In search of optimal disjunctions in Mixed Integer Linear Programming

Complexity and Computational Experiments

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Department of Industrial and Systems Engineering
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MCS-ANL, 2008

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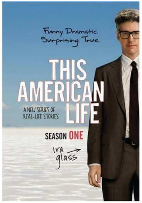

NEOS
Server for Optimization

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00:10 on point.ie.lehigh.edu
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Thank you for having me in Chicago

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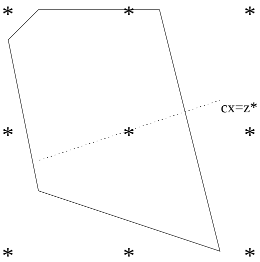
COR@L
COMPUTATIONAL OPTIMIZATION RESEARCH AT LEHIGH

Solving MILPs with Branch and Bound

$$\text{MILP: } \min_{x \in S} cx$$

$$S: \begin{array}{rcl} Ax & \geq & b \\ x & \in & \mathbb{Z}^d \times \mathbb{R}^{n-d}, \end{array}$$



Solving MILPs with Branch and Bound

$$\text{MILP: } \min_{x \in S} cx \qquad \text{LP: } \min_{x \in \mathcal{P}} cx$$

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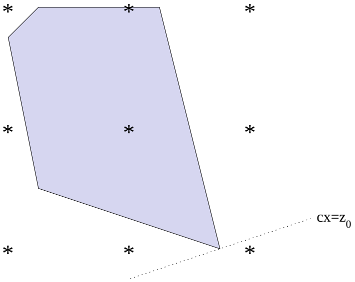

$$\text{LB} = z_0$$

Solving MILPs with Branch and Bound

MILP: $\min_{x \in S} cx$

$$S: \begin{array}{rcl} Ax & \geq & b \\ x & \in & \mathbb{Z}^d \times \mathbb{R}^{n-d}, \end{array}$$

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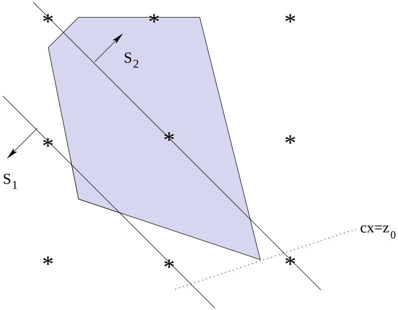

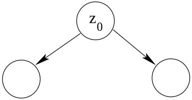

$\text{LB} = z_0,\ x^* \in S_1 \cup S_2$

Solving MILPs with Branch and Bound

MILP: $\min_{x \in S} cx$

LP: $\min_{x \in \mathcal{P}} cx$

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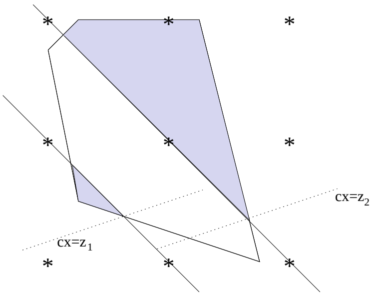

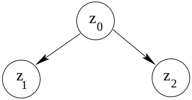

$$\text{LB} = \min(z_1, z_2)$$

Solving MILPs with Branch and Bound

MILP: $\min_{x \in S} cx$

LP: $\min_{x \in \mathcal{P}} cx$

$$S : \begin{array}{rcl} Ax & \geq & b \\ x & \in & \mathbb{Z}^d \times \mathbb{R}^{n-d}, \end{array}$$

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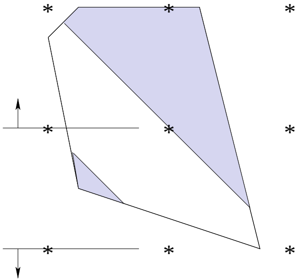

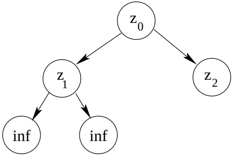

LB = z_2

Solving MILPs with Branch and Bound

MILP: $\min_{x \in S} cx$

LP: $\min_{x \in \mathcal{P}} cx$

$$S: \begin{array}{rcl} Ax & \geq & b \\ x & \in & \mathbb{Z}^d \times \mathbb{R}^{n-d}, \end{array}$$

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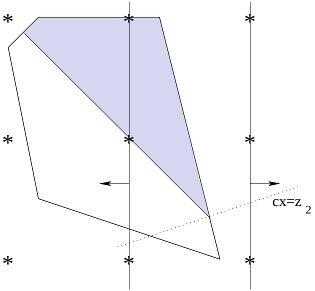

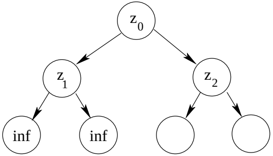

LB = z_2

Solving MILPs with Branch and Bound

MILP: $\min_{x \in S} cx$

LP: $\min_{x \in \mathcal{P}} cx$

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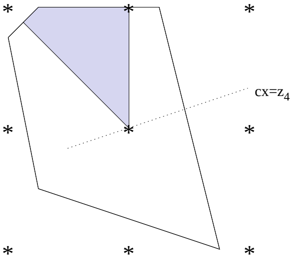
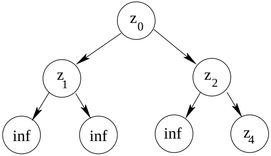
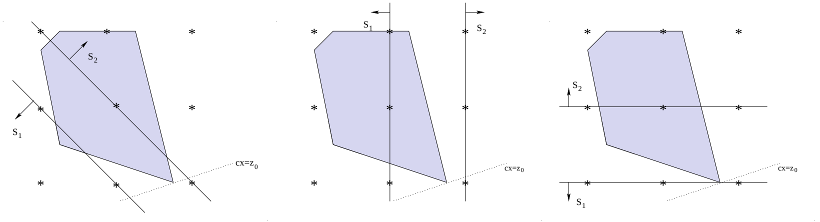




LB = z_4 = UB

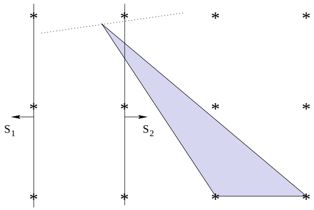
The Branching Decision



- Any subsets S_1, S_2 such that $S \subseteq (\mathcal{P} \cup S_1) \cup (\mathcal{P} \cap S_2)$ could be used to partition.
- We restrict ourselves to partitions s.t. S_1, S_2 are "polyhedral".
- Assume that we only create at most two partitions in each iteration.
- There are possibly infinite ways of creating such partitions.
- A *good* partition is one that can be solved *easily*.
- How should we select such a partition? How should we divide?

The Branching Decision

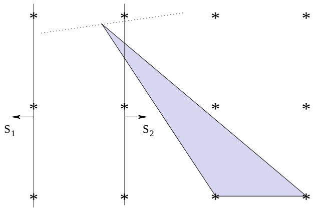
- Most commonly used branching rule is $x_i \leq \pi_0 \vee x_i \geq \pi_0 + 1$ for an $i \in \{1, \ldots, d\}$.
- e.g. $S_1 = \{x | x_1 \leq 0\}, S_2 = \{x | x_1 \geq 1\}$.
- This is called Variable Disjunction. Also denoted as: $x_1 \leq 0 \vee x_1 \geq 1$.



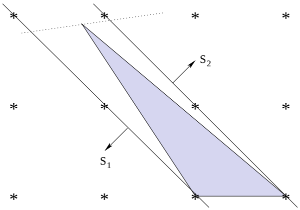
A variable disjunction

The Branching Decision

- Most commonly used branching rule is $x_i \leq \pi_0 \vee x_i \geq \pi_0 + 1$ for an $i \in \{1, \dots, d\}$.
- e.g. $S_1 = \{x | x_1 \leq 0\}, S_2 = \{x | x_1 \geq 1\}$.
- This is called Variable Disjunction. Also denoted as: $x_1 \leq 0 \vee x_1 \geq 1$.
- Disjunctions like $x_1 + x_2 \leq 4 \vee x_1 + x_2 \geq 5$ are also valid.
- A General Disjunction is of the form $\pi x \leq \pi_0 \vee \pi x \geq \pi_0 + 1$, where $(\pi, \pi_0) \in \mathbb{Z}^d \times 0^{n-d} \times \mathbb{Z}$.



A variable disjunction



A general disjunction

Previous Work

Branching on Variables:

- Finding the minimum size tree is $\mathcal{NP}-$Hard. (Liberatore, 2000)
- Heuristics for improving bounds: Strong branching, Pseudo-cost branching, Reliability branching (Experiments by Linderoth and Savelsbergh, 1999, Achterberg et al., 2005)
- Heuristics for feasible solutions (Patel and Chinneck, 2007)
- Heuristics for conflict analysis and resolution (Chvátal, 1997, Achterberg, 2007)
- Heuristics based on thin directions (Derpich and Vera, 2006)

Branching on General Disjunctions:

- Trees polynomial in size, when dimension is fixed (Lenstra 1983, other works by Aardal, Lovasz, Lenstra, Pataki).
- Greedy heuristic by Owen and Mehrotra, 2001.
- Local branching heuristic for feasibility (Fischetti and Lodi, 2003).
- Heuristics based on Gomory Cuts (Karamanov and Cornuéjols, 2007, Cornuéjols et. al., 2008).
- SOS-1 and SOS-2 (Beale, 1970).

Branching on General Disjunctions

$$\text{MILP:} \quad \begin{aligned} z^* &= \min cx \\ \text{s.t. } Ax &\geq b \\ x &\in \mathbb{Z}^n \end{aligned}$$

$$\text{LP:} \quad \begin{aligned} z^*_{LP} &= \min cx \\ \text{s.t. } Ax &\geq b \\ x &\in \mathbb{R}^n \end{aligned}$$

$$\begin{aligned} z^*_L = \min cx & \\ \text{s.t. } Ax &\geq b \\ \pi x &\leq \pi_0 \\ x &\in \mathbb{R}^n \end{aligned} \qquad\qquad \begin{aligned} z^*_R = \min cx & \\ \text{s.t. } Ax &\geq b \\ \pi x &\geq \pi_0 + 1 \\ x &\in \mathbb{R}^n \end{aligned}$$

New lower bound, $z_l = \min\{z^*_L, z^*_R\}$

Objective

Find $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that $z_l = \min\{z^*_L, z^*_R\}$ is maximized

Complexity

Optimization Problem

Find $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that $z_l = \min\{z_L^*, z_R^*\}$ is maximized

- For branching on variables, this can be solved in polynomial-time.
- Is there a similar result when considering general disjunctions?
- When $(\pi, \pi_0) \in \{0, 1\}^{n+1}$?

Decision Problem

Given a MILP and K, does there exist $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that $z_l = \min\{z_L^*, z_R^*\} > K$

Decision Problem (2)

Given a MILP, does there exist $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that LP relaxation of each branch is infeasible.

Complexity Results

Decision Problem (2)

Given a MILP, does there exist $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that LP relaxation of each branch is infeasible.

Above problem is:

- $\mathcal{NP}-$complete in general (Reduction from Number Partitioning Problem)
- $\mathcal{NP}-$complete when $\pi \in \{0, 1\}^n$
- $\mathcal{NP}-$complete when $\pi \in \{-1, 0, 1\}^n$
- $\mathcal{NP}-$complete when $(\pi, \pi_0) \in \{0, 1\}^{n+1}$
- $\mathcal{NP}-$complete when $x \in \{0, 1\}^n$ and either of above four conditions hold (Reduction from 1-IN-3SAT)

Optimization Problem

Find $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that $z_l = \min\{z_L^*, z_R^*\}$ is maximized.
$\mathcal{NP}-$hard

Problem formulation: Maximum bound improvement

$$\begin{aligned} z_L^* = \min\ & cx \\ \text{s.t. } Ax &\geq b \\ \pi x &\leq \pi_0 \\ x &\in \mathbb{R}^n \end{aligned} \qquad \begin{aligned} z_R^* = \min\ & cx \\ \text{s.t. } Ax &\geq b \\ \pi x &\geq \pi_0 + 1 \\ x &\in \mathbb{R}^n \end{aligned}$$

Decision Problem

Given a MILP and K, does there exist $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that $z_l = \min\{z_L^*, z_R^*\} > K$?

Suppose $(\hat{\pi}, \hat{\pi}_0)$ is one such disjunction. Then the following systems must be infeasible

$$\begin{aligned} Ax &\geq b \\ cx &\leq K \\ \hat{\pi} x &\leq \hat{\pi}_0 \\ x &\in \mathbb{R}^n \end{aligned} \qquad \begin{aligned} Ax &\geq b \\ cx &\leq K \\ \hat{\pi} x &\geq \hat{\pi}_0 + 1 \\ x &\in \mathbb{R}^n \end{aligned}$$

Problem formulation: Maximum bound improvement

$$
\begin{aligned}
z_L^* = \min\, & cx \\
\text{s.t. } Ax & \geq b \\
\pi x & \leq \pi_0 \\
x & \in \mathbb{R}^n
\end{aligned}
\qquad\qquad
\begin{aligned}
z_R^* = \min\, & cx \\
\text{s.t. } Ax & \geq b \\
\pi x & \geq \pi_0 + 1 \\
x & \in \mathbb{R}^n
\end{aligned}
$$

Decision Problem

Given a MILP and K, does there exist $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that $z_l = \min\{z_L^*, z_R^*\} > K$?

Suppose $(\hat{\pi}, \hat{\pi}_0)$ is one such disjunction. Then the following systems must be infeasible

$$
\begin{aligned}
Ax & \geq b \\
cx & \leq K \\
\hat{\pi} x & \leq \hat{\pi}_0 \\
x & \in \mathbb{R}^n
\end{aligned}
\qquad\qquad
\begin{aligned}
Ax & \geq b \\
cx & \leq K \\
\hat{\pi} x & \geq \hat{\pi}_0 + 1 \\
x & \in \mathbb{R}^n
\end{aligned}
$$

How to deal with bilinear terms $\hat{\pi} x$?

Problem formulation: Maximum bound improvement

If $(\hat{\pi}, \hat{\pi}_0)$ is the required disjunction,

$$\begin{array}{rcl} Ax & \geq & b \\ cx & \leq & K \\ \hat{\pi}x & \leq & \hat{\pi}_0 \\ x & \in & \mathbb{R}^n \end{array} \quad \& \quad \begin{array}{rcl} Ax & \geq & b \\ cx & \leq & K \\ \hat{\pi}x & \geq & \hat{\pi}_0 + 1 \\ x & \in & \mathbb{R}^n \end{array}$$

must be infeasible

Problem formulation: Maximum bound improvement

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must be infeasible

$$\begin{array}{rcl} z_1^* & = & \min \hat{\pi}x \\ Ax & \geq & b \\ cx & \leq & K \\ x & \in & \mathbb{R}^n \end{array} \quad \& \quad \begin{array}{rcl} z_2^* & = & \min -\hat{\pi}x \\ Ax & \geq & b \\ cx & \leq & K \\ x & \in & \mathbb{R}^n \end{array}$$

must have $z_1^* > \hat{\pi}_0$, $z_2^* > -\hat{\pi}_0 - 1$

Problem formulation: Maximum bound improvement

If $(\hat{\pi}, \hat{\pi}_0)$ is the required disjunction,

$$\begin{aligned} Ax &\geq b \\ cx &\leq K \\ \hat{\pi}x &\leq \hat{\pi}_0 \\ x &\in \mathbb{R}^n \end{aligned} \quad \& \quad \begin{aligned} Ax &\geq b \\ cx &\leq K \\ \hat{\pi}x &\geq \hat{\pi}_0 + 1 \\ x &\in \mathbb{R}^n \end{aligned}$$

must be infeasible

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must have $z_1^* > \hat{\pi}_0$, $z_2^* > -\hat{\pi}_0 - 1$

$$\begin{aligned} z_1^* = \max pb &- s_L K \\ pA - s_L c &= -\hat{\pi} \\ p, s_L &\geq 0 \end{aligned} \quad \& \quad \begin{aligned} z_2^* = \max qb &- s_R K \\ qA - s_R c &= -\hat{\pi} \\ q, s_R &\geq 0 \end{aligned}$$

(as above)

Problem formulation: Maximum bound improvement

If $(\hat{\pi}, \hat{\pi}_0)$ is the required disjunction,

$$\begin{array}{rcl} z_1^* & = & \min \hat{\pi} x \\ Ax & \geq & b \\ cx & \leq & K \\ x & \in & \mathbb{R}^n \end{array} \quad \& \quad \begin{array}{rcl} z_2^* & = & \min -\hat{\pi} x \\ Ax & \geq & b \\ cx & \leq & K \\ x & \in & \mathbb{R}^n \end{array}$$

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(as above)

$$\begin{array}{rcl} pb - s_L K & > & \hat{\pi}_0 \\ pA - s_L c & = & -\hat{\pi} \\ p, s_L & \geq & 0 \end{array} \quad \& \quad \begin{array}{rcl} qb - s_R K & > & -\hat{\pi}_0 - 1 \\ qA - s_R c & = & -\hat{\pi} \\ q, s_R & \geq & 0 \end{array}$$

must be feasible

Problem formulation: Maximum bound improvement

If $(\hat{\pi}, \hat{\pi}_0)$ is the required disjunction,

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$$\begin{array}{rcl} pb - s_L K & > & \hat{\pi}_0 \\ pA - s_L c & = & -\hat{\pi} \\ p, s_L & \geq & 0 \end{array} \quad \& \quad \begin{array}{rcl} qb - s_R K & > & -\hat{\pi}_0 - 1 \\ qA - s_R c & = & -\hat{\pi} \\ q, s_R & \geq & 0 \end{array} \quad \begin{array}{l} \text{must be} \\ \text{feasible} \end{array}$$

$$\begin{array}{rcl} pb - s_L K & > & \hat{\pi}_0 \\ pA - s_L c & = & -\hat{\pi} \\ qb - s_R K & > & -\hat{\pi}_0 - 1 \\ qA - s_R c & = & -\hat{\pi} \\ q, s_R, p, s_L & \geq & 0 \end{array}$$

Problem formulation: Maximum bound improvement

If $(\hat{\pi}, \hat{\pi}_0)$ is the required disjunction,

$$\begin{array}{rcl} z_1^* = \max pb & - & s_L K \\ pA - s_L c & = & -\hat{\pi} \\ p, s_L & \geq & 0 \end{array} \quad \& \quad \begin{array}{rcl} z_2^* = \max qb & - & s_R K \\ qA - s_R c & = & -\hat{\pi} \\ q, s_R & \geq & 0 \end{array} \quad \text{(as above)}$$

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$$\begin{array}{rcl} pb - s_L K & > & \pi_0 \\ pA - s_L c & = & -\pi \\ qb - s_R K & > & -\pi_0 - 1 \\ qA - s_R c & = & -\pi \\ q, s_R, p, s_L & \geq & 0 \\ (\pi, \pi_0) & \in & \mathbb{Z}^n \end{array}$$

Problem formulation: Maximum bound improvement

If $(\hat{\pi}, \hat{\pi}_0)$ is the required disjunction,

$$\begin{array}{rcl} Ax & \geq & b \\ cx & \leq & K \\ \hat{\pi}x & \leq & \hat{\pi}_0 \\ x & \in & \mathbb{R}^n \end{array} \quad \& \quad \begin{array}{rcl} Ax & \geq & b \\ cx & \leq & K \\ \hat{\pi}x & \geq & \hat{\pi}_0 + 1 \\ x & \in & \mathbb{R}^n \end{array}$$

must be infeasible

$$\begin{array}{rcl} pb - s_L K & > & \hat{\pi}_0 \\ pA - s_L c & = & -\hat{\pi} \\ qb - s_R K & > & -\hat{\pi}_0 - 1 \\ qA - s_R c & = & -\hat{\pi} \\ q, s_R, p, s_L & \geq & 0 \end{array}$$

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$(\hat{\pi}, \hat{\pi}_0)$ is the required disjunction $\iff$ it satisfies the last MILP.

Problem formulation: Maximum bound improvement

$$
\begin{aligned}
pb - s_L K &> \pi_0 \\
pA - s_L c &= -\pi \\
qb - s_R K &> -\pi_0 - 1 \\
qA - s_R c &= -\pi \\
q, s_R, p, s_L &\geq 0 \\
(\pi, \pi_0) &\in \mathbb{Z}^n
\end{aligned}
$$

- $2n + 2$ constraints, $2m + n + 3$ variables.
- Tells us if we can increase the bound to K by branching at the current node.
- Does not tell us the maximum such value of K.
- We do a binary search over a range of K and solve the above MILP in each iteration.

Problem formulation: Thin Directions

Similar approach has been used before:

- Cut generating LPs (Balas)
- Separation of a given point from the the *Split Closure* (Caprara, 2003, Balas 2007)
- Finding thinnest directions?

Problem formulation: Thin Directions

Similar approach has been used before:

- Cut generating LPs (Balas)
- Separation of a given point from the the *Split Closure* (Caprara, 2003, Balas 2007)
- Finding thinnest directions?

Thin Directions

- Thickness of the LP relaxation polytope $\mathcal{P}$ along a direction $\hat{\pi}$ is:

$$\max \hat{\pi}y - \hat{\pi}x, \quad x, y \in \mathcal{P}$$

- The width of $\mathcal{P}$, $w(\mathcal{P})$, is minimum such width over all $\hat{\pi}$:

$$w(Q) = \min_{\pi} \max_{x,y \in \mathcal{P}} (\pi y - \pi x) \;\; s.t. \;\; \pi \in \mathbb{Z}^d \times \{0\}^{n-d}, \pi \neq \mathbf{0}.$$

Problem formulation: Thin Directions

$$w(Q) = \min_{\pi} \max_{x,y \in \mathcal{P}} (\pi y - \pi x) \quad s.t. \quad \pi \in \mathbb{Z}^d \times \{0\}^{n-d}, \pi \neq \mathbf{0}.$$

For a fixed $(\hat{\pi}, \hat{\pi}_0)$, the inner LP is:

$$\begin{aligned} \max \hat{\pi} y - \hat{\pi} x \\ \text{s.t. } Ax \geq b \\ Ay \geq b \end{aligned}$$

Dual of the inner LP is:

$$\begin{aligned} \min -qb - pb \\ \text{s.t. } pA - \hat{\pi} = 0 \\ qA + \hat{\pi} = 0 \\ p, q \geq 0 \end{aligned}$$

Over all π:

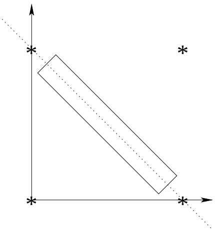
$$\begin{aligned} \min -qb - pb \\ pA - \pi = 0 \\ qA + \pi = 0 \\ p, q \geq 0 \\ \pi \in \mathbb{Z}^n \times \{0\}^{n-d}, \pi \neq \mathbf{0}. \end{aligned}$$

Optional constraint:

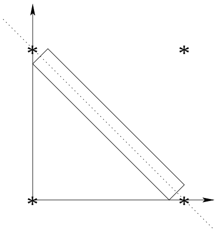
$$\pi_0 < \pi x^* < \pi_0 + 1, \quad \pi_0 \in \mathbb{Z}$$

More on Thin Directions

- Problem of finding the thinnest direction is $NP-$hard. Remains hard even if the given LP polytope is a simplex (Sebö, 1999).
- Branching on thinnest direction does not necessarily result in the smallest tree (Examples by Krishnamoorthy, 2008).
- Branching on the thinnest direction need not make the child nodes infeasible.
- If a general branching disjunction makes the children infeasible then $w(\mathcal{P}) < 1$. Converse is not true.



Branching along thinnest direction gives a feasible LP.



$w(\mathcal{P}) < 1$ but children not infeasible for any disjunction.

Computational Experiments

- Objective: Study the effect of branching on *best* general hyperplanes on the size of branch-and-bound tree.
- Ignore time taken to solve.
- Used ILOG CPLEX-10.2 MILP solver.
- Compared against strong branching (our own callback).
- Simple branch-and-bound (no cuts, preprocessing, heuristics).
- Provided best known upper bound to the solver.
- To perform experiments in reasonable time:
  - Imposed a limit of $-M \leq \pi_i \leq M$ (M is a fixed parameter).
  - Imposed a constraint of $|\sum_i \pi_i| \leq k$ (k is a fixed parameter).
  - Each MILP solved for finding a branching hyperplane had time limit t seconds (t is a fixed parameter).
  - Limit time on branching on a node to $8t$ seconds
  - If more than 18 hours spent on branching, then switch to strong branching on variables.
  - Total time limit of 20 hours for each instance.

Instances

30 Instances chosen from MIPLIB-3, MIPLIB-2003 and Mittlemann's collection.

Instance	Cons	Vars	Ints	Bins
10teams	231	2025	1800	1800
aflow30a	479	842	421	421
bell3a	123	133	71	39
blend2	274	353	264	231
egout	98	141	55	55
fiber	363	1298	1254	1254
flugpl	18	18	11	0
gen	780	870	150	144
gesa2	1392	1224	408	240
gesa2_o	1248	1152	672	336
gt2	29	188	24	0
harp2	112	2993	2993	2993
khb05250	101	1350	24	24
l152lav	97	1989	1989	1989
lseu	28	89	89	89

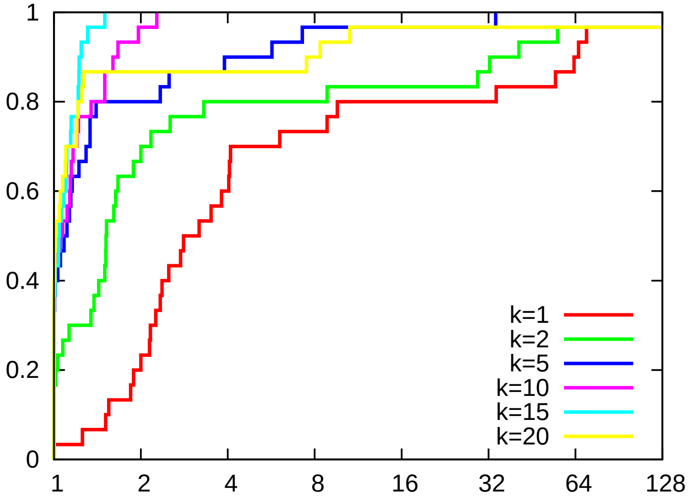
Instance	Cons	Vars	Ints	Bins
mod008	6	319	319	319
neos6	1037	8768	8340	8340
nug08	913	1632	1632	0
nw04	36	87482	87482	87482
p0548	176	548	548	548
pp08aCUTS	246	240	64	64
qnet1	503	1541	1417	1288
qnet1_o	456	1541	1417	1288
ran10x26	297	520	260	260
ran12x21	286	504	502	502
ran13x13	196	338	169	169
rout	291	556	315	300
stein45	331	45	45	45
vpm1	234	378	168	168
vpm2	234	378	168	168

First Experiment

Instance	N_1	N_5	r_5	N_{10}	r_{10}	N_{15}	r_{15}	N_{20}	r_{20}
10teams	115	28	4.11	18	6.39	12	9.58	12	9.58
aflow30a	36634	19485	1.88	20388	1.8	24112	1.52	20271	1.81
bell3a	16387	8771	1.87	588	27.87	259	63.27	259	63.27
blend2	304	231	1.32	188	1.62	165	1.84	209	1.45
egout	2246	554	4.05	572	3.93	676	3.32	558	4.03
fiber	18412	7612	2.42	3039	6.06	3358	5.48	3324	5.54
flugpl	394	6	65.67	10	39.4	6	65.67	6	65.67
gen	100	100	1	100	1	100	1	100	1
gesa2	33526	21664	1.55	21849	1.53	21849	1.53	21778	1.54
gesa2_o	98550	24435	4.03	24661	4	24661	4	24661	4
gt2	340	10	34	12	28.33	10	34	12	28.33
harp2	432010	174656	2.47	183306	2.36	174454	2.48	179130	2.41
khb05250	738	594	1.24	588	1.26	614	1.2	618	1.19
l152lav	60	32	1.88	28	2.14	34	1.76	30	2
lseu	4058	226	17.96	78	52.03	58	69.97	58	69.97
mod008	2840	296	9.59	102	27.84	68	41.76	52	54.62
neos6	5989	2131	2.81	2131	2.81	2131	2.81	2131	2.81
nug08	14	4	3.5	6	2.33	6	2.33	5	2.8
nw04	30	16	1.88	12	2.5	12	2.5	12	2.5
p0548	1050	466	2.25	566	1.86	565	1.86	565	1.86
pp08aCUTS	1301300	147271	8.84	166943	7.79	168905	7.7	231527	5.62
qnet1	42	24	1.75	20	2.1	22	1.91	18	2.33
qnet1_o	154	94	1.64	77	2	80	1.93	92	1.67
ran10x26	68449	23309	2.94	24716	2.77	23704	2.89	21520	3.18
ran12x21	494558	219967	2.25	208948	2.37	225980	2.19	212910	2.32
ran13x13	124716	74699	1.67	57825	2.16	66008	1.89	58789	2.12
rout	219322	65201	3.36	61806	3.55	61226	3.58	57673	3.8
stein45	31086	21238	1.46	20594	1.51	20601	1.51	20601	1.51
vpm1	263111	145	1814.56	32	8222.22	20	13155.55	5929	44.38
vpm2	273994	77504	3.54	67014	4.09	69515	3.94	73687	3.72

Number of nodes (N_k) in tree and the ratio $r_k = \frac{N_1}{N_k}$ for selected instances when $t = 1000s$, $M = 1$, k is varied.

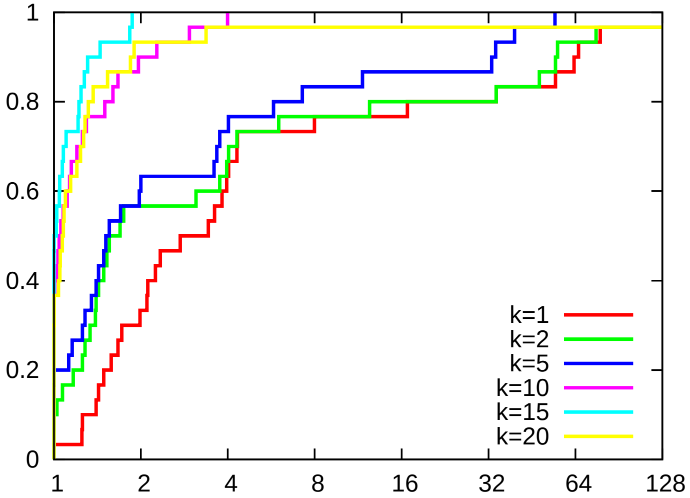
*Using performance profiles**



Performance profile for number of subproblems when $t = 1000s$ is fixed and k is varied. $M = 1$.

*Dolan and Moré, 2002

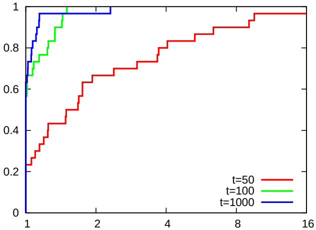
*Using performance profiles**



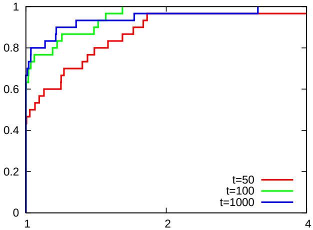
Performance profile for number of subproblems when $t = 50s$ is fixed and k is varied. $M = 1$.

*Dolan and Moré, 2002

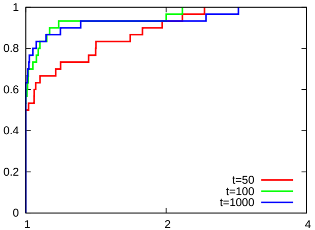
Dependence on t



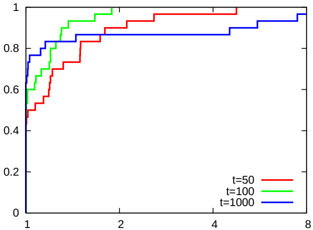
(a) $k = 5$



(b) $k = 10$



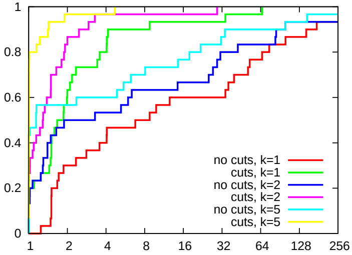
(c) $k = 15$



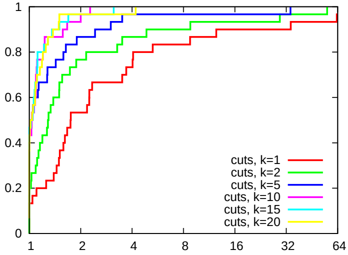
(d) $k = 20$

Performance profile for number of subproblems when k is fixed and t (in seconds) is varied.

In presence of cuts



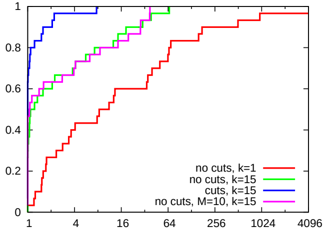
(e) $t = 100$ seconds



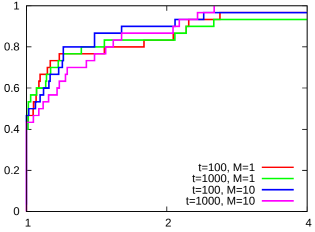
(f) $t = 100$ seconds

Figure: Performance profile for number of subproblems when cuts are added to the original problem. t is fixed and k is varied.

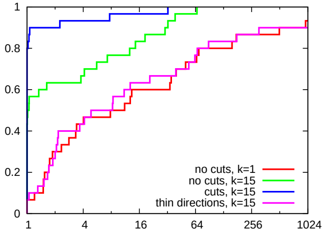
Higher coefficients and thin directions



M is increased to 10, $t = 100s$

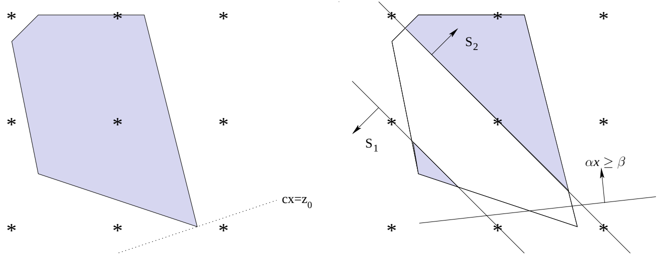


Effects of changing t, M



Comparison of effect of branching on *thin* directions against other criteria, $t = 100$ seconds

Valid Inequalities



- $\alpha x \geq \beta$ is valid because it is valid for $\mathcal{P} \cup S_1$ and for $\mathcal{P} \cup S_2$.
- Such inequalities are called Split Inequalities (Balas).
- C-G, GMI, MIR, Lift-and-project, k-cuts are special cases of these.
- These can be generated from disjunctions used for branching.
- Conversely, disjunctions used for generating these inequalities can be used for branching.

Rank of Split Inequalities

An Elementary Split Inequality

$\alpha x \geq \beta$ is said to be an elementary split inequality for $\mathcal{P}$ if it is a valid inequality for both $\mathcal{P} \cup \{x | \pi x \leq \pi_0\}$ and $\mathcal{P} \cup \{x | \pi x \geq \pi_0 + 1\}$.

Elementary Split Closure of $\mathcal{P}$ ($\mathcal{P}^1$):

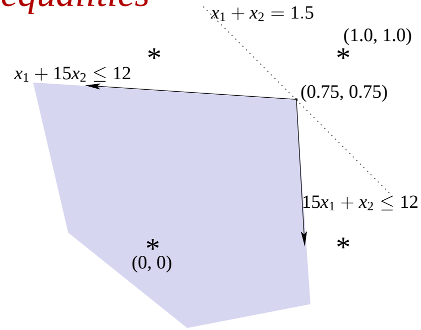
The closure of $\mathcal{P}$ and all of its elementary split inequalities.

Rank-1 cuts:

Inequalities that are valid for $\mathcal{P}^1$ but not for $\mathcal{P}$.

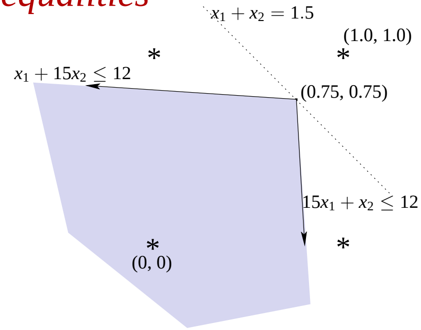
- Similarly, we can define rank-2 cuts etc.
- Separation of an arbitrary point from the elementary closure is $\mathcal{NP}-$complete (Caprara and Letchford, 2003).
- Equivalently, optimization over elementary split closure is $\mathcal{NP}-$hard.
- However, every rank-1 split inequality is not an elementary split inequality.

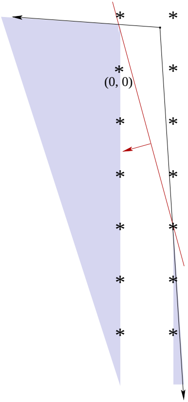
Split inequalities

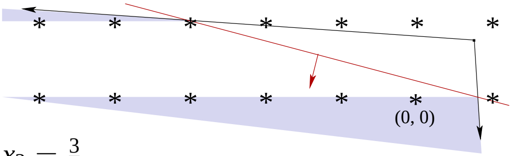


$$\min_{x \in \mathcal{P}} x_1 + x_2 = \frac{3}{2},$$

Split inequalities

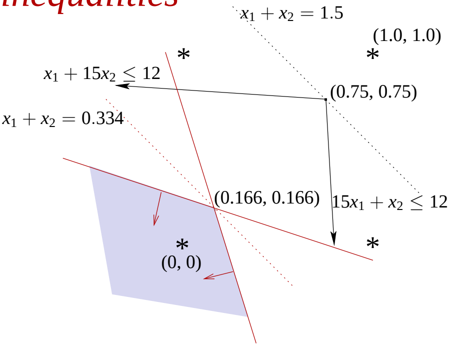


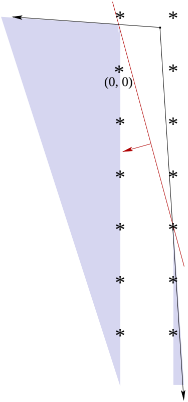


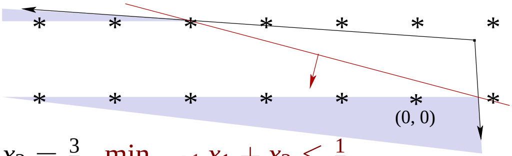


$$\min_{x \in \mathcal{P}} x_1 + x_2 = \frac{3}{2},$$

Split inequalities

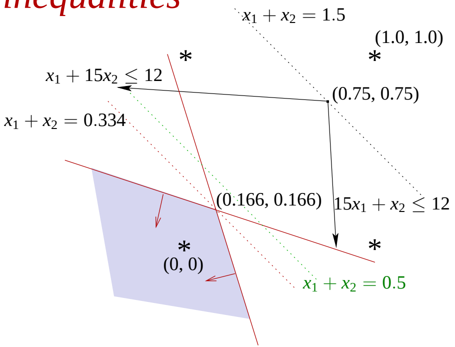


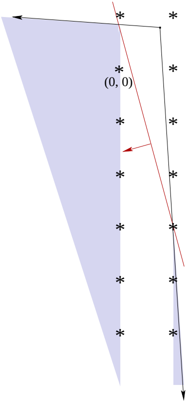


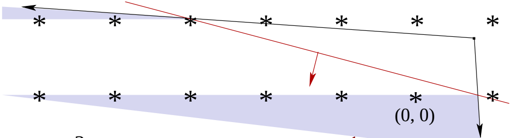


$$\min_{x \in \mathcal{P}} x_1 + x_2 = \frac{3}{2}, \ \min_{x \in \mathcal{P}^1} x_1 + x_2 \leq \frac{1}{3},$$

Split inequalities



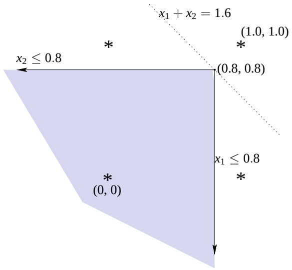




$\min_{x \in \mathcal{P}} x_1 + x_2 = \frac{3}{2}$, $\min_{x \in \mathcal{P}^1} x_1 + x_2 \leq \frac{1}{3}$, $\min x_1 + x_2$ after adding any single elementary split inequality ≥ 0.5 (believe me) .

Chvátal-Gomory Cuts

- If $\alpha x \geq \beta$ is a valid inequality for $\mathcal{P}$ and $\alpha \in \mathbb{Z}^n$.
- Then $\alpha x \geq \lceil \beta \rceil$ is a C-G cut.
- The associated disjunction is $\alpha x \leq \lceil \beta \rceil - 1 \vee \alpha x \geq \lceil \beta \rceil$.
- The LP associated with one of the disjunctive sets is infeasible.
- "The disjunction and the cut are identical".
- Similar definitions can be used for elementary C-G cut, elementary closure and rank.
- Separtion from elementary closure of C-G cuts is $\mathcal{NP}-$complete. Optimization is $\mathcal{NP}-$hard (Eisenbrand, 1999).



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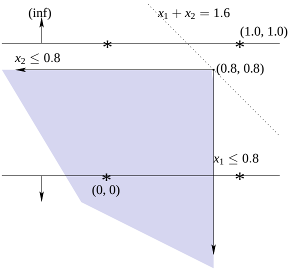

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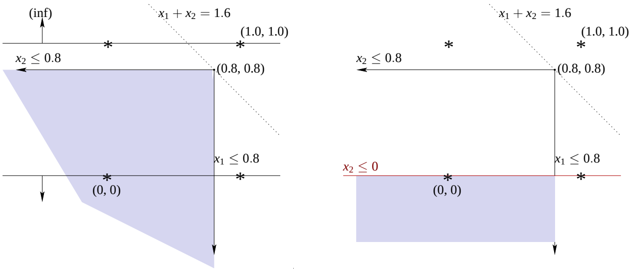
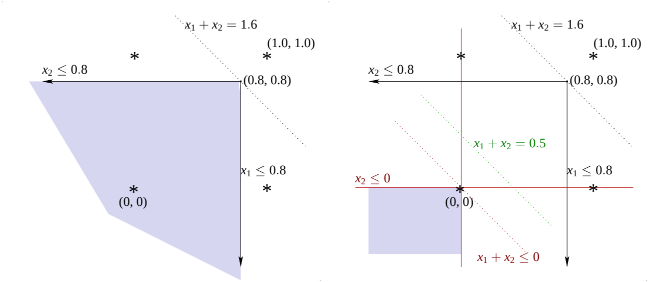


Similar story



$\min_{x \in \mathcal{P}} x_1 + x_2 = \frac{4}{5}$, $\min_{x \in \mathcal{P}^1_{CG}} x_1 + x_2 = 0$, $\min x_1 + x_2$ after adding any single elementary C-G inequality ≥ 0.5 (believe me) .

What does all this mean?

- Some faces of split-closure may not be elementary split inequalities.
- Some faces of C-G closure may not be elementary C-G inequalities.
- Some faces of C-G closure may not be elementary split inequalities.

Consider three problems

1. Given an inequality $\alpha x \geq \beta$, is this an elementary split inequality?
2. Given an inequality $\alpha x \geq \beta, \alpha \in \mathbb{Z}^n$, is this an elementary C-G inequality?
3. Given a polyhedron $\mathcal{P}$ and $K \in \mathbb{R}$, does there exist an elementary split inequality such that the LP bound after adding it is at least K?

What does all this mean?

- Some faces of split-closure may not be elementary split inequalities.
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- Some faces of C-G closure may not be elementary split inequalities.

Consider three problems

1. Given an inequality $\alpha x \geq \beta$, is this an elementary split inequality? $\mathcal{NP}-$Complete.
2. Given an inequality $\alpha x \geq \beta, \alpha \in \mathbb{Z}^n$, is this an elementary C-G inequality? $\mathcal{P}$.
3. Given a polyhedron $\mathcal{P}$ and $K \in \mathbb{R}$, does there exist an elementary split inequality such that the LP bound after adding it is at least K? $\mathcal{NP}-$Complete.

Concluding remarks

- We formulated the problem of finding a general disjunction that maximizes the lower bound.
- The problem is shown to be $\mathcal{NP}-$hard in the general case and also for several specific restrictions.
- Computational experiments show that these disjunctions can reduce significantly the size of branch-and-bound tree.
- Leads to interesting results for the problem of generating "best" split inequalities and for that of deciding if an inequality is a split inequality.

Available at optimization-online:

- A.M. and T.K. Ralphs, *Experiments with Branching using General Disjunctions*, Proceedings of the Eleventh INFORMS Computing Society Meeting, 2009, To Appear.
- A.M. and T.K. Ralphs, *On the complexity of selecting branching disjunctions for integer programming*, Submitted.