Capacitated Vehicle Routing and Some Related Problems

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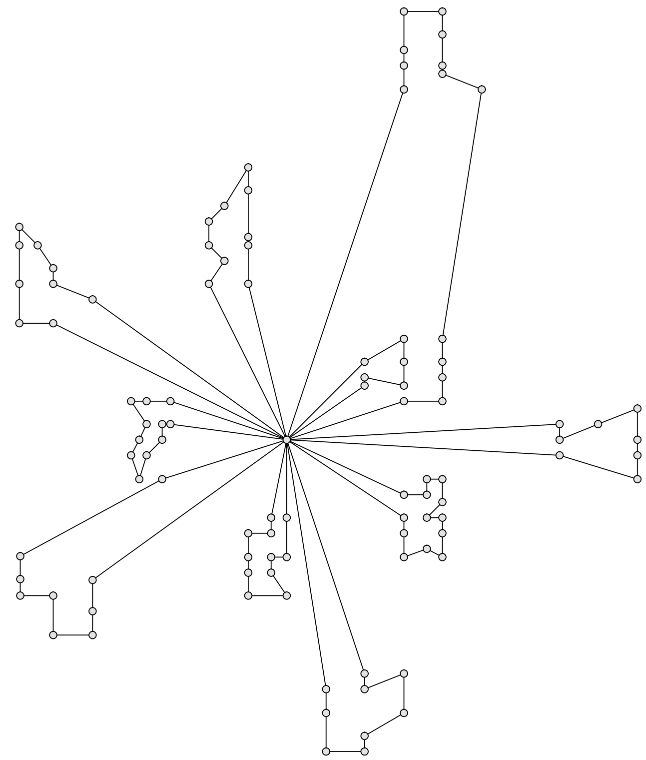
Outline of Talk

- Introduction to Integer Programming
  - Modeling
  - Solutions Techniques
- Introduction to Vehicle Routing
- Modeling Issues
- Related Problems
- Complexity
- Valid Inequalities
- Implementation
- Computational Issues and Results
- Future Directions



Mathematical Programming Models

- What does *mathematical programming* mean?
- Programming here means "planning."
- Literally, these are "mathematical models for planning."
- Also called *optimization models*.
- Essential elements
  - Decision variables
  - Constraints
  - Objective Function
  - Parameters and Data

Forming a Mathematical Programming Model

- The general form of a *mathematical programming model* is:

$$\begin{array}{lcl} min\ f(x) & & \\ s.t.\ \ g_i(x) & \left\{ \begin{array}{c} \leq \\ = \\ \geq \end{array} \right\} & b_i \\ x & \in & X \end{array}$$

  $X \subseteq \mathbb{R}^n$ is an (implicitly defined) set that may be discrete.

- A *mathematical programming problem* is a problem that can be expressed using a mathematical programming model (called the *formulation*).
- A single mathematical programming problem can be represented using many different formulations (important).

Types of Mathematical Programming Models

- The type of mathematical programming model is determined mainly by
  - The form of the objective and the constraints.
  - The form of the set X.
- In this talk, we consider linear models.
  - The objective function is linear.
  - The constraints are linear.
  - Linear models are specified by cost vector $c \in \mathbb{R}^n$, constraint matrix $A \in \mathbb{R}^{m \times n}$, and right-hand side vector $b \in \mathbb{R}^m$ and have the form

$$\begin{aligned} min\, c^T x & \\ s.t. \quad Ax &\geq b \\ x &\in X \end{aligned}$$

Linear Models

- Generally speaking, linear models are easier to solve than more general types of models.
- If $X = \mathbb{R}^n$, the model is called a *linear program* (LP).
- Linear programming models can be solved effectively.
- If some of the variables in the model are required to take on integer values, the model is called a *mixed integer linear programs* (MILPs).
- MILPs can be extremely difficult to solve in practice.

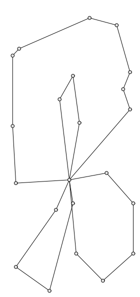
Modeling with Integer Variables

- Why do we need integer variables?
- If a variable represents the quantity of a physical resource that only comes in discrete units, then it must be assigned an integer value.
- We can use *0-1 (binary) variables* for a variety of other purposes.
  - Modeling yes/no decisions.
  - Enforcing disjunctions.
  - Enforcing logical conditions.
  - Modeling fixed costs.
  - Modeling piecewise linear functions.
- The simplest form of ILP is a *combinatorial optimization problem* (COP), where all variables are binary.

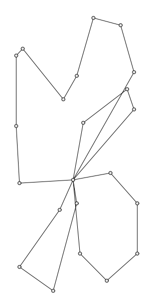
Combinatorial Optimization

- A *combinatorial optimization problem* $CP = (E, \mathcal{F})$ consists of
  - A *ground set* E,
  - A set $\mathcal{F} \subseteq 2^E$ of *feasible solutions*, and
  - A *cost function* $c \in \mathbb{Z}^E$ (optional).
- The *cost* of $S \in \mathcal{F}$ is $c(S) = \sum_{e \in S} c_e$.
- A *subproblem* is defined by $\mathcal{S} \subseteq \mathcal{F}$.

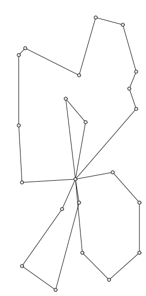
- Problem: Find a least cost member of $\mathcal{F}$.



Cost 1100



Cost 1105



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Example: Perfect Matching Problem

- We are given a set N of n people that need to paired in teams of two.
- Let c_{ij} represent the "cost" of the team formed by persons i and j.
- We wish to minimize the overall cost of the pairings.
- The nodes represent the people and the edges represent pairings.
- We have $x_{ij} = 1$ if i and j are matched, $x_{ij} = 0$ otherwise.
- To simplify the presentation, we assume that $x_{ij} = 0$ if $i \geq j$.

$$
\begin{aligned}
min \quad & \sum_{\{i,j\} \in N \times N} c_{ij} x_{ij} \\
s.t. \quad & \sum_{j \in N} x_{ij} = 1, \qquad \forall i \in N \\
& x_{ij} \in \{0,1\}, \ \forall \{i,j\} \in N \times N, i < j.
\end{aligned}
$$

Fixed-charge Problems

- In many instances, there is a fixed cost and a variable cost associated with a particular decision.
- Example: Fixed-charge Network Flow Problem (FCNFP)
  - We are given a directed graph $G = (N, A)$ and a demand/supply at each node.
  - There is a fixed cost c_{ij} associated with building arc (i, j).
  - There is also a variable cost d_{ij} for each unit of flow along arc (i, j).
  - We want to minimize the sum of these two costs while meeting demand.
- Example: Cable-Trench Problem (CTP)
  - A FCNFP with only one supply node (the communications hub).
  - All other nodes must be connected to the hub by a cable.
  - The fixed cost is the cost of digging the trenches.
  - The variable cost is the cost of laying the cable.

How Do We Solve These Problems?

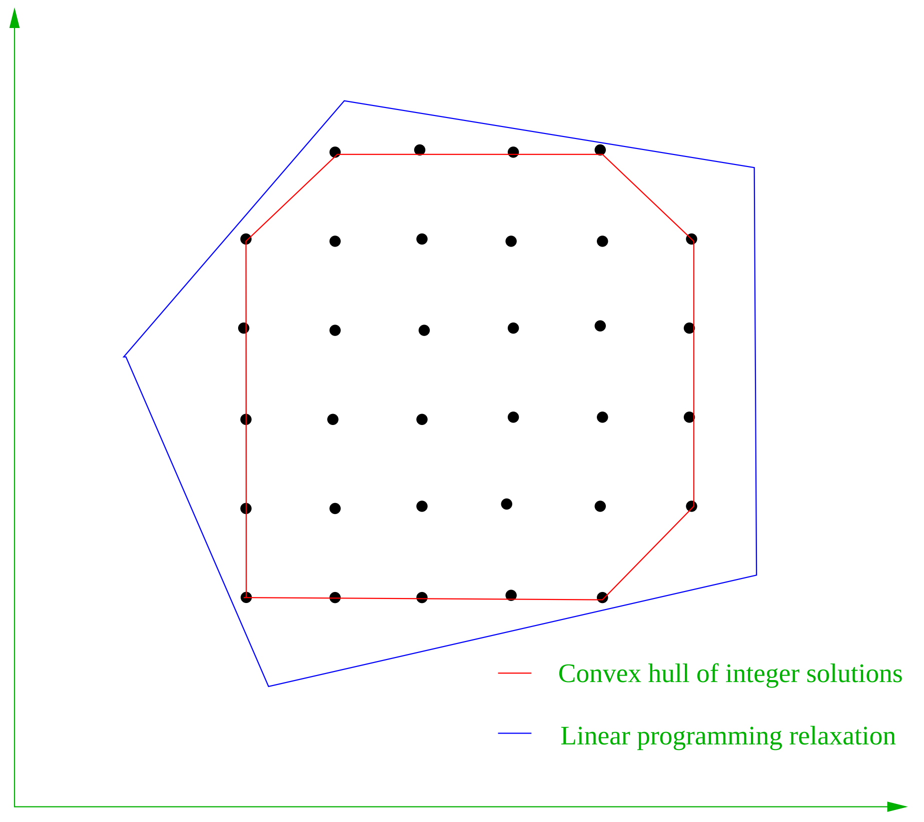
- *Implicit enumeration* techniques try to enumerate the solution space in an intelligent way.
- The most common algorithm of this type is *branch and bound*.
- Suppose F is the set of feasible solutions for some MILP and we wish to solve $\min_{x \in F} c^T x$.
- Consider a partition of F into subsets $F_1, \ldots F_k$. Then

$$\min_{x \in F} c^T x = \min_{1 \le i \le k} \{\min_{x \in F_i} c^T x\}$$

.

- Idea: If we can't solve the original problem directly, we might be able to solve the smaller *subproblems* recursively.
- Dividing the original problem into subproblems is called *branching*.
- Taken to the extreme, this scheme is equivalent to complete enumeration.
- We avoid complete enumeration primarily by deriving bounds on the value of an optimal solution to each subproblem.

The Geometry of Integer Programming



Bounding

- A *relaxation* of an ILP is an auxiliary mathematical program for which
  - the feasible region contains the feasible region for the original ILP, and
  - the objective function value of each solution to the original ILP is not increased.
- Types of Relaxations
  - Continuous relaxations
    * Most common continuous relaxation is the *LP relaxation*.
    * Obtained by dropping some or all of the integrality constraints.
    * Easy to solve.
    * Initial bounds weak, but can be strengthened with valid inequalities.
    * Other relaxations are possible using semi-definite programming, for instance.
  - Combinatorial relaxations
    * Obtained by dropping some of the linear constraints.
    * Violation of these constraints can then penalized in the objective function (*Lagrangian relaxation*)
    * Bound strength depends on what constraints are dropped.

Branch and Bound Algorithm

- We maintain a queue of *active* subproblems initially containing just the *root subproblem*.
- We choose a subproblem from the queue and solve a relaxation of it to obtain a *bound*.
- The result is one of the following:

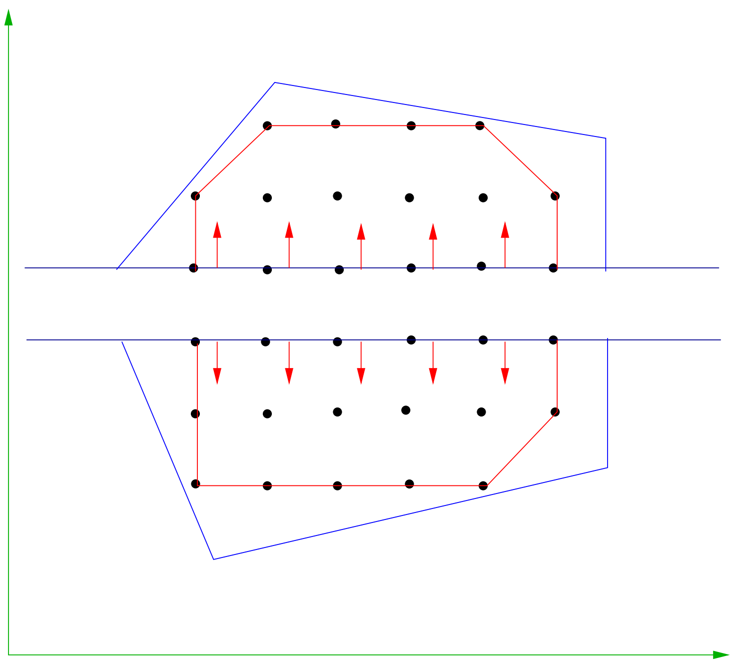
1. The relaxation is infeasible $\Rightarrow$ subproblem is infeasible.
2. We obtain a feasible solution for the MILP $\Rightarrow$ subproblem solved (new upper bound??).
3. We obtain an optimal solution to the relaxation that is not feasible for the MILP $\Rightarrow$ lower bound.

- In the first two cases, we are finished.
- In the third case, we compare the lower bound to the global upper bound.
  - If it exceeds the upper bound, we discard the subproblem.
  - If not, we branch and add the resulting subproblems to the queue.

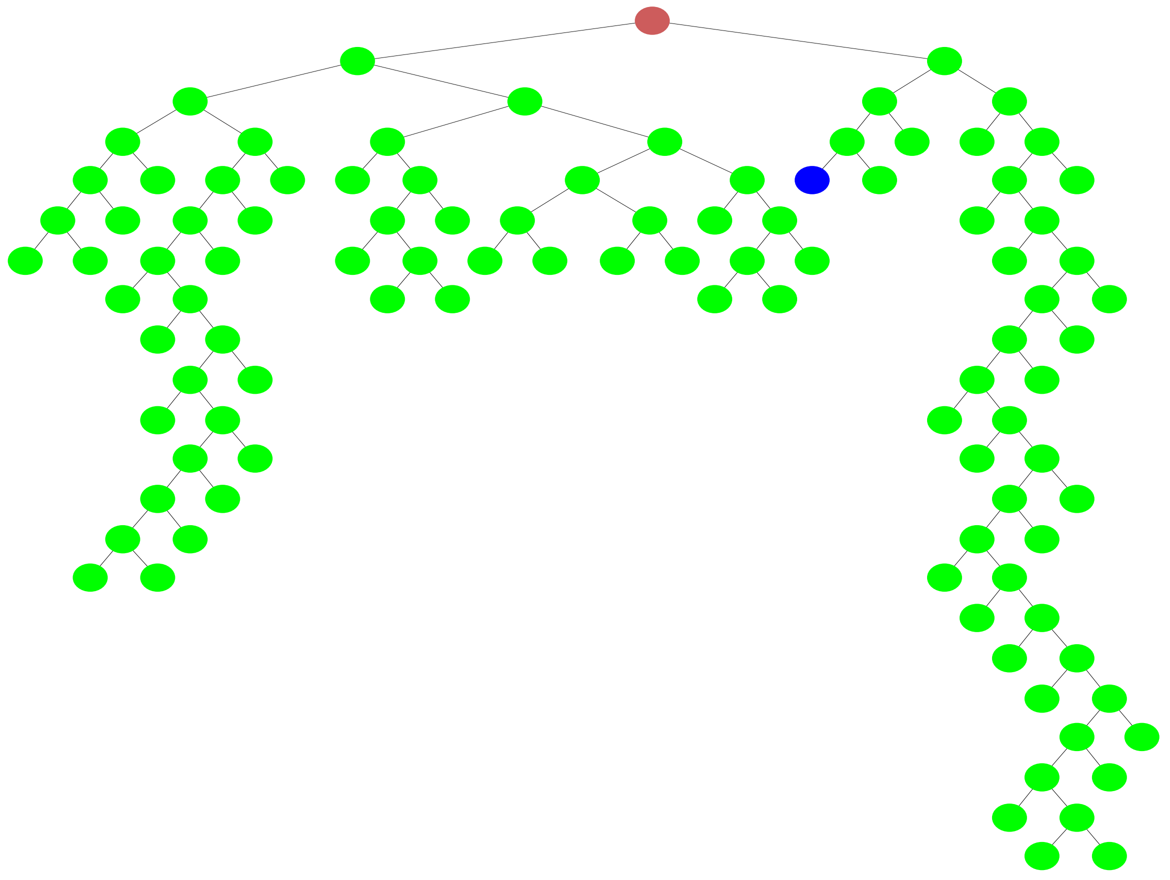
Branching

Branching involves partitioning the feasible region with hyperplanes such that:

- All optimal solutions are in one of the members of the partition.
- The solution to the current relaxation is not in any of the members of the partition.



Branch and Bound Tree



The Vehicle Routing Problem

The VRP is a combinatorial problem whose *ground set* is the edges of a graph $G(V, E)$. Notation:

- V is the set of customers and the depot (0).
- d is a vector of the customer demands.
- k is the number of routes.
- C is the capacity of a truck.

A feasible solution is composed of:

- a partition $\{R_1, \ldots, R_k\}$ of V such that $\sum_{j \in R_i} d_j \leq C,\ 1 \leq i \leq k;$
- a permutation σ_i of $R_i \cup \{0\}$ specifying the order of the customers on route i.

Classical Formulation for the VRP

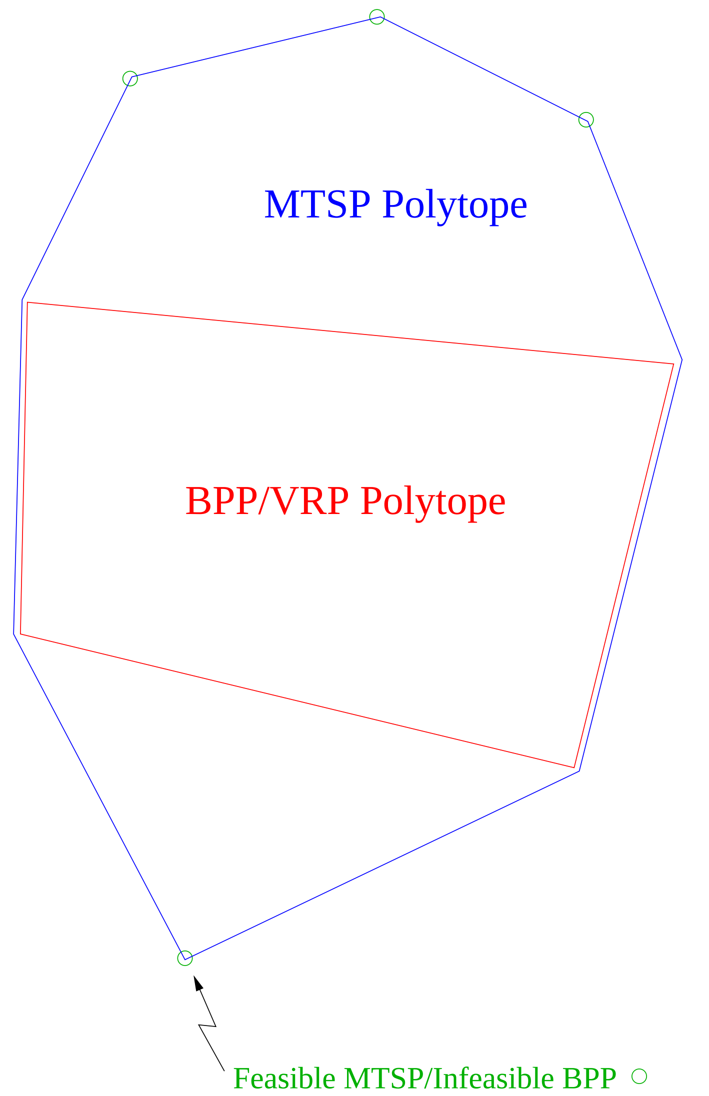
IP Formulation:

$$\begin{array}{rcll} \sum_{j=1}^{n} x_{0j} & = & 2k & \\ \sum_{j=1}^{n} x_{ij} & = & 2 & \forall i \in V \setminus \{0\} \\ \sum_{\substack{i \in S \\ j \notin S}} x_{ij} & \geq & 2b(S) & \forall S \subset V \setminus \{0\},\ |S| > 1. \end{array}$$

$b(S)$ = lower bound on the number of trucks required to service S (normally $\lceil (\sum_{i \in S} d_i)/C \rceil$).

If $C = \sum_{i \in S} d_i$, then we have the Multiple Traveling Salesman Problem.

Alternatively, if the edge costs are all zero, then we have the Bin Packing Problem.


MTSP Polytope
BPP/VRP Polytope
Feasible MTSP/Infeasible BPP

How Hard is the VRP?

- Test Set
  - TSPLIB/VRPLIB
  - Augerat's repository
  - Available at `BranchAndCut.org/VRP`
- Largest VRP instance solved: F-n135-k7
- Largest TSP instance solved: d15112
- Smallest VRP instance unsolved (as of '01): B-n50-k8
- Time to solve B-n50-k8 as an MTSP: .1 sec
- Why the gap?

Standard Approach

- Standard approaches treat the VRP in much the same way as the TSP.
  - Most known valid inequalities are generalizations from the TSP.
  - Branching rules are also generalizations from the TSP.
- However, the TSP does not seem to be the right template.
- It is the packing, not the routing that makes the problem difficult.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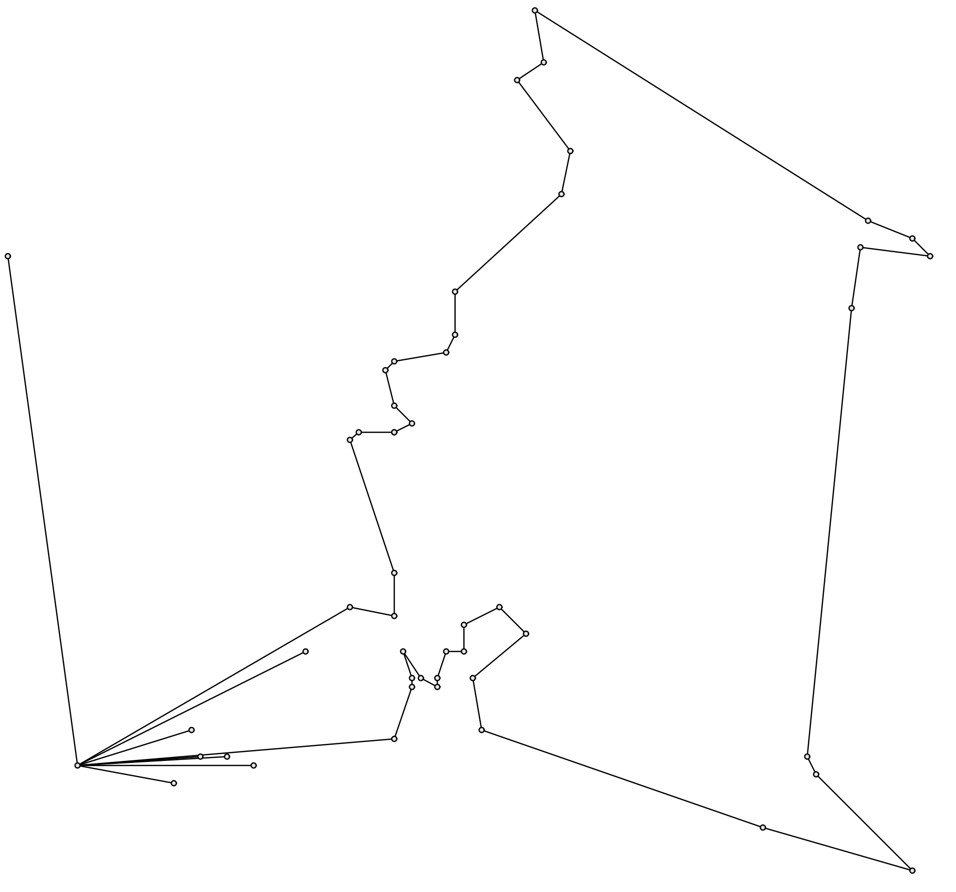

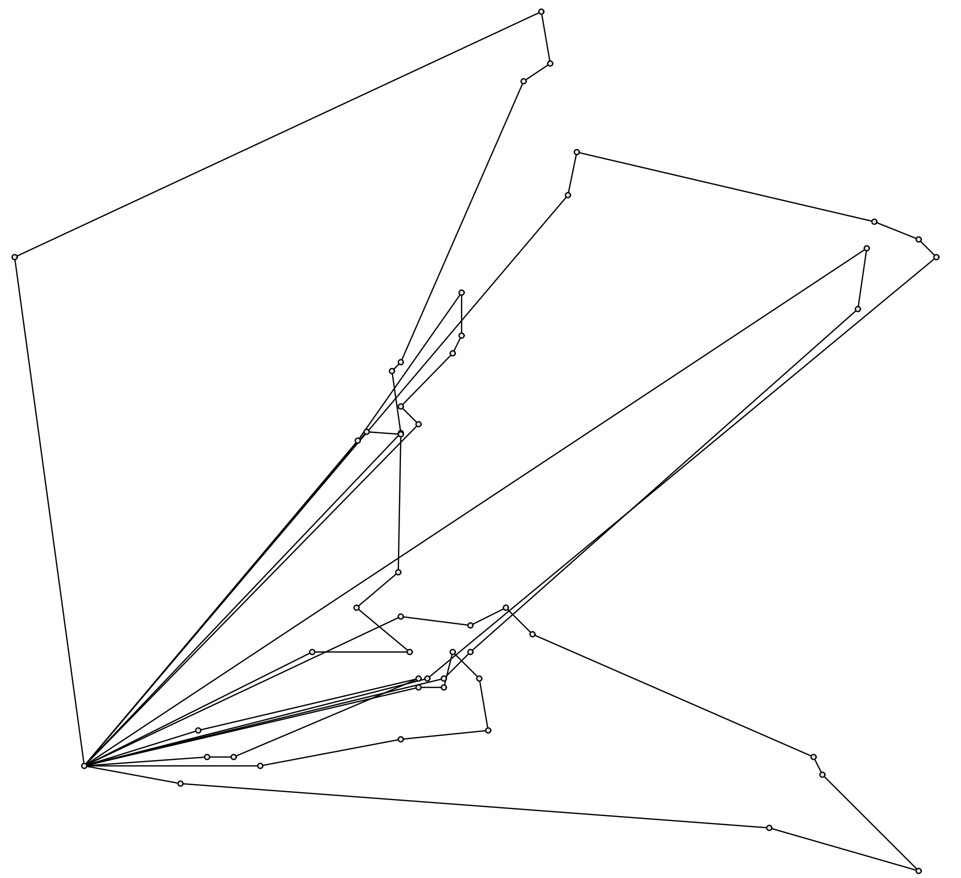

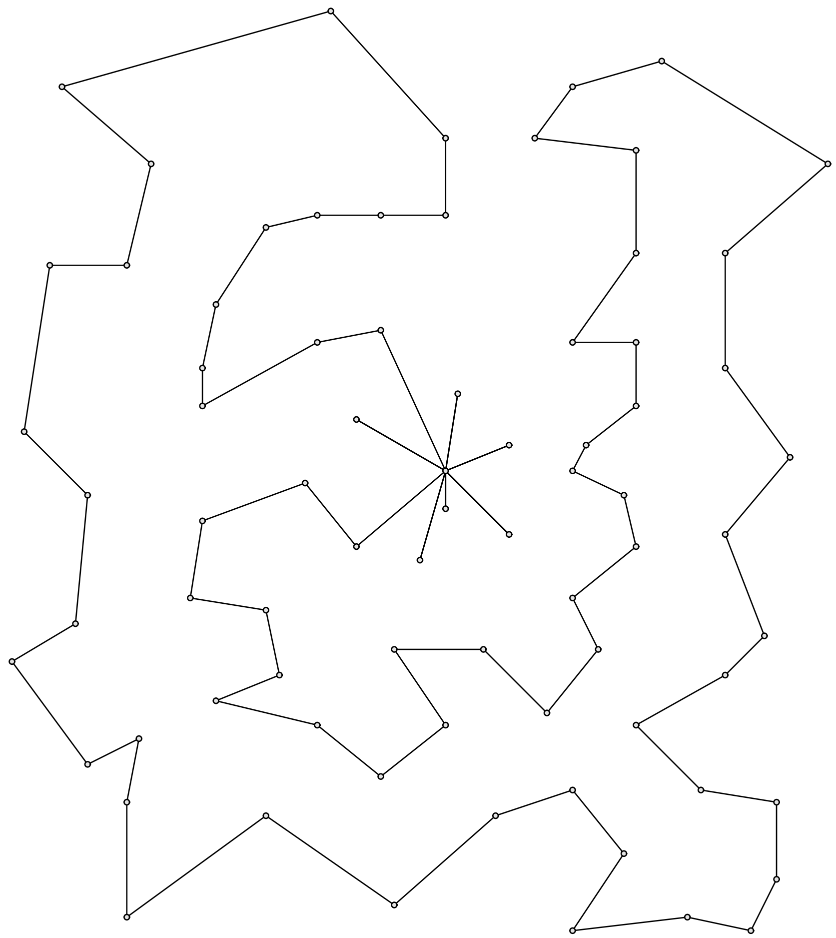

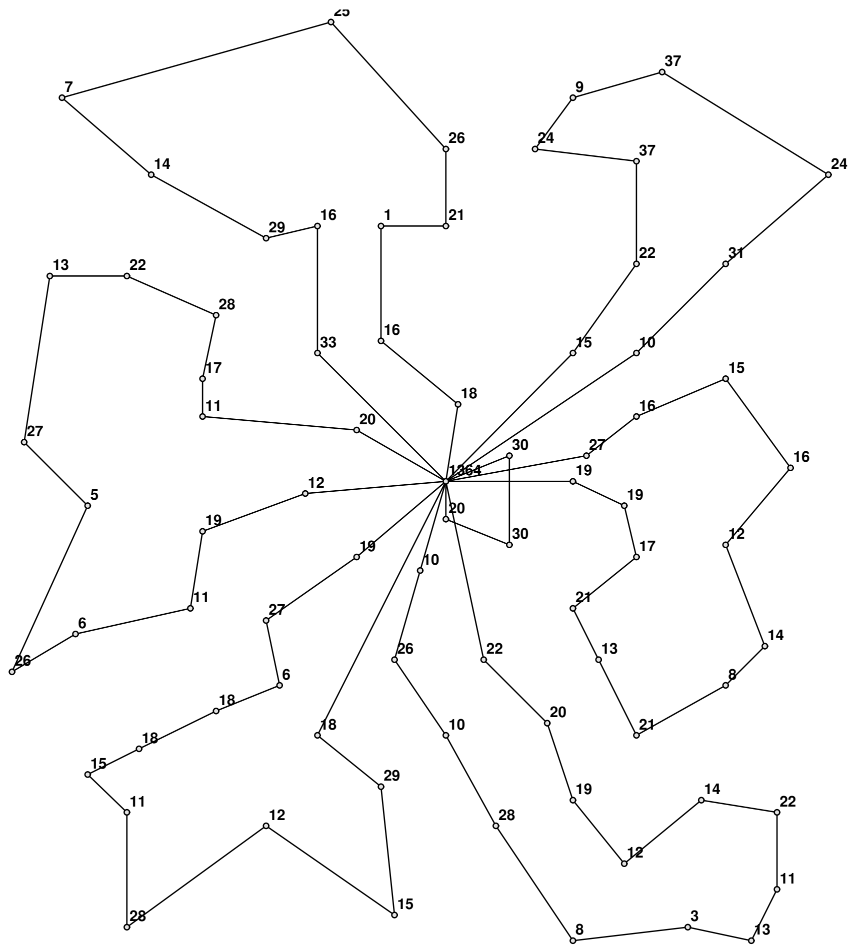
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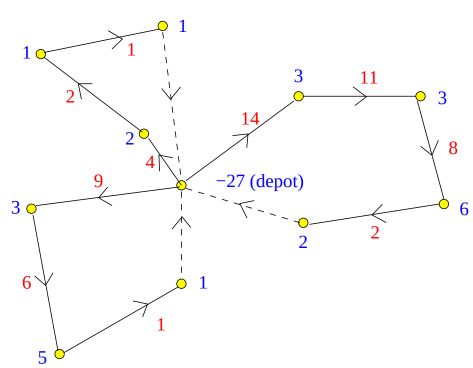
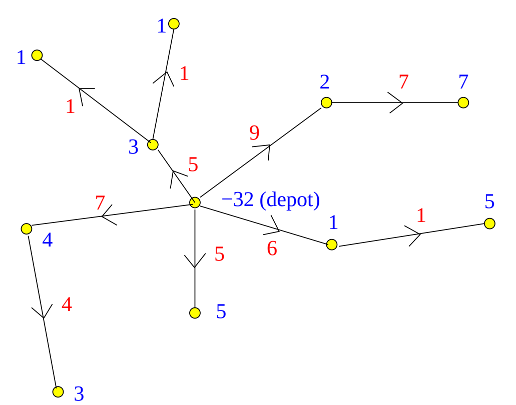
What Makes the VRP Difficult?

- It is the intersection of two difficult problems.
  - Traveling Salesman Problem (Routing)
  - Bin Packing Problem (Packing)
- We don't have an effective, polynomially sized relaxation.
- Current approaches treat it as a routing problem.
- We know very little about the packing aspect.
- We need a different template.
- Idea: Consider flow-based formulations.



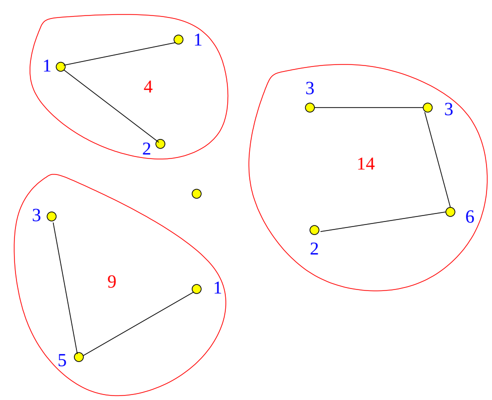
Node Routing

- We are given an undirected graph $G = (V, E)$.
  - The nodes represent supply/demand points.
- We consider problems with one supply point (the *depot*).
- A *node routing* is a directed subgraph G' of G satisfying the following properties:
  - G' is (weakly) connected.
  - The in-degree of each non-depot node is 1.



Capacitated Node Routing

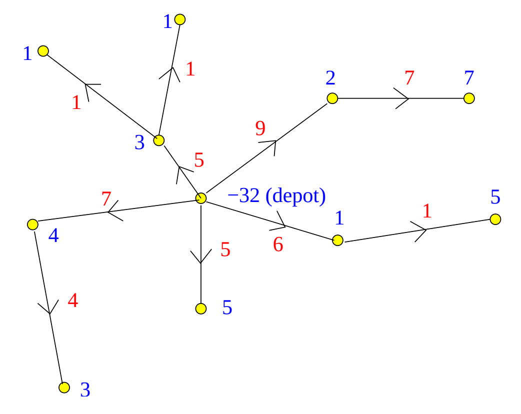
- A *capacitated node routing* is one in which the demand in each component of $G' \setminus \{0\}$ is $\leq C$.
- Feasible solutions are bin packings.
- This restriction is easily modeled using a flow-based formulation.
- With capacities, we can model the VRP and the Capacitated Spanning Tree Problem (CSTP).



Optimal Node Routing

- Properties of a node routing.
  - It is a spanning arborescence plus (possibly) some edges returning to the depot.
  - There is a unique path from the depot to each demand point.
- We wish to construct a least cost routing.

- Cost Measures
  - Lengths of all edges in G'.
  - Length of all paths from the depot.
  - Linear combination of these two.



IP Formulation

IP formulation for this routing problem:

$$
\begin{aligned}
Min \quad & \sum_{(i,j)\in A} \gamma\, c_{ij}x_{ij} + \tau\, c_{ij}f_{ij} & \\
s.t. \quad & x(\delta(V \setminus \{i\})) = 1 & \forall i \in V \setminus \{0\} \\
& f(\delta(V \setminus \{i\})) - f(\delta(\{i\})) = d_i & \forall i \in V \setminus \{0\} \\
& 0 \leq f_{ij} \leq C x_{ij} & \forall (i,j) \in A \\
& x_{ij} \in \{0,1\} & \forall (i,j) \in A
\end{aligned}
$$

where:

- x_{ij}, x_{ji} (*fixed-charge variables*) indicate whether $\{i, j\}$ is in the routing *and its orientation*.
- f_{ij} (*flow variable*) represents demand flow from i to j.

Complexity

- This node routing problem is NP-hard even in the uncapacitated case (fixed-charge network flow problem).
- Polynomially solvable special cases.
  - $\tau = 0 \Rightarrow$ Minimum Spanning Tree Problem.
  - $\gamma = 0 \Rightarrow$ Shortest Paths Tree Problem.
  - Note that demands are irrelevant.
- Other special cases.
  - $\tau = 0 \Rightarrow$ Capacitated Spanning Tree Problem.
  - $\tau, \gamma > 0 \Rightarrow$ Cable-Trench Problem.
  - $\tau = 0$ and $x(\delta(\{i\})) = 1 \Rightarrow$ Traveling Salesman Problem.
  - $\tau > 0$ and $x(\delta(\{i\})) = 1 \Rightarrow$ Variable Cost TSP.
  - $x(\delta(V \setminus \{0\})) = x(\delta(\{0\})) = k \Rightarrow$VRP.

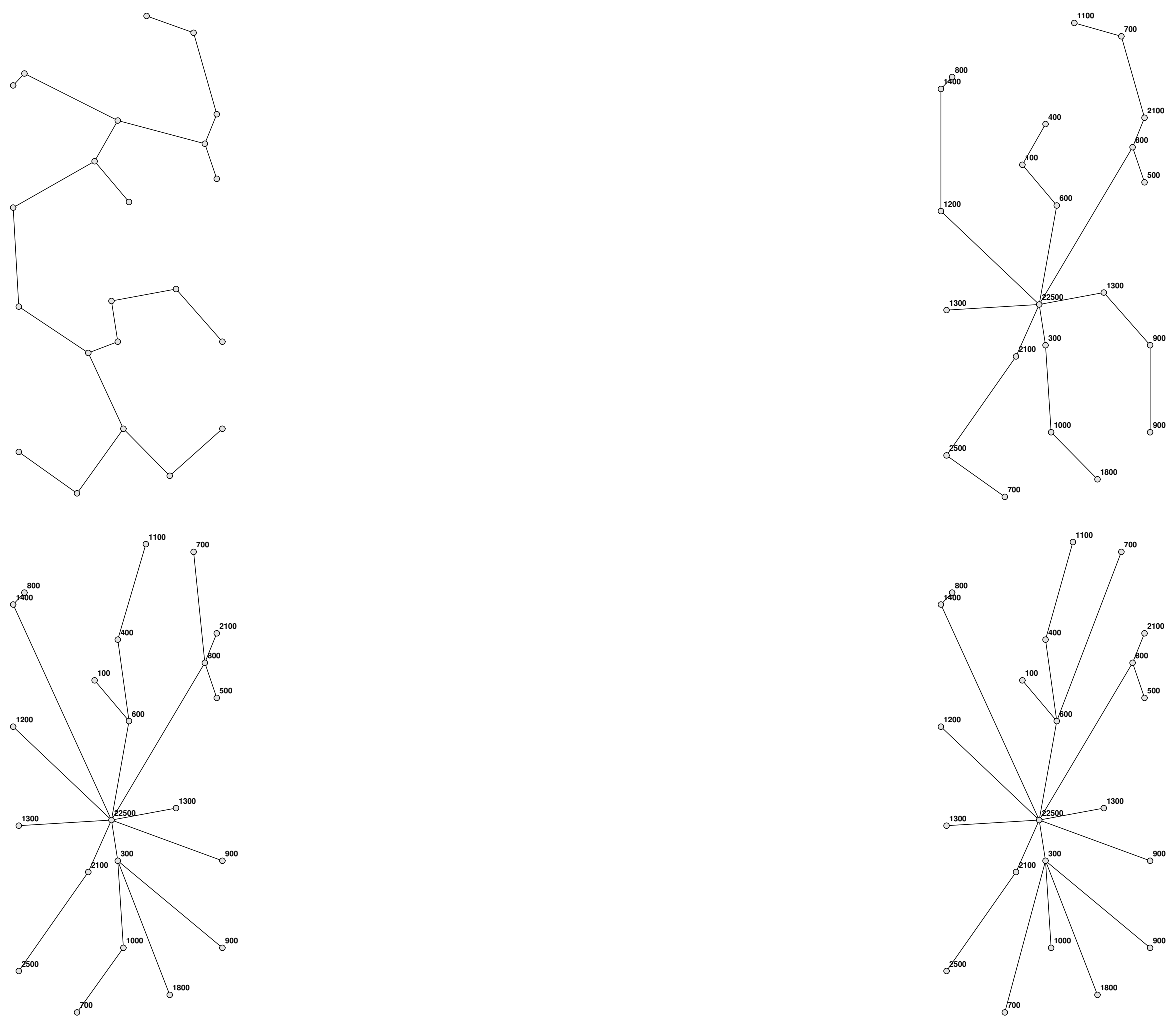


Figure 1: Optimal uncpacitated spanning trees with increasing τ/γ ratios

Figure 2: Uncapacitated vs. capacitated spanning trees ($\tau = 0$)

Connection to Other Models

- There are connections to many well-studied models that may provide better templates.
- The basic model can be seen as an instance of the Fixed-charge Network Flow Problem.
- Removing the upper bounds on the fixed-charge variables yields the Capacitated Network Design Problem.
- We have already mentioned several other related combinatorial models.
- We are looking to make stronger connections among these varied areas of the literature.

Valid Inequalities

- Note that any inequalities valid for the TSP, VRP, or CSTP have counterparts here.
- Many can be strengthened by taking advantage of the directed formulation.
- Fractional Capacity Constraints

$$\sum_{i \notin S,\ j \in S} x_{ij} \geq d(S)/C,\ 0 \notin S$$

- Multi-star Inequalities

$$\sum_{i \notin S,\ j \in S} x_{ij} \geq d(S)/C + \frac{\sum_{i \notin S,\ j \in S} x_{ji} d_i}{C},\ 0 \notin S$$

Valid Inequalities

- Rounded Capacity Constraints

$$\sum_{i \notin S,\ j \in S} x_{ij} \geq \lceil d(S)/C \rceil$$

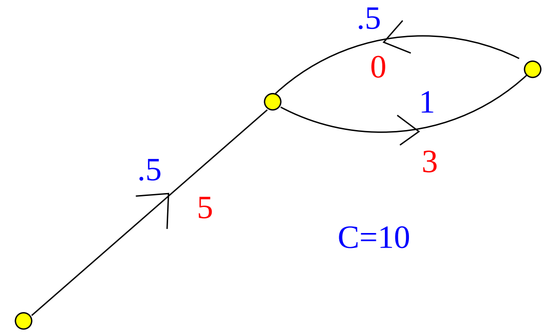
- Generalized, framed capacity constraints
- Combs, Hypo-tours, Clique Clusters
- Path-bin inequalities

Flow Linking

- Note that only the edge variables are required to be integral.
- We use the flow variables to force integrality of the edge variables through *flow linking constraints*.
  - Flow Linking Constraints

$$f_{ij} \leq (C - d_i)x_{ij} \Leftrightarrow x_{ij} \geq \frac{f_{ij}}{C - d_i}$$

$$f_{ij} - \sum_{k \neq j} f_{jk} \leq x_{ij} d_j$$

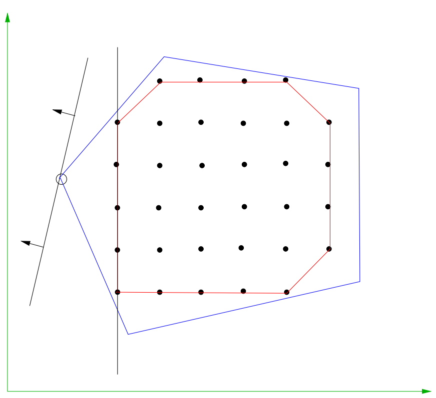


  - Edge Cuts

$$x_{ij} + x_{ji} \leq 1$$

Separation

- The fractional capacity constraints and multi-star inequalities are automatically satisfied.
- Flow linking constraints and edge cuts can be included explicitly or separated in polynomial time.
- Separating rounded capacity constraints is NP-complete, but can be done effectively.
- Other classes must be separated using heuristic procedures.



Solver Implementation

- The implementation uses SYMPHONY, a parallel framework for branch, cut, and price (relative of COIN/BCP).
- In SYMPHONY, the user supplies:
  - the initial LP relaxation,
  - separation subroutines,
  - feasibility checker, and
  - other optional subroutines.
- SYMPHONY handles everything else.
- The source code and documentation are available from `www.BranchAndCut.org`

Computation: Formulation Issues

- The new formulation is polynomial in size and yields stronger relaxations initially, but there are drawbacks.
- For the VRP, the formulation creates symmetry.
- It also seems to make branching less effective.
- There is a related "undirected" formulation which uses one fixed-charge variable per edge.
  - This formulation is smaller and performs much better for the VRP.
  - For the CSTP and CTP, however, the undirected formulation is extremely weak.

Computation: Comparison of Models

- So far, the presence of the flow variables does not seem to help.
- Capacitating the model does increase difficulty significantly.
- Consider relaxations of the VRP.
  - The TSP is very easy relative to the VRP.
  - The CSTP is not much easier than the VRP.
- Versions of these models with positive variable (flow) costs are extremely difficult.
  - Is this due to the upper bound or lower bound?
  - The flow linking constraints are important for these models.

	TSP		*CSTP*		*VRP*	
problem	Tree Size	CPU sec	Tree Size	CPU sec	Tree Size	CPU sec
eil13	1	0.00	13	0.09	1	0.00
eil22	1	0.11	2	0.10	1	0.02
eil33	1	0.02	69	3.97	2	0.44
bayg29	1	0.12	1	0.04	4	0.32
bays29	1	0.17	15	1.12	5	0.55
ulysses16.tsp	1	0.00	1	0.03	1	0.01
ulysses22.tsp	1	0.00	1	0.06	1	0.03
gr17	1	0.01	5	0.05	1	0.01
gr21	1	0.00	1	0.02	1	0.03
gr24	1	0.02	5	0.27	4	0.40
fri26	1	0.02	1	0.07	8	0.39
swiss42	1	0.02	35	3.66	10	2.45
att48	2	0.30	92	5.04	193	30.10
gr48	2	1.38	1	0.07	16	4.17
hk48	1	0.19	209	22.88	45	21.19
eil51	1	0.16	77	15.11	11	10.79
A − n32 − k5	1	0.02	1	0.07	2	0.20
A − n33 − k5	3	0.81	3	0.21	7	0.90
A − n34 − k5	6	2.06	4	0.40	9	2.63
A − n36 − k5	1	0.03	52	5.17	51	7.95
A − n37 − k5	1	0.03	5	0.22	11	0.97
A − n38 − k5	1	0.10	1	0.13	111	21.80
A − n39 − k5	1	0.30	11	0.99	480	310.92
A − n44 − k6	3	1.72	586	84.08	1185	1525.78
A − n45 − k6	2	0.27	47	6.19	133	145.59
A − n46 − k7	1	1.25	3	0.20	2	1.95
A − n48 − k7	2	2.01	775	507.41	1949	1620.57
A − n53 − k7	1	0.62	115	19.99	619	881.05
B − n31 − k5	1	0.01	3	0.63	1	0.08
B − n38 − k6	1	0.04	5	0.56	14	1.73
B − n39 − k5	1	0.03	188	9.67	1	0.05
B − n41 − k6	1	0.08	216	18.96	20	2.89
B − n43 − k6	1	0.09	1	0.36	138	34.92
B − n45 − k5	1	0.09	22	1.13	18	5.81
B − n51 − k7	1	0.36	1	0.13	129	32.48
B − n52 − k7	1	0.19	1	0.20	26	0.76
B − n56 − k7	1	0.10	38	2.20	1	0.29
Total	50	12.73	2606	711.48	5211	4670.22

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Figure 3: Preliminary Computational Results

Conclusions and Future Directions

- We have established interesting connections to other well-studied models.
- The TSP does not seem to be the right template to follow.
- We have yet to take full advantage of the information provided by the flow variables.
- Better flow linking seems to be the key.
- We also need some new branching rules.
- The connection to the network design literature needs to be explored.
- We are also considering decomposition-based methods.