Branch, Constrain, and Price Algorithms for Large-scale Discrete Optimization

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Outline of Talk

- Introduction to Branch, Cut, and Price (BCP)
- Frameworks for BCP
- The Abstract Library for Parallel Search
- Implementation Issues for BCP
  - Parallel Scalability
  - Data Handling
- Advanced Algorithms
- What's Available

LP-based Branch and Bound

- Consider problem P:

$$\begin{aligned} & min\, c^T x \\ & s.t. \quad Ax \leq b \\ & \qquad\quad x_i \in \mathbf{Z}\, \forall\, i \in I \end{aligned}$$

  where $(A, b) \in \mathbf{R}^{m \times n+1}, c \in \mathbf{R}^n$.

- Let $\mathcal{P} = conv\{x \in \mathbf{R}^n : Ax \leq b, x_i \in \mathbf{Z}\, \forall\, i \in I\}$.

- Basic Algorithmic Approach
  - Use LP relaxations to produce lower bounds.
  - Branch using hyperplanes.

- Basic Algorithmic Elements
  - A method for producing and tightening the LP relaxations.
  - A method for branching.

Branch, Cut, and Price

- Weyl-Minkowski
  - $\exists(\bar{A}, \bar{b}) \in \mathbf{R}^{\bar{m} \times n+1}$ s.t. $\mathcal{P} = \{x \in \mathbf{R}^n : \bar{A}x \leq \bar{b}\}$
  - We want the solution to $min\{c^T x : \bar{A}x \leq \bar{b}\}$.
  - Solving this LP isn't practical (or necessary).
- BCP Approach
  - Form LP relaxations using submatrices of $\bar{A}$.
  - The submatrices are defined by sets $\mathcal{V} \subseteq [1..n]$ and $\mathcal{C} \subseteq [1..\bar{m}]$.
  - Forming/managing these relaxations efficiently is one of the primary challenges of BCP.

The Challenge of BCP

- The efficieny of BCP depends heavily on the size (number of rows and columns) and tightness of the LP relaxations.
- Tradeoff
  - Small LP relaxations $\Rightarrow$ faster LP solution.
  - Big LP relaxations $\Rightarrow$ better bounds.
- The goal is to keep relaxations small while not sacrificing bound quality.
- We must be able to easily move constraints and variables in and out of the *active* set.
- This means dynamic generation and deletion.

An Object-oriented Approach

- The rows/columns of a static LP are called *constraints* and *variables*.
- What do these terms mean in a dynamic context?
- Conceptual Definitions
  - Constraint: A mapping

$$f_i^c(\mathcal{C}) : 2^{[1..n]} \longrightarrow \mathbf{R}^{|\mathcal{C}|}$$

    generating coefficients for the submatrix C.
  - Variable: A mapping

$$f_j^v(\mathcal{V}) : 2^{[1..\bar{m}]} \longrightarrow \mathbf{R}^{|\mathcal{V}|}$$

    generating coefficients for the submatrix C.
  - Subproblem: A subset $(\mathcal{C}, \mathcal{V})$ of $[1..n] \times [1..\bar{m}]$.
- To construct a subproblem, an initial *core relaxation* is needed.
- From the core, we can build up other relaxations using the mappings.

Frameworks for BCP

- Concept: Provide a *framework* in which the user has only to define constraints, variables, and a core.
  - Branch and bound ⇒ core only
  - Branch and cut ⇒ core plus constraints
  - Branch and price ⇒ core plus variables
  - Branch, cut, and price ⇒ the whole caboodle
- Existing BCP frameworks
  - SYMPHONY (parallel)
  - COIN/BCP (parallel)
  - ABACUS (sequential)
- Other frameworks
  - PICO, PUBB, BoB, PPBB-Lib (branch and bound)
  - MINTO (branch and cut)

The ALPS Project

- In partnership with IBM and the COIN-OR project.
- Multi-layered C++ class library for implementing scalable, parallel tree search algorithms.
- Design is fully generic.
  - Abstract notions from BCP.
  - Support for implementing general search algorithms.
  - Support for any bounding scheme.
  - No assumptions on problem/algorithm type.
- Increased parallel scalability.
- Support for large-scale, data-intensive applications (BCP).
- Support for advanced methods not available in commercial codes.

ALPS Design

Modular library design with minimal assumptions in each layer.

ALPS Abstract Library for Parallel Search

- manages the search tree.
- prioritizes based on quality.

BiCePS Branch, Constrain, and Price Software

- manages the data.
- adds notion of primal and dual objects.
  - dual objects are functions of primal objects.
  - objects have bounds, a value, and a price.
  - objective function is a dual object without bounds.

BLIS BiCePS Linear Integer Solver

- assumes linear constraints and a linear objective.
- utilizes LP relaxations.

ALPS: Abstract Library for Parallel Search

Properties of a search tree node:

- status: `candidate`, `processed`, `branched`, `fathomed`.
- quality: a numerical priority (below threshold $\Rightarrow$ fathomed).

Operations on the search tree nodes:

- create children (branch).
- remove a node (recursively: remove a subtree).

ALPS: Abstract notions.

Procedural abstractions:

- process
  - status `candidate` $\longrightarrow$ processed/fathomed.
- branch
  - status `processed` $\longrightarrow$ `branched`.
  - create children (`candidate` nodes) and add to queue.

Data management abstraction:

- differencing scheme: node description can be stored with respect to parent.
- explicit description can be extracted, relative description can be created.

BiCePS: Branch, Constrain, and Price Software

Adds the notion of objects in terms of Lagrangean duality:

- primal objects or variables have associated
  - value: must lie between the primal object's bounds.
  - reduced cost: the partial derivative of the objective function.
- dual objects or constraints are functions of the primal objects and have associated
  - value: the Lagrange multipliers.
  - slack: the distance from the bounds of the evaluated function of the dual object.
- objective: a function of the primal objects (a dual object without bounds, value, and slack).

Primal and dual solutions: the objects with their associated values.

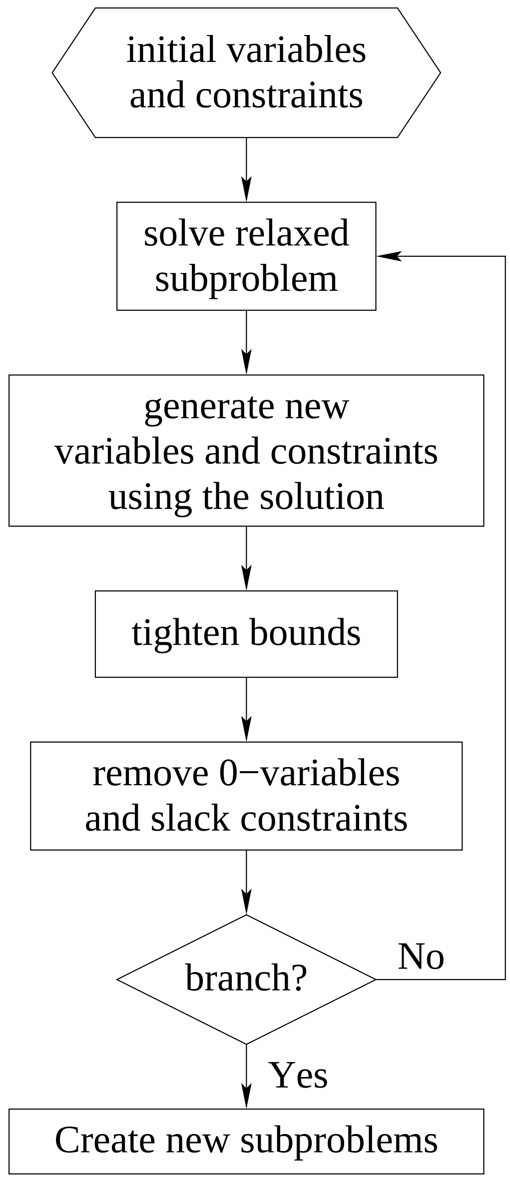
BiCePS: Processing a subproblem

A subproblem is a set of objects with an objective.

Processing a subproblem

- solve a relaxation.
- generate new objects.
- tighten bounds.
- remove objects with value 0.

If all else fails or when desired, branch.



BiCePS: Branching

Traditional branching: Choose $\hat{x}_j$ fractional.

Children: $x_j \leq \lfloor \hat{x}_j \rfloor$ and $x_j \geq \lceil \hat{x}_j \rceil$.

General branching:

- Add new objects.
- Change object bounds.
- Children must cover the feasible region.

Example: y_i binary variable and $y_i = 0 \Rightarrow a^T x \leq \beta$.

Children: $y_i = 1$ and $\{y_i = 0$ and $a^T x \leq \beta\}$.

(this avoids using the big M method)

Strong branching:

- "Pre-solve" with multiple candidates to estimate bound.
- Pick the best.

BLIS: BiCePS Linear Integer Solver

A concretization of BiCePS specifying the bounding relaxation to be used.

Defines:

- the representation of the objects: columns/rows.
- the relaxation to be used: LP relaxation.
- the objective function: linear.

Leaves the notion of the relaxation solver abstract by using Open Solve Interface.

Scalability

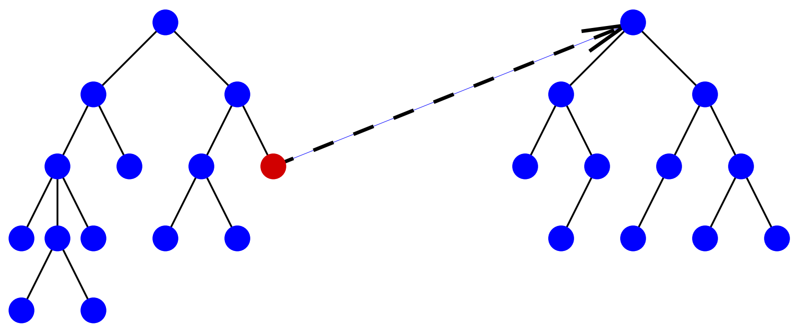
- Parallel System: Parallel algorithm + parallel architecture [Kumar and Gupta '94].
- Scalability: How well a parallel system takes advantage of increased computing resources.
- Fixed problem size: Efficiency decreases with more processors (Amdahl's Law) [Amdahl '67].
- Fixed number of processors: Efficiency increases with problem size.
- Isoefficiency: The rate problem size must be increased to maintain a fixed efficiency [Kumar and Rao '87].

Scalability Issues for Parallel Search

- Grain size
- Decentralization
- Synchronous vs. asynchronous messaging
- Ramp-up/ramp-down time

Scalability: Increased granularity

Work unit is a subtree.



Advantages:

- less communication.
- more compact storage via differencing.

Disadvantage:

- load balancing is more difficult.

Scalability: Master - Hubs - Workers Paradigm

Master

- has global information (node quality and distribution).
- balances load between hubs.
- balances quantity *and* quality.

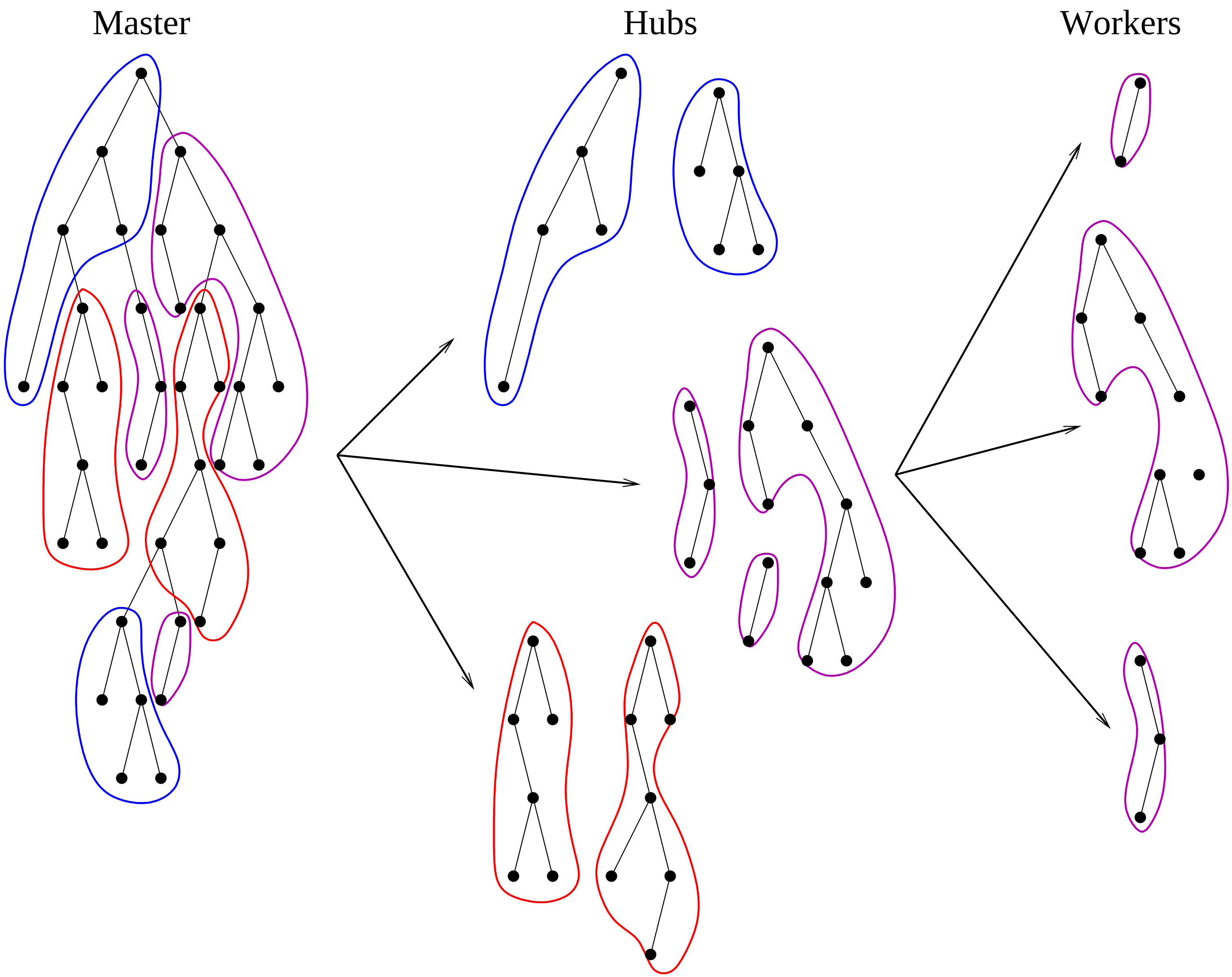
Hubs

- manage collections of subtrees (may not have full descriptions)
- balances load between workers

Workers

- processes one subtree.
- hub can interrupt.
- sends branch and quality information to hub.

Scalability: Master - Hubs - Workers Paradigm



Scalability: Asynchronous messaging

Possible communication bottlenecks:

- Too many messages.
  - avoided by the increased task granularity.
  - master-hub-worker paradigm also contributes.
- Too much synchronization (handshaking)
  - almost no handshaking.
  - must take place when a worker finishes exploring a subtree.

Scalability: Ramp-up/Ramp-down

- Ramp-up time: Time until all processors have useful work to do.
- Ramp-down time: Time during which there is not enough work for all processors.
- Controlling Ramp-up/ramp-down
  - Use different branching rules.
  - Hub instructs workers when to change rules.

Data Handling Issues

- Focused on data-intensive applications.
- Need to deal with huge numbers of objects.
- Need compact storage.
- Need to avoid duplication (generation and storage).

Data Handling: Object Representation

Each object has three representations:

- the user's representation.
  - information to generate the realization.
  - core, indexed, or algorithmic.
- the realization in the solver.
  - (projected) matrix row
  - (projected) matrix column
- the encoded representation.
  - for identification and transfer between processors.

Data Handling: Encoding / Decoding

Encodable objects: Any object that is sent between processes.

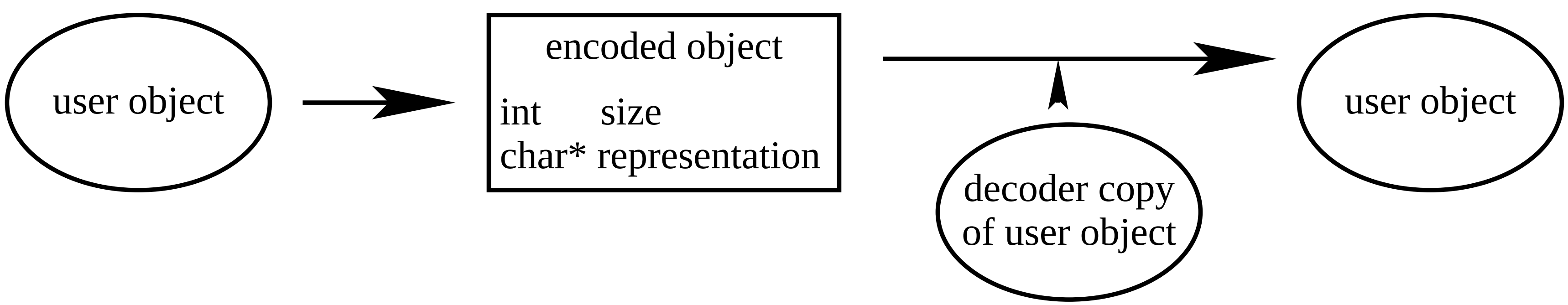
Question: How to encode/decode objects the framework does not know about?

Encoding: Easy $\Rightarrow$ use virtual methods.

Decoding: Catch-22

- can't invoke constructor for unknown type.
- can't invoke decode method without an object.

Solution: "Register" encodable objects.



Data Handling: Tracking Lists of Objects

Goal is memory conservation: no unnecessary object storage.

Implementation:

1. Object arrives in encoded form with hash value.
2. Object is looked up in hash map.
3. If it does not exist, then it is inserted.
4. A pointer to the unique copy in the hash map is added to the list.

Primary uses:

- storing variables and constraints—object can be active in multiple search nodes, but only one copy will be stored.
- object pools.

Data Handling: Object Pools

- Share objects across nodes in the tree.
- Object pools allow generated objects to be shared.
  - Cut pool is one example.
- Quality measures (slack or reduced cost, tree level, touches) ensure only the most effective objects are utilized.
- Object encoding is used to ensure that objects are stored only once.

Enhancements

- Domain decomposition
- Multi-phase methods
- Fault tolerance
- On-the-fly reconfiguration

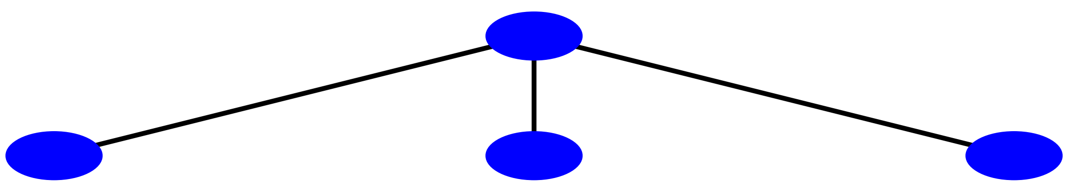
Domain Decomposition

Useful when primal and dual objects can be partitioned such that

- primal objects of a group do not interact with dual objects of other groups.
- the objective object is f-separable in terms of the primal variables in the groups. f can be additive (traditional MILP, stochastic programming), multiplicative (MIQP), etc.

After decomposing solve the children recursively, propagating intermediate results (thus creating new upper bound or fathoming the whole subtree).

$$[l, u] = [f(l_1, l_2, l_3), f(u_1, u_2, u_3)]$$



$[l_1, u_1]$ $[l_2, u_2]$ (l_3, u_3)

Multi-phase Methods

- Solve the problem on a subset of the variables, resulting in
  - a good upper bound,
  - a collection of good cuts,
  - an explored search tree.
- Price the remaining variables and propagate survivors down the tree, repeating pricing periodically.
- Unfathomed leaves are entered into the candidate list for the next phase.
- Difficulty: reproducing the search tree.

Fault Tolerance

Recovering from process failure (hardware or software)

- Worker $\Rightarrow$ easy.
  - only the processed subtree is lost.
  - the hub can reassign the work.
- Object pool $\Rightarrow$ easy.
  - has no effect on correctness.
  - can be restarted.
- Hub $\Rightarrow$ hard.
  - all managed subtrees are lost.
  - other hubs must discard subtrees whose parents were on the dead hub.
  - workers must be reassigned to another hub.
- Master
  - can attempt to restart from saved data.

On-the-fly Reconfiguration

- On the fly reconfiguration (planned "fault") — restricted use of processors.
- Messages from external source instructing master to
  - offload work from a process and kill it—work is redistributed.
  - to start new processes and redistribute work.

What's Available

- SYMPHONY: C library for implementing BCP
  - User fills in stub functions.
  - Supports shared or distributed memory.
  - Documentation and source code available `www.BranchAndCut.org`.
- COIN/BCP: C++ library for implementing BCP
  - User derives classes from library.
  - Documentation and source code available `www.coin-or.org`.
- ALPS/BiCePS/BLIS
  - In early development.
  - Sequential version by year end.
- The COIN-OR repository `www.coin-or.org`