Generalized Benders' Algorithm for Mixed Integer Bilevel Linear Optimization

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Outline

Outline

Mixed Integer Bilevel Linear Optimization

- $x \in X \subseteq \mathbb{Z}_+^{r_1} \times \mathbb{R}_+^{n_1 - r_1}$ are the *first-level variables*
- $y \in Y \subseteq \mathbb{Z}_+^{r_2} \times \mathbb{R}^{n_2 - r_2}$ are the *second-level variables*

Mixed Integer Bilevel Linear Optimization Problem

$$\min \left\{ cx + d^1 y \mid x \in X, y \in \mathcal{P}_1(x) \cap \mathcal{P}_2(x) \cap Y, d^2 y \leq \phi(b^2 - A^2 x) \right\}, \tag{MIBLP}$$

where

$$\mathcal{P}_1(x) = \left\{ y \in \mathbb{R}_+^{n_2} \mid G^1 y \geq b^1 - A^1 x \right\},$$

$$\mathcal{P}_2(x) = \left\{ y \in \mathbb{R}_+^{n_2} \mid G^2 y \geq b^2 - A^2 x \right\},$$

$$\phi(\beta) = \min\{d^2 y \mid G^2 y \geq \beta, y \in Y\} \ \forall \beta \in \mathbb{R}^{m_2}.$$

Assumption: The first-level variables participating in the second-level problem are all integer variables.

Mixed Integer Bilevel Linear Optimization

Alternatively, an *MIBLP* can also be described as following.

Mixed Integer Bilevel Linear Optimization Problem

$$\min\left\{cx + \Xi(x) \mid x \in X\right\}, \tag{MIBLP-RF}$$

where Ξ is a *risk function* that encodes the part of the objective function that depends on the response to x in the second level.

Risk Function

$$\Xi(x) = \min\left\{d^1 y \mid y \in \mathcal{P}_1(x) \cap \mathcal{P}_2(x) \cap Y, d^2 y \leq \phi(b^2 - A^2 x)\right\}. \tag{RF}$$

Benders' Principle (Linear Optimization)

$$
\begin{aligned}
z_{\mathrm{LP}} &= \min_{(x,y)\in\mathbb{R}^n_+} \left\{ c'x + c''y \;\middle|\; A'x + A''y \geq b \right\} \\
&= \min_{x\in\mathbb{R}^{n'}_+} \left\{ c'x + \phi(b - A'x) \right\},
\end{aligned}
$$

where

$$
\begin{aligned}
\phi(\beta) = \quad & \min\ c''y \\
& \text{s.t. } A''y \geq \beta \\
& y \in \mathbb{R}^{n''}_+
\end{aligned}
$$

Basic strategy:

- ϕ is the *value function* of a linear optimization problem.
- It is *piecewise linear* and *convex*.
- It is iteratively approximated by generating *lower-bounding functions*.

Lower-Bounding Functions

A *lower-bounding function* (LBF) $\underline{\phi} : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{\pm\infty\}$ is a function such that

$$\underline{\phi}(\beta) \leq \phi(\beta) \quad \forall \beta \in \mathbb{R}^m$$

An LBF is *strong* at the given value b if

$$\underline{\phi}(b) = \phi(b)$$

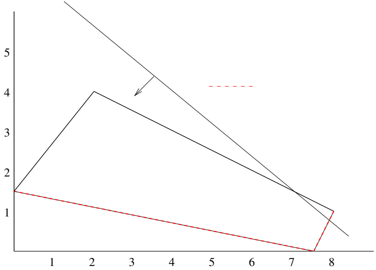
Example-1 [Moore and Bard, 1990]

$$z_{LP} = \min_{x \in \mathbb{R}_+} x + \phi(b - A'x),$$

where

$$\phi(\beta) = \min \quad y \quad \text{s.t.} \begin{bmatrix} -20 \\ -2 \\ 1 \\ 10 \end{bmatrix} y \geq \beta \quad y \in \mathbb{R}_+$$

with $b = \begin{bmatrix} -30 \\ -10 \\ -15 \\ 15 \end{bmatrix}, A' = \begin{bmatrix} 25 \\ -1 \\ -2 \\ 2 \end{bmatrix}.$



Benders' Algorithm

Master problem:

$$\begin{aligned} z_{LP} = \quad & \min\ c'x + \theta \\ & \text{s.t. } \theta \geq (b - A'x)v^i\ \forall i \\ & 0 \geq (b - A'x)d^j\ \forall j \\ & x \in \mathbb{R}_+^{n'} \end{aligned}$$

Subproblem:

$$\begin{aligned} \phi(b - A'\bar{x}) = \quad & \min\ c''y \\ & \text{s.t. } A''y \geq b - A'\bar{x} \\ & y \in \mathbb{R}_+^{n''} \end{aligned}$$

0. **Initialize**
   - Let $x^1 := \arg\min\{c'x | x \in \mathbb{R}_+^{n'}\}$, $\theta^1 := -\infty$ and $k := 1$.
1. **Update the LBF**
   a) Solve the *subproblem* for the RHS $b - A'x^k$. Construct an LBF $\underline{\phi}$.
   b) Check if $\theta^k = \phi(b - A'x^k)$. If yes, STOP. $x^* := x^k$.
2. **Solve the master problem**
   a) Add $\underline{\phi}$ to the *master problem*.
   b) Solve it to obtain an optimal solution (x^{k+1}, θ^{k+1}).
   c) Set $k := k + 1$. Go to Step 1.

Benders' Algorithm

Master problem:

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   - c) Set $k := k + 1$. Go to Step 1.

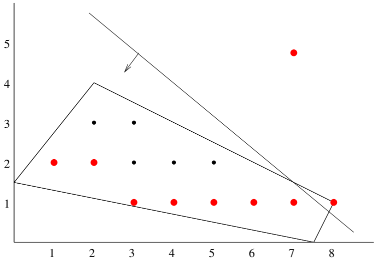
Example-2 [Moore and Bard, 1990]

$$z_{IP} = \min_{x \in \mathbb{Z}_+} x + \phi(b - A'x),$$

where

$$\phi(\beta) = \min \quad y$$
$$\text{s.t.} \begin{bmatrix} -20 \\ -2 \\ 1 \\ 10 \end{bmatrix} y \geq \beta$$
$$y \in \mathbb{Z}_+$$

with $b = \begin{bmatrix} -30 \\ -10 \\ -15 \\ 15 \end{bmatrix}, A' = \begin{bmatrix} 25 \\ -1 \\ -2 \\ 2 \end{bmatrix}.$



Essential Components of Benders' Algorithm

- A *master problem*
- A *subproblem* which is relatively easier to solve
- An *algorithm* for solving the subproblem which can generate *LBFs*
- A practical & efficient way to *strengthen the value function approximation*

Outline

Generalized Benders' for MIBLP

Mixed Integer Bilevel Linear Optimization Problem

$$\min \{cx + \Xi(x) \mid x \in X\}, \qquad \text{(MIBLP-RF)}$$

where

Risk Function

$$\Xi(x) = \min \left\{d^1 y \mid y \in \mathcal{P}_1(x) \cap \mathcal{P}_2(x) \cap Y, d^2 y \le \phi(b^2 - A^2 x)\right\}. \qquad \text{(RF)}$$

A variable partitioning scheme:

- x in the master problem
- y in the subproblem

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A variable partitioning scheme:

- x in the master problem
- y in the subproblem

Generalized Benders' for MIBLP

Master problem:

$$
\begin{aligned}
z_{MILP} = \quad & \min\ cx + \theta \\
& \text{s.t. } \theta \geq \Xi(x) \\
& x \in X
\end{aligned}
$$

Subproblem:

$$
\begin{aligned}
\Xi(\bar{x}) = \min\ & d^1 y \\
\text{s.t. } & G^1 y \geq b^1 - A^1 \bar{x} \\
& y \in \{\arg\min\ d^2 w \\
& \quad \text{s.t. } G^2 w \geq b^2 - A^2 \bar{x} \\
& \quad w \in Y\}
\end{aligned}
$$

- Master problem is an MILP
- Subproblem is an *MIBLP*
- θ represents the approximation of the (bilevel) risk function
- An LBF is generated in each iteration

Essential Components of Benders' Algorithm

- A master problem ✓
- A subproblem which is relatively easier to solve ✓
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Quick Detour: Bounding Functions of an MILP

Consider the following MILP.

$$z_{MILP} = \min cx$$
$$\text{s.t. } Ax = b$$
$$x \geq 0$$
$$x \in \mathbb{Z}^r \times \mathbb{R}^{n-r}$$

Question: How to construct strong lower- and *upper-bounding functions* (UBFs)?

LBFs from Branch-and-Bound Algorithm [Güzelsoy, 2009]

Let T be the set of leaf nodes of the tree. Then, at a leaf node $t \in T$

Primal problem:

$$\begin{aligned} \min\ & c^\top x \\ \text{s.t. } & Ax = b, \\ & l^t \leq x \leq u^t, x \geq 0 \end{aligned}$$

Dual problem:

$$\begin{aligned} \max\ & \{\eta^t b + \underline{\eta}^t l^t + \bar{\eta}^t u^t\} \\ \text{s.t. } & \eta^t A + \underline{\eta}^t + \bar{\eta}^t \leq c^\top \\ & \underline{\eta} \geq 0, \bar{\eta} \leq 0 \end{aligned}$$

We obtain the following *strong* LBF:

$$\min_{t \in T} \{\eta^{*t} b + \underline{\eta^{*}}^{t} l^t + \bar{\eta^{*}}^{t} u^t\}$$

Note: This can be further strengthened by considering *non-leaf nodes*.

Example-3

Figure: Single LBF

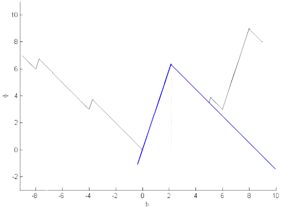
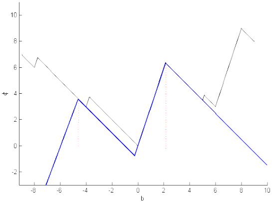


Figure: Multiple LBFs



UBFs of an MILP

Theorem 1 (Güzelsoy [2009]) *Let $K \in N := \{1, \ldots, n\}$, $s_i \in \mathbb{R}_+ \; \forall i \in K$ be given, and define the function $\bar{\phi} : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ such that*

$$\bar{\phi}(\beta) = \sum_{i \in K} c_i s_i + \phi_{N \setminus K}(\beta - \sum_{i \in K} A_i s_i) \quad \forall \beta \in \mathbb{R}^m,$$

where A_i is the i^{th} column of A and

$$\begin{aligned} \phi_{N \setminus K}(\beta) = \quad & \min \sum_{i \in N \setminus K} c_i x_i \\ & s.t. \sum_{i \in N \setminus K} A_i x_i = \beta \\ & \quad x_i \in \mathbb{Z}_+ \; \forall i \in I, \; x_i \in \mathbb{R}_+ \; \forall i \in C. \end{aligned}$$

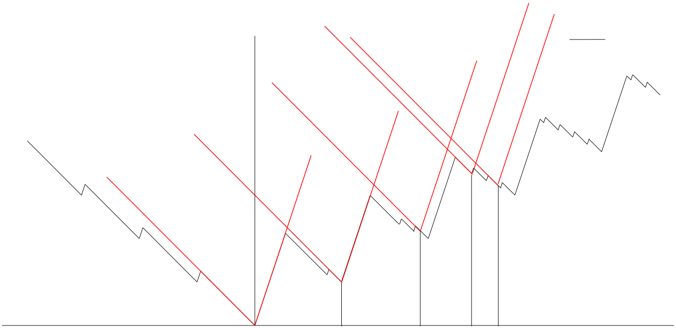
Then, $\bar{\phi}(\beta) \geq \phi(\beta) \; \forall \beta \in \mathbb{R}^m$, if $s_i \in \mathbb{Z}_+ \; \forall i \in I \cap K$ and $s_i \in \mathbb{R}_+ \; \forall i \in C \cap K$.

UBFs of an MILP

- A UBF can be constructed by considering a *restriction* of the MILP.
- $\bar{\phi}(b) = \phi(b)$ if and only if $s_i = x_i^* \; \forall i \in K$.
- Consider a *single column restriction* in the extreme case.

Example-4

Figure: Multiple UBFs



Why Discuss MILPs Here?

Answer: Because bilevel subproblem *for the given $x = \bar{x}$* is nothing but an MILP!

Bilevel subproblem:

$$\begin{aligned}\Xi(\bar{x}) = \min\ & d^1 y \\ \text{s.t. } & G^1 y \geq b^1 - A^1 \bar{x} \\ & y \in \{\arg\min\ d^2 w \\ & \qquad \text{s.t. } G^2 w \geq b^2 - A^2 \bar{x} \\ & \qquad w \in Y\}\end{aligned}$$

Single level reformulation:

$$\begin{aligned}\Xi(\bar{x}) = \min\ & d^1 y \\ \text{s.t. } & G^1 y \geq b^1 - A^1 \bar{x} \\ & d^2 y \leq \phi(b^2 - A^2 \bar{x}) \\ & G^2 y \geq b^2 - A^2 \bar{x} \\ & y \in Y\end{aligned}$$

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where

$$\begin{aligned}\phi(\beta) = \min\ & d^2 w \\ \text{s.t. } & G^2 w \geq \beta \\ & w \in Y\end{aligned}$$

LBFs for the Bilevel Subproblem

- Single level reformulation is an MILP.
- The branch-and-bound algorithm can be used.
- Hence, the required strong LBF:

$$\underline{\Xi}(\bar{x}) = \min_{t \in T} \{ {\eta_1^*}^t (b^1 - A^1 \bar{x}) + {\eta_2^*}^t \phi(b^2 - A^2 \bar{x}) + {\eta_3^*}^t (b^2 - A^2 \bar{x}) + {\underline{\eta}^*}^t l^t + {\bar{\eta}^*}^t u^t \}$$

Essential Components of Benders' Algorithm

- A master problem ✓
- A subproblem which is relatively easier to solve ✓
- An *algorithm* for solving the subproblem which can generate *LBFs* ✓
- A practical & efficient way to *strengthen the value function approximation*

Challenges So Far

The required LBF:

$$\underline{\Xi}(x) = \min_{t \in T} \{ {\eta_1^*}^t (b^1 - A^1 x) + {\eta_2^*}^t \phi(b^2 - A^2 x) + {\eta_3^*}^t (b^2 - A^2 x) + {\underline{\eta}^*}^t l^t + {\bar{\eta}^*}^t u^t \}$$

Challenge-1: Ensuring the validity of LBF for all values of x

Challenge-2: Ensuring that the LBF stays strong at *all previous iterates* of x

Our Approach

Iterative Improvement of the Second-Level Value Function ϕ

- Construct a strong UBF $\bar{\phi}$ for the second-level value function ϕ.
- Use $\bar{\phi}$ for constructing $\underline{\Xi}$ for the subproblem.
- Strengthen $\bar{\phi}$ iteratively to ensure that it is strong at all previous iterates of x.

Note: Construction of *(bilevel) risk function* and *second-level value function*!

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Details of Our Approach

- $\bar{\phi}$ can be constructed from continuous restriction of the second-level MILP.

Second-level MILP *for* $x = \bar{x}$:

$$\phi(b^2 - A^2\bar{x}) = \min d^2 y \quad \text{s.t. } G^2 y = b^2 - A^2\bar{x}, \quad y \in Y$$

Continuous restriction:

$$\phi_C(b^2 - A^2\bar{x}) = \min d_C^2 y_C \quad \text{s.t. } G_C^2 y_C = b^2 - A^2\bar{x} - G_I^2 y_I^*, \quad y_C \geq 0$$

- We have the following UBF where η^* is the optimal dual solution.

$$\bar{\phi} = (b^2 - A^2 x - G_I^2 y_I^*)\eta^*$$

- Note: From sensitivity analysis, above $\bar{\phi}$ is valid

$$\forall\, x \in X \cap \{x \geq 0 \mid (G_B^2)^{-1}(b^2 - A^2 x - G_I^2 y_I^*) \geq 0\},$$

where B is the index set corresponding to basis columns of G_C^2.

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Continuous restriction:

$$\phi_C(b^2 - A^2\bar{x}) = \min\ d^2_C y_C \quad \text{s.t. } G^2_C y_C = b^2 - A^2\bar{x} - G^2_I y^*_I,\quad y_C \geq 0$$

- We have the following UBF where η^* is the optimal dual solution.

$$\bar{\phi} = (b^2 - A^2 x - G^2_I y^*_I)\eta^*$$

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$$\forall\ x \in X \cap \{x \geq 0 \mid (G^2_B)^{-1}(b^2 - A^2 x - G^2_I y^*_I) \geq 0\},$$

where B is the index set corresponding to basis columns of G^2_C.

LBFs for the Bilevel Subproblem

The LBF:

$$\underline{\Xi}(x) = \min_{t \in T} \{ {\eta_1^*}^t (b^1 - A^1 x) + {\eta_2^*}^t \bar{\phi} + {\eta_3^*}^t (b^2 - A^2 x) + {\underline{\eta}^*}^t l^t + {\bar{\eta}^*}^t u^t \},$$

where

$$\bar{\phi} = \begin{cases} (b^2 - A^2 x - G_I^2 y_I^*)\eta^* & \text{if } (G_B^2)^{-1}(b^2 - A^2 x - G_I^2 y_I^*) \geq 0 \\ \infty & \text{otherwise} \end{cases}$$

Updated Master Problem

The master problem:

$$
\begin{aligned}
z_{MILP} = \quad & \min cx + \theta \\
& \text{s.t. } \theta \geq \min_{t \in T} \{ \eta_1^{*t} (b^1 - A^1 x) + \eta_2^{*t} \bar{\phi} + \eta_3^{*t} (b^2 - A^2 x) + \underline{\eta^*}^t l^t + \bar{\eta^*}^t u^t \}, \\
& \quad \bar{\phi} = \begin{cases} (b^2 - A^2 x - G_I^2 y_I^*) \eta^* & \text{if } (G_B^2)^{-1} (b^2 - A^2 x - G_I^2 y_I^*) \geq 0 \\ \infty & \text{otherwise} \end{cases}, \\
& \quad x \in X
\end{aligned}
$$

Can be linearlized further by adding binary variables and big-M parameters...

Essential Components of Benders' Algorithm

- A master problem ✓
- A subproblem which is relatively easier to solve ✓
- An *algorithm* for solving the subproblem which can generate *LBFs* ✓
- A practical & efficient way to *strengthen the value function approximation* ✓

Outline

Computational Framework

- The *Mixed Integer Bilevel Solver* (MibS) [DeNegre et al., 2017] implements the branch-and-cut algorithm for solving MIBLPs using software available from the Computational Infrastructure for Operations Research (COIN-OR) repository.
- We coded our algorithm in the MibS for now.
- We use CPLEX for solving the master problem, the second-level problem, and other auxiliary problems.
- We use SYMPHONY for solving the subproblem and obtaining required dual information from branch-and-bound nodes.
- We use SoPlex's exact algorithmic feature for solving the continuous restriction.

Computational Challenges

The master problem:

$$z_{MILP} = \min cx + \theta$$

$$\text{s.t. } \theta \geq \min_{t \in T} \{ {\eta_1^*}^t (b^1 - A^1 x) + {\eta_2^*}^t \bar{\phi} + {\eta_3^*}^t (b^2 - A^2 x) + {\underline{\eta}^*}^t l^t + {\bar{\eta}^*}^t u^t \},$$

$$\bar{\phi} = \begin{cases} (b^2 - A^2 x - G_I^2 y_I^*)\eta^* & \text{if } (G_B^2)^{-1}(b^2 - A^2 x - G_I^2 y_I^*) \geq 0 \\ \infty & \text{otherwise} \end{cases},$$

$$x \in X$$

Computational Challenges

The master problem:

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\begin{aligned}
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\text{s.t. } & \theta \geq \min_{t \in T}\{\eta_1^{*t}(b^1 - A^1 x) + \eta_2^{*t}\bar{\phi} + \eta_3^{*t}(b^2 - A^2 x) + \underline{\eta}^{*t} l^t + \bar{\eta}^{*t} u^t\}, \\
& \bar{\phi} = \begin{cases} (b^2 - A^2 x - G_I^2 y_I^*)\eta^* & \text{if } (G_B^2)^{-1}(b^2 - A^2 x - G_I^2 y_I^*) \geq 0 \\ \infty & \text{otherwise} \end{cases}, \\
& x \in X
\end{aligned}
$$

Computational Challenges

The master problem:

$$z_{MILP} = \min\ cx + \theta$$

$$\text{s.t. } \theta \geq \min_{t \in T} \{ {\eta_1^*}^t (b^1 - A^1 x) + {\eta_2^*}^t \bar{\phi} + {\eta_3^*}^t (b^2 - A^2 x) + \underline{\eta^*}^t l^t + \bar{\eta^*}^t u^t \}, \longleftarrow \underline{M}$$

$$\longrightarrow \bar{M}\ \ \bar{\phi} = \begin{cases} (b^2 - A^2 x - G_I^2 y_I^*) \eta^* & \text{if } (G_B^2)^{-1} (b^2 - A^2 x - G_I^2 y_I^*) \geq 0 \\ \infty & \text{otherwise} \end{cases},$$

$$x \in X$$

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The master problem:

$$z_{MILP} = \min\ cx + \theta$$

$$\text{s.t. } \theta \geq \min_{t \in T}\{{\eta_1^*}^t(b^1 - A^1x) + {\eta_2^*}^t\bar{\phi} + {\eta_3^*}^t(b^2 - A^2x) + {\underline{\eta}^*}^t l^t + {\bar{\eta}^*}^t u^t\}, \longleftarrow \underline{M}$$

$$\longrightarrow \bar{M}\ \bar{\phi} = \begin{cases} (b^2 - A^2x - G_I^2 y_I^*)\eta^* & \text{if } (G_B^2)^{-1}(b^2 - A^2x - G_I^2 y_I^*) \geq 0 \\ \infty & \text{if } (G_B^2)^{-1}(b^2 - A^2x - G_I^2 y_I^*) < 0 \end{cases},$$

$$x \in X$$

Solving Continuous Restriction with SoPlex

- Solve the continuous restriction with SoPlex's exact solving feature
- Obtain basis matrix in rational form
- Rewrite the condition $(G_B^2)^{-1}(b^2 - A^2x - G_I^2 y_I^*) < 0$ to an equivalent condition with all integer coefficients
- Then, apply the condition $(\bar{G}_B^2)^{-1}(b^2 - A^2x - G_I^2 y_I^*) \leq -1$

Outline

Conclusions and Future Work

- A *generalized Benders' decomposition* algorithm for solving MIBLPs is developed and presented.
- The *(bilevel) risk function* is constructed via *lower-bounding functions*.
- The *second-level value function* is constructed via *upper-bounding functions*.
- Implementation of the algorithm is in-progress.
- The efficiency of the algorithm can further be improved, e.g., via *warm starting of MILPs*.

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Thank You!

Questions?

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