{R}^{\mathcal{B}}_{+}} \left\{ c^\top \left(\sum_{s \in \mathcal{B}} s\lambda_s \right) \;\middle|\; A'' \left(\sum_{s \in \mathcal{B}} s\lambda_s \right) \geq b'', \sum_{s \in \mathcal{B}} \lambda_s = 1 \right\}$$

- Closely related to *volume* algorithm and *bundle* methods

Outline

Decomposition Software

There have been a number of efforts to create frameworks supporting the implementation of decomposition-based branch and bound.

Column Generation Frameworks

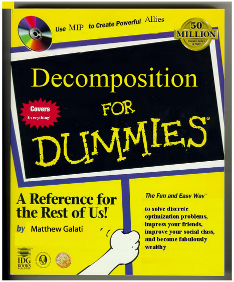
- ABACUS [Jünger and Thienel(2012)]
- SYMPHONY [Ralphs et al.(2012)Ralphs, Ladányi, Güzelsoy, and Mahajan]
- COIN/BCP [Ladányi(2012)]

Generic decomposition frameworks

- BaPCod [Vanderbeck(2012)]
  - Dantzig-Wolfe
  - Automatic reformulation,
  - Generic cuts
  - Generic branching
- GCG [Gamrath and Lübbecke(2012)]
  - Dantzig-Wolfe
  - Automatic hypergraph-based decomposition
  - Automatic reformulation,
  - Generic cut generation
  - Generic branching

Shameless Self Promotion

- The use of decomposition methods in practice is hindered by a number of serious drawbacks.
  - *Implementation is difficult*, usually requiring development of sophisticated customized codes.
  - Choosing an algorithmic strategy requires *in-depth knowledge* of theory and strategies are *difficult to compare empirically*.
  - The powerful techniques modern solvers use to solve integer programs are *difficult to integrate* with decomposition-based approaches.
- DIP and CHiPPS are two frameworks that together allow for easier implementation of decomposition approaches.
  - CHiPPS (COIN High Performance Parallel Search Software) is a flexible library hierarchy for implementing parallel search algorithms.
  - DIP (Decomposition for Integer Programs) is a framework for implementing decomposition-based bounding methods.
  - DIP with CHiPPS is a full-blown branch-and-cut-and-price framework in which details of the implementation are hidden from the user.
- DIP can be accessed through a modeling language or by providing a model with notated structure.



DIP Framework: Implementation

COmputational INfrastructure for Operations Research
Have some DIP with your CHiPPS?

- **DIP** was built around data structures and interfaces provided by COIN-OR
- The **DIP** framework, written in C++, is accessed through two user interfaces:
  - Applications Interface: `DecompApp`
  - Algorithms Interface: `DecompAlgo`
- **DIP** provides the bounding method for branch and bound
- **ALPS** (Abstract Library for Parallel Search) provides the framework for tree search
  - `AlpsDecompModel : public AlpsModel`
    - a wrapper class that calls (data access) methods from `DecompApp`
  - `AlpsDecompTreeNode : public AlpsTreeNode`
    - a wrapper class that calls (algorithmic) methods from `DecompAlgo`

DIP Framework: API

- The base class `DecompApp` provides an interface for user to define the application-specific components of their algorithm
  - Define the model(s)
    - `setModelObjective(double * c)`: define c
    - `setModelCore(DecompConstraintSet * model)`: define $\mathcal{Q}''$
    - `setModelRelaxed(DecompConstraintSet * model, int block)`: define $\mathcal{Q}'$ [optional]
  - `solveRelaxed()`: define a method for $\mathrm{OPT}(\mathcal{P}', c)$ [optional, if $\mathcal{Q}'$, **CBC** is built-in]
  - `generateCuts()`: define a method for $\mathrm{SEP}(\mathcal{P}', x)$ [optional, **CGL** is built-in]
  - `isUserFeasible()`: is $\hat{x} \in \mathcal{P}$? [optional, if $\mathcal{P} = \mathrm{conv}(\mathcal{P}' \cap \mathcal{Q}'' \cap \mathbb{Z})$]
  - All methods have appropriate defaults but are `virtual` and may be overridden.
- The base class `DecompAlgo` provides the shell (init / master / subproblem / update).
  - Each of the methods described has derived default implementations `DecompAlgoX : public DecompAlgo` which are accessible by any application class, allowing full flexibility.
  - New, hybrid or extended methods can be easily derived by overriding the various subroutines, which are called from the base class.

DIP Framework: Feature Overview

- **One interface** to all algorithms: **CP, DW, LD, PC, RC**
- **Automatic reformulation** allows users to specify methods in the compact (original) space.
- **Integrate different decomposition methods**
  - Can utilize **CGL** cuts in all algorithms (separate from original space).
  - Can utilize *structured separation* (efficient algorithms that apply only to vectors with special structure (integer) in various ways).
  - Can separate from $\mathcal{P}'$ using subproblem solver (DC).
- **Integerate multiple bounding methods**
  - Column generation based on *multiple/nested relaxations* can be easily defined and employed.
  - Bounds based on *multiple model/algorithm* combinations.
- **Use of generic MILP solution technology**
  - Using the mapping $\hat{x} = \sum_{s \in \mathcal{E}} s\hat{\lambda}_s$ we can import any generic MILP technique to the PC/RC context.
  - Use generic MILP solver to solve subproblems.
  - Hooks to define branching methods, heuristics, etc.

DIP Framework: Feature Overview (cont.)

- **Performance enhancements**
  - Detection and removal of columns that are close to parallel
  - Basic dual stabilization (Wentges smoothing)
  - Redesign (and simplification) of treatment of master-only variables.
  - Branching can be enforced in subproblem or master (when oracle is MILP)
  - Ability to stop subproblem calculation on gap/time and calculate LB (can branch early)
  - For oracles that provide it, allow multiple columns for each subproblem call
- **Algorithms for generating initial columns**
  - Solve $\mathrm{OPT}(\mathcal{P}', c + r)$ for random perturbations
  - Solve $\mathrm{OPT}(\mathcal{P}_N)$ heuristically
  - Run several iterations of LD or DC collecting extreme points
- **Choice of master LP solver**
  - Dual simplex after adding rows or adjusting bounds (warm-start dual feasible)
  - Primal simplex after adding columns (warm-start primal feasible)
  - Interior-point methods might help with stabilization vs extremal duals

DIP Generic Decomposition-based MILP Solver

- Many difficult MILPs have a block structure, but this structure is not part of the input (MPS) or is not exploitable by the solver.
- In practice, it is common to have models composed of independent subsystems coupled by global constraints.
- The result may be models that are highly symmetric and difficult to solve using traditional methods, but would be easy to solve if the structure were known.

$$\begin{pmatrix} A''_1 & A''_2 & \cdots & A''_\kappa \\ \hline A'_1 & & & \\ & A'_2 & & \\ & & \ddots & \\ & & & A'_\kappa \end{pmatrix}$$

- MILPBlock provides a black-box solver for applying integrated methods to generic MILP
- Input is an MPS/LP and a *block file* specifying structure.
- Optionally, the block file can be automatically generated using the hypergraph partitioning algorithm of HMetis.
- This is the engine underlying DIPPY.

Outline

Modeling Systems

- In general, there are not many options for expressing block structure directly in a modeling language.
- Part of the reason for this is that there are also not many software frameworks that can exploit this structure.
- One substantial exception is GAMS, which offers the Extended Mathematical Programming (EMP) Language.
- With EMP. it is possible to directly express multi-level and multi-stage problems in the modeling language.
- For other modeling languages, it is possible to manually implement decomposition methods using traditional underlying solvers.
- Here, we present a modeling language interface to DIP that provides the ability to express block structure and exploit it within DIP.

DipPy

- DipPy provides an interface to DIP through the modeling language PuLP.
- PuLP is a modeling language that provides functionality similar to other modeling languages.
- It is built on top of Python so you get the full power of that language for free.
- PuLP and DipPy are being developed by Stuart Mitchell and Mike O'Sullivan in Auckland and are part of COIN.
- Through DipPy, a user can
  - Specify the model and the relaxation, including the block structure.
  - Implement methods (coded in Python) for solving the relaxation, generating cuts, custom branching.
- With DipPy, it is possible to code a customized column-generation method from scratch in a few hours.
- This would have taken months with previously available tools.

Example: Generalized Assignment Problem

- The problem is to find a minimum cost assignment of n tasks to m machines such that each task is assigned to one machine subject to capacity restrictions.
- A binary variable x_{ij} indicates that machine i is assigned to task j. $M = 1, \ldots, m$ and $N = 1, \ldots, n$.
- The cost of assigning machine i to task j is c_{ij}

Generalized Assignment Problem (GAP)

$$
\begin{array}{llll}
\min & \displaystyle\sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} & & \\
& \displaystyle\sum_{j \in N} w_{ij} x_{ij} \quad \le & b_i & \forall i \in M \\
& \displaystyle\sum_{i \in M} x_{ij} \quad = & 1 & \forall j \in N \\
& x_{ij} \quad \in & \{0,1\} & \forall i,j \in M \times N
\end{array}
$$

GAP in DipPy

Creating GAP model in DipPy

```
prob = dippy.DipProblem("GAP", LpMinimize)

# objective
prob += lpSum(assignVars[m][t] * COSTS[m][t] for m, t in MACHINES_TASKS), "min"

# machine capacity (knapsacks, relaxation)
for m in MACHINES:
    prob.relaxation[m] +=
    lpSum(assignVars[m][t] * RESOURCE_USE[m][t] for t in TASKS) <= CAPACITIES[m]

# assignment
for t in TASKS:
    prob += lpSum(assignVars[m][t] for m in MACHINES) == 1

prob.relaxed_solver = relaxed_solver

dippy.Solve(prob)
```

GAP in DipPy

Solver for subproblem for GAP in DipPy

```
def relaxed_solver(prob, machine, redCosts, convexDual):
    # get tasks which have negative reduced
    task_idx = [t for t in TASKS if redCosts[assignVars[machine][t]] < 0]
    vars     = [assignVars[machine][t] for t in task_idx]
    obj      = [-redCosts[assignVars[machine][t]] for t in task_idx]
    weights  = [RESOURCE_USE[machine][t] for t in task_idx]

    z, solution = knapsack01(obj, weights, CAPACITIES[machine])
    z = -z

    # get sum of original costs of variables in solution
    orig_cost  = sum(prob.objective.get(vars[idx]) for idx in solution)
    var_values = [(vars[idx], 1) for idx in solution]

    dv = dippy.DecompVar(var_values, z-convexDual, orig_cost)

    # return, list of DecompVar objects
    return [dv]
```

GAP in DipPy

DipPy Auxiliary Methods

```
def solve_subproblem(prob, index, redCosts, convexDual):
   ...
   z, solution = knapsack01(obj, weights, CAPACITY)
   ...
   return []
prob.relaxed_solver = solve_subproblem
def knapsack01(obj, weights, capacity):
    ...
    return c[n-1][capacity], solution
def first_fit(prob):
    ...
    return bvs
def one_each(prob):
   ...
   return bvs
prob.init_vars = first_fit
def choose_antisymmetry_branch(prob, sol):
   ...
   return ([], down_branch_ub, up_branch_lb, [])
prob.branch_method = choose_antisymmetry_branch
def generate_weight_cuts(prob, sol):
    ...
    return new_cuts
prob.generate_cuts = generate_weight_cuts
def heuristics(prob, xhat, cost):
    ...
    return sols
prob.heuristics = heuristics
dippy.Solve(prob, {
    'doPriceCut': '1',
})
```

To Infinity and Beyond...

- Separable subproblems (Important!)
  - Identical subproblems (symmetry)
  - Parallel solution of subproblems
  - Automatic detection
  - Cuts and branching?
- Use of generic MILP solution technology
  - Incorporation of advanced branchng techniques (how to do strong branching)
  - Gomory cuts (crossover?)
  - Use generic MILP solver to generate multiple columns in each iteration.
- Primal Heuristics
  - For block-angular case, at end of each node, a simplt heuristic is to solve with $\lambda \in \mathbb{Z}$
  - Used in *root node* by Barahona and Jensen ('98), we extend to tree
  - A number of other heuristics have been proposed.
- Dual stabilization
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  - Interior-point methods might help with stabilization vs extremal duals
  - Can we use volume or bundle along with an exact LP solver?
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  - How much diving do we do?
- **Nested pricing and solution methods**
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Where Is All This Going?

- Decomposition methods are important in practice (see Mike Trick!), but have proven difficult to utilize in practice.
- There is renewed interest in making these method acessible to general users.
  - Computational frameworks are being developed that employ these methods "generically."
  - Modeling language support is emerging that allows users to express structure that can be exploited.
- All of this capability is still early in the development stages.
- There will need to be an evolution similar to what happened when generic MILP solvers generalized problem-specific techniques.
- There are LOADS of questions to be answered and research to be done.

THANKS FOR LISTENING!

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