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On Families of Quadratic Surfaces Having Fixed Intersections with Two Hyperplanes *†

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Abstract

We investigate families of quadrics all of which have the same intersection with two given hyperplanes. The cases when the two hyperplanes are parallel and when they are nonparallel are discussed. We show that these families can be described with only one parameter and describe how the quadrics are transformed as the parameter changes. This research was motivated by an application in mixed integer conic optimization. In that application, we aimed to characterize the convex hull of the union of the intersections of an ellipsoid with two half-spaces arising from the imposition of a linear disjunction.

1 Introduction

This paper is motivated by efforts to extend the disjunctive procedure of Mixed Integer Linear Optimization (MILO) [1] to Mixed Integer Second Order Conic Optimization (MISOCO). We first

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introduce the following MISOCO problem

$$\begin{aligned} \text{minimize: } & c^\top w \\ \text{subject to: } & Aw = b \\ & w \in \mathcal{K} \\ & w \in \mathbb{Z}^d \times \mathbb{R}^{\ell - d}, \end{aligned} \tag{MISOCO}$$

where $A \in \mathbb{R}^{m\times\ell}$, $c \in \mathbb{R}^\ell$, $b \in \mathbb{R}^m$, and the rows of A are linearly independent. Additionally, we have that $\mathcal{K} = \mathbb{L}_1^{\ell_1} \times \cdots \times \mathbb{L}_k^{\ell_k}$, where $\mathbb{L}^{\ell_i} = \{w^i | w_1^i \geq \|w^i_{2:\ell_i}\|\}$ $i = 1, \ldots, k$ and $\sum_{i=1}^k \ell_i = \ell$. Here, the notation $w_{2:\ell}$ refers to the vector formed by the components 2 to ℓ of vector w. This problem can be solved using a branch-and-cut algorithm. In particular, this algorithm uses the continuous relaxation of (MISOCO) to bound its objective function. This relaxation is a second order cone optimization problem [2, 3]. The bounding process may be improved if the formulation of (MISOCO) is strengthened. This can be achieved with the addition of valid inequalities to (MISOCO). Some of these inequalities can be derived if we relax the constraint $x \in \mathcal{K}$ in the continuous relaxation (MISOCO) to consider only one Lorentz cones in $\mathcal{K}$ [4]. For this reason, in this paper we focus primarily on the feasible set of the continuous relaxation of (MISOCO) when $k = 1$. The projection of this feasible set into the affine space defined by the linear constraints of (MISOCO) is a quadric of the form

$$\hat{\mathcal{Q}} = \{w \in \mathbb{R}^\ell \mid w^\top P w + 2p^\top w + \rho \leq 0\},$$

where $P \in \mathbb{R}^{\ell\times\ell}$ is a given symmetric matrix, $p \in \mathbb{R}^\ell$ is a given vector, and $\rho \in \mathbb{R}$ is some scalar. Such quadrics are the central objects of study in this paper.

During the past decade, the study of MISOCO has gained significant attention in the mathematical optimization community. A common approach used in the literature for tackling this problem is to extend some of the techniques developed for MILO to MISOCO, see, e.g., [4, 5, 6, 7, 8, 9]. One particularly successful technique used in MILO is the disjunctive procedure of Balas [1]. The generalization of the disjunctive procedure to MISOCO is the motivation for our interest in quadrics having fixed intersection with two given hyperplanes. Let $\mathcal{A} = \{w \in \mathbb{R}^\ell \mid a_1^\top w \leq \alpha_1\}$, $\mathcal{B} = \{w \in \mathbb{R}^\ell \mid a_2^\top w \geq \alpha_2\}$ for $a_1, a_2 \in \mathbb{R}^\ell$, $\alpha_1, \alpha_2 \in \mathbb{R}$, be given half spaces where (a_1, α_1) and (a_2, α_2) are not scalar multiples of each other. Additionally, assume that $\hat{\mathcal{Q}} \cap \mathcal{A} \cap \mathcal{B} = \emptyset$, $\hat{\mathcal{Q}} \cap \mathcal{A} \neq \emptyset$, and $\hat{\mathcal{Q}} \cap \mathcal{B} \neq \emptyset$. We are interested in analyzing the set $\hat{\mathcal{Q}} \cap (\mathcal{A} \cup \mathcal{B})$, which results from adding the constraint $w \in \mathcal{A} \cup \mathcal{B}$ (a disjunction) to (MISOCO). Observe that the set $\hat{\mathcal{Q}} \cap (\mathcal{A} \cup \mathcal{B})$ is not a convex set, but its *convex hull* is a convex set containing the feasible set of (MISOCO).

Consider the hyperplanes $\mathcal{A}^= = \{w \in \mathbb{R}^\ell | a_1^\top w = \alpha_1\}$ and $\mathcal{B}^= = \{w \in \mathbb{R}^\ell | a_2^\top w = \alpha_2\}$. In [4] it is shown that the convex hull of $\hat{\mathcal{Q}} \cap (\mathcal{A} \cup \mathcal{B})$ can be obtained by intersecting $\hat{\mathcal{Q}}$ with either a convex cone or a convex cylinder that has the same intersection with the hyperplanes $\mathcal{A}^=$ and $\mathcal{B}^=$ as $\hat{\mathcal{Q}}$. The existence of such a cone or cylinder was conjectured at an early stage of this research. This paper contributes to answering these questions by explicitly parameterizing the family of quadrics that passes through the two intersections, in both the parallel and nonparallel cases. We provide a detailed analysis of the quadrics in this family, and show that they always contain a cone or cylinder that has the same intersection with the hyperplanes $\mathcal{A}^=$ and $\mathcal{B}^=$ as $\hat{\mathcal{Q}}$. Additionally, this analysis presents explicit and efficiently computable formulas for these quadratic cones or cylinders, enabling their efficient use in solving MISOCO problems.

The motivation for this study differs from what has been the common focus in the previous literature about quadrics. In particular, among solid body modelers, some of the common motivations for these studies are performing boundary evaluations, generating images, and calculating mechanical properties [10]. Thus, the focus has been on the study of the intersection of two or more quadrics in dimension three, see e.g., [10, 11, 12, 13, 14, 15]. Our goal is to show the existence of $n-$dimensional quadratic cones or cylinders used to derive conic constraints to tighten the description of the feasible set of (MISOCO).

Throughout the paper, a quadric $\{w \in \mathbb{R}^\ell \mid w^\top P w + 2p^\top w + \rho \leq 0\}$, is represented by the triplet (P, p, ρ), for a matrix $P \in \mathbb{R}^{\ell\times\ell}$, a vector $p \in \mathbb{R}^\ell$, and a scalar $\rho \in \mathbb{R}$. A hyperplane $\{w \in \mathbb{R}^\ell \mid a^\top w = \alpha\}$ is represented by the pair (a, α), for a given $a \in \mathbb{R}^\ell$ and $\alpha \in \mathbb{R}$, and we assume w.l.o.g. that $\|a\| = 1$. For the sake of clarity, these objects will be defined in the context of each section of the paper. Additionally, for a matrix $P \in \mathbb{R}^{\ell\times\ell}$, $P \succ 0$ denotes that P is positive definite and $P \succeq 0$ denotes that P is positive semi-definite. We have that $P = P^{1/2}P^{1/2}$ and $P^{-1} = P^{-1/2}P^{-1/2}$, where $P^{1/2}$ is the unique symmetric square root of a positive definite matrix P. To simplify notation, we define the vector $u_a = P^{-1/2}a$ for a given vector a. On the other hand, when we have the indexed vectors $a_i, i = 1, 2$, we will use the notation $u_i = P^{-1/2}a_i$.

The rest of the paper is structured as follows. In §2, some background material is presented on the geometry of quadrics. The two main sections of the paper are §3 and §4, which deal with the intersections of a quadric with parallel and nonparallel hyperplanes, respectively. In §5 we discuss the scope of the results considering general quadrics. Finally, we present some conclusions and discuss directions of ongoing research in §6.

2 Background

This section discusses some results needed for the analysis developed in §3 and §4. We start by defining the shapes of the quadrics that are considered in this paper. Then, we give some results about the intersection of quadrics and hyperplanes that are used later for proving the main theorems presented in §3 and §4. Readers interested in more general results about the intersection of quadrics in $\mathbb{R}^n$ can consult Cox et al. [16, Chapter 8], and Harris [17, lecture 22]. Finally, for the fundamental results about eigenvalues and quadratic forms used in this section, the interested reader is referred to [18].

2.1 Shapes of a Quadric with at Most One Non-positive Eigenvalue

Here we identify the shapes of the quadrics containing the set

$$\mathcal{G} = \{w \in \mathbb{R}^\ell \mid Aw = b, w \in \mathbb{L}^\ell\},$$

which is the intersection of an affine space and a Lorentz cone. $\mathcal{G}$ is the feasible set of a relaxation of (MISOCO), where the integrality constraint is relaxed and $k = 1$. Let $w^0 \in \mathcal{G}$, and $H_{\ell\times n}$ be a matrix with orthogonal columns that form a basis for the null space of A, where $n = \ell - m$. We assume w.l.o.g. that $\|H_i\| = 1$, $i = 1, \ldots, n$, where H_i is the i-th column of H. Thus, we obtain the identity $\{w \in \mathbb{R}^n \mid Aw = b\} = \{w \in \mathbb{R}^n \mid w = w^0 + Hx, \exists x \in \mathbb{R}^n\}$. Now, let

$$J = \begin{bmatrix} -1 & 0 \\ 0 & I \end{bmatrix}$$

and let us relax the constraint $w \in \mathbb{L}^\ell$ to $w^\top J w \leq 0$. Substituting $w = w^0 + Hx$, we obtain

$$(w^0 + Hx)^\top J(w^0 + Hx) \leq 0 \tag{2.1}$$

$$x^\top H^\top JHx + 2(w^0)^\top JHx + (w^0)^\top Jw^0 \leq 0. \tag{2.2}$$

Define $P = H^\top JH$, $p = H^\top Jw^0$, and $\rho = (w^0)^\top Jw^0$, then from (2.2) we obtain the constraint

$$x^\top Px + 2p^\top x + \rho \leq 0. \tag{2.3}$$

Hence, from (2.1)–(2.3) we have that for every member of $\mathcal{G}$ there is a corresponding element of the quadric

$$\mathcal{Q} = \{x \in \mathbb{R}^n \mid x^\top Px + 2p^\top x + \rho \leq 0\}. \tag{2.4}$$

Additionally, for every $\bar{x} \in \mathcal{Q}$ such that $(w^0 + H\bar{x})_1 \geq 0$ there exists a corresponding element in $\mathcal{G}$. Finally, observe that if $P \succ 0$, the set $\mathcal{Q}$ is an ellipsoid, which is a convex set. In this case, we obtain from (2.3), the inequality $w_1^0(w^0 + Hx)_1 \geq 0$, and the convexity of $\mathcal{Q}$, that for every $\bar{x} \in \mathcal{Q}$ there exists a corresponding element in $\mathcal{G}$

In order to limit the set of shapes to be considered in this paper, we now present a characteristic of the quadric $\mathcal{Q}$.

Lemma 2.1. *The matrix P defining the quadric $\mathcal{Q}$ that contains the projection of $\mathcal{G}$ into the affine space $\{w \in \mathbb{R}^\ell \mid Aw = b\}$ has at most one negative eigenvalue.*

Proof. Recall that the columns H_i, $i = 1, \dots, n$ of H are unit length and orthogonal. Thus, the matrix P has the form $P = I - 2h_{1:}h_{1:}^\top$, where $h_{1:}^\top$ is the first row of H. The rank one matrix $2h_{1:}h_{1:}^\top$ has at most one nonzero eigenvalue. Hence, P has at most one negative eigenvalue. □

We can now define the shapes needed for the analysis of a MISOCO problem. If P is assumed invertible, we can rewrite the defining inequality (2.4) as

$$\left(x + P^{-1}p\right)^\top P \left(x + P^{-1}p\right) \leq p^\top P^{-1}p - \rho. \tag{2.5}$$

Thus, either $P \succ 0$ or P is indefinite with exactly one negative eigenvalue (ID1). One can see that the shape of the quadric $\mathcal{Q}$ is determined by two quantifiers: the inertia of matrix P and the quantity $p^\top P^{-1}p - \rho$. The possible shapes of the quadric are summarized in the following table:

	$p^\top P^{-1}p - \rho$		
	> 0	$= 0$	< 0
P is PD	ellipsoid	point	empty set
P is ID1	hyperboloid of one sheet	cone	hyperboloid of two sheets

Table 2.1: Shapes of the quadric $\mathcal{Q}$ when the matrix P is non-singular.

In all of these cases, either the center of the ellipsoid or the intersection of the asymptotes of the hyperboloids is at $-P^{-1}p$.

Now, let P be positive semi-definite but not positive definite, i.e., the smallest eigenvalue of P is 0. Then, there are two cases:

Case 1: If there is a vector x_c such that $Px_c = -p$, then $\mathcal{Q}$ is:

- **empty**, if $x_c^\top P x_c - \rho < 0$;
- **a line through** x_c in the direction of the eigenvector of the zero eigenvalue of P, if $x_c^\top P x_c - \rho = 0$;
- **a cylinder** with its center line through x_c in the direction of the eigenvector of the zero eigenvalue of P, if $x_c^\top P x_c - \rho > 0$.

Case 2: If there is no vector x_c such that $Px_c = -p$, then $\mathcal{Q}$ is a **paraboloid**.

Finally, if $P \succ 0$ and the quadric $\mathcal{Q}$ is not single point, $\mathcal{Q}$ can be transformed to a unit hypersphere $\{y \in \mathbb{R}^n \mid \|y\|^2 \leq 1\}$ using the affine transformation

$$y = \frac{P^{1/2}(x + P^{-1}p)}{\sqrt{\|u_p\|^2 - \rho}}. \tag{2.6}$$

Observe that this transformation preserves the inertia of P, hence the classification of the quadric is not changed. Additionally, observe that if we apply the same transformation to two parallel hyperplanes, the resulting hyperplanes are still parallel. Hence, throughout the paper, if $P \succ 0$, we assume w.l.o.g. that the quadric $\mathcal{Q}$ is a unit hypersphere centered at the origin.

2.2 Intersections of Quadrics and Hyperplanes

The following lemma enables us to characterize when the intersection of an ellipsoid and a hyperplane is nonempty.

Lemma 2.2. *The intersection of an ellipsoid $\{x \in \mathbb{R}^n \mid x^\top P x + 2p^\top x + \rho \leq 0\}$, for a given matrix $P \in \mathbb{R}^{n\times n}$ such that $P \succ 0$, a vector $p \in \mathbb{R}^n$ and a scalar $\rho \in \mathbb{R}$, with a hyperplane $\{x \in \mathbb{R}^n | a^\top x = \alpha\}$, for some $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, is nonempty if and only if*

$$\left(\alpha + u_a^\top u_p\right)^2 \leq \|u_a\|^2 \left(\|u_p\|^2 - \rho\right).$$

Proof. Let $g : \mathbb{R}^n \to \mathbb{R}$ be defined by $g(x) = x^\top P x + 2p^\top x + \rho, \forall x \in \mathbb{R}^n$. We are interested in characterizing $x \in \mathbb{R}^n$ for which $g(x) = 0$ and $\nabla g(x) = \lambda a$ for some $\lambda \in \mathbb{R}$, i.e., the points at which the hyperplane touches the ellipsoid. The gradient of g is defined by $\nabla g(x) = 2Px + 2p, \forall x \in \mathbb{R}^n$. Then, solving the system $2Px + 2p = \lambda a$ for x, we obtain the solution

$$\hat{x} = \frac{P^{-1}(\lambda a - 2p)}{2}.$$

From the condition $g(\hat{x}) = 0$, we obtain $\lambda^2 \|u_a\|^2 / 4 - \|u_p\|^2 + \rho = 0$, which is a quadratic equation in λ. The roots of this equation are

$$\lambda^+ = 2\sqrt{\frac{\|u_p\|^2 - \rho}{\|u_a\|^2}} \qquad \text{and} \qquad \lambda^- = -2\sqrt{\frac{\|u_p\|^2 - \rho}{\|u_a\|^2}}.$$

Thus, the solutions are

$$\hat{x} = -P^{-1}p \pm P^{-1}a\sqrt{\frac{\|u_p\|^2 - \rho}{\|u_a\|^2}}.$$

Hence, we can compute the two extreme values of α for which a hyperplane (a, α) is tangent to an ellipsoid (P, p, ρ). The values are

$$\alpha = -u_a^\top u_p \pm \sqrt{\|u_a\|^2 \left(\|u_p\|^2 - \rho\right)}.$$

Observe that the set $\{x \in \mathbb{R}^n \mid g(x) \leq 0\}$ is feasible only if $\lambda^+ \leq \lambda \leq \lambda^-$, which gives the bounds in the lemma. □

3 Intersections with Parallel Hyperplanes

In this section we investigate the intersection of an ellipsoid with two parallel hyperplanes. Consider a quadric $\mathcal{Q}$ represented by (P, p, ρ) for some matrix $P \in \mathbb{R}^{n\times n}$, a vector $p \in \mathbb{R}^n$, and $\rho \in \mathbb{R}$. Let $\mathcal{A}^=$ and $\mathcal{B}^=$ two hyperplanes represented by (a, α_1) and (a, α_2) respectively, for some $a \in \mathbb{R}^n$ and $\alpha_1, \alpha_2 \in \mathbb{R}$, where $\|a\| = 1$, and $\alpha_1 \neq \alpha_2$. Additionally, assume that the intersections $\mathcal{Q} \cap \mathcal{A}^=$ and $\mathcal{Q} \cap \mathcal{B}^=$ are nonempty. We first present a theorem that characterizes a family of quadrics having the same intersection with the hyperplanes $\mathcal{A}^=$ and $\mathcal{B}^=$ as the quadric $\mathcal{Q}$. Then, we analyze this family when $P \succ 0$, to show that there is always a quadric that satisfies the definition of either a cone or a cylinder given in §2.1.

3.1 The Family of Quadrics with Fixed Parallel Planar Sections

First, we recall the definition of a pencil of quadrics given in [15].

Definition 3.1. *Consider two given quadrics represented by (P_1, p_1, ρ_1) and (P_2, p_2, ρ_2), for $P_1, P_2 \in \mathbb{R}^{n\times n}$, $p_1, p_2 \in \mathbb{R}^n$ and $\rho_1, \rho_2 \in \mathbb{R}$. The family of quadrics $\{\mathcal{F}(\tau) \mid \tau \in \mathbb{R}\}$ is called a* pencil of quadrics*, where $\mathcal{F}(\tau)$ is represented by $\hat{P}(\tau) = P_1 + \tau P_2$, $\hat{p}(\tau) = p_1 + \tau \tilde{p}_2$, and $\hat{\rho}(\tau) = \rho_1 + \tau\tilde{\rho}_2$.*

Now we characterize a family of quadrics having the same intersection with two hyperplanes $\mathcal{A}^=$ and $\mathcal{B}^=$ as the quadric $\mathcal{Q}$.

Theorem 3.2. *Consider a quadric $\mathcal{Q}$ represented by (P, p, ρ) and two parallel hyperplanes $\mathcal{A}^=$ and $\mathcal{B}^=$, represented by (a, α_1) and (a, α_2) respectively. The uni-parametric family of quadrics having the same intersection with $\mathcal{A}^=$ and $\mathcal{B}^=$ as the quadric $\mathcal{Q}$ is defined by the pencil of quadrics $\{\mathcal{F}(\tau) \mid \tau \in \mathbb{R}\}$, where $\mathcal{F}(\tau)$ is represented by $P(\tau) = P + \tau\tilde{P}$, $p(\tau) = p + \tau\tilde{p}$, and $\rho(\tau) = \rho + \tau\tilde{\rho}$, with*

$$\tilde{P} = aa^\top, \quad \tilde{p} = -\frac{(\alpha_1 + \alpha_2)}{2}\, a, \quad \tilde{\rho} = \alpha_1\alpha_2.$$

Proof. Consider the set $\mathcal{A}^= \cup \mathcal{B}^=$, which can be described as

$$\{x \in \mathbb{R}^l \quad | \quad (a^\top x - \alpha_1)(a^\top x - \alpha_2) = 0\},$$

and observe that

$$(a^\top x - \alpha_1)(a^\top x - \alpha_2) = x^\top aa^\top x - (\alpha_1 + \alpha_2)a^\top x + \alpha_1\alpha_2 = 0. \tag{3.1}$$

Now, let

$$\tilde{P} = aa^\top, \qquad \tilde{p} = -\frac{(\alpha_1 + \alpha_2)}{2}a, \qquad \tilde{\rho} = \alpha_1\alpha_2.$$

Then, the set of solutions of equation (3.1) can be written as a quadric surface $\tilde{\mathcal{Q}}$ represented by $(\tilde{P}, \tilde{p}, \tilde{\rho})$. Now, consider the pencil $\{\mathcal{F}(\tau) \mid \tau \in \mathbb{R}\}$, where $\mathcal{F}(\tau)$ is represented by $\hat{P}(\tau) = P + \tau\tilde{P}$, $\hat{p}(\tau) = p + \tau\tilde{p}$, and $\hat{\rho}(\tau) = \rho + \tau\tilde{\rho}$. Let $\bar{x}$ be a given vector satisfying $\bar{x}^\top\tilde{P}\bar{x} + 2\tilde{p}^\top\bar{x} + \tilde{\rho} = 0$. Then, for $\tau \in \mathbb{R}$ we have $\bar{x} \in \mathcal{F}(\tau)$ if and only if

$$\bar{x}^\top(P + \tau\tilde{P})\bar{x} + 2(p + \tau\tilde{p})^\top\bar{x} + (\rho + \tau\tilde{\rho}) = \bar{x}^\top P\bar{x} + 2p^\top\bar{x} + \rho \leq 0.$$

Hence, we have $\bar{x} \in \mathcal{F}(\tau) \cap (\mathcal{A}^= \cup \mathcal{B}^=)$ if and only if $\bar{x} \in \mathcal{Q} \cap (\mathcal{A}^= \cup \mathcal{B}^=)$ for $\tau \in \mathbb{R}$. □

3.2 Classification of the Family $\{\mathcal{F}(\tau) \mid \tau \in \mathbb{R}\}$

In what follows we assume that the quadric $\mathcal{Q}$ is an ellipsoid, i.e., $P \succ 0$. We can assume w.l.o.g. that the quadric $\mathcal{Q}$ is not a single point, i.e., $\|u_p\|^2 - \rho > 0$, since otherwise $\alpha_1 = \alpha_2$. Now, recall the assumption from the affine transformation (2.6) that $\mathcal{Q}$ is a unit hypersphere centered at the origin. Also, recall that two parallel hyperplanes will remain parallel under the affine transformation (2.6). Then, in this case we have a representation of $\mathcal{F}(\tau)$ defined by

$$P(\tau) = I + \tau aa^\top, \quad p(\tau) = -\tau\frac{\alpha_1 + \alpha_2}{2}a, \quad \rho(\tau) = -1 + \tau\alpha_1\alpha_2. \tag{3.2}$$

It is possible to characterize the behavior of the family $\{\mathcal{F}(\tau) | \tau \in \mathbb{R}\}$ in (3.2) as a function of parameter τ. First, we need a result on the inertia of $P(\tau)$.

Lemma 3.3. *The matrix $P(\tau)$ can be classify as a function of parameter τ as follows:*

- *$P(\tau) \succ 0$ if $\tau > -1$,*
- *$P(\tau) \succeq 0$ but not $P(\tau) \succ 0$ if $\tau = -1$,*
- *$P(\tau)$ is ID1 if $\tau < -1$.*

Proof. The eigenvalues of

$$P(\tau) = \left(I + \tau aa^\top\right),$$

are known [19, 20] to be 1 with multiplicity $n - 1$, and $1 + \tau\|a\|^2$. This proves the lemma. □

We can identify two cases to analyze: $P(\tau)$ is non-singular; $P(\tau)$ singular. In the following sections, we analyze these two cases separately.

3.2.1 $P(\tau)$ is Non-singular

If $\tau \neq -1$, we obtain from Lemma 3.3 that $P(\tau)$ is non-singular, which relates to the cases in Table 2.1 in the background section. Hence, if there exists a τ for which $p(\tau)^\top P(\tau)^{-1}p(\tau) - \bar{\rho}(\tau) = 0$, then $\mathcal{F}(\tau)$ is a cone.

We use the Sherman-Morrison-Woodbury formula [18] to compute the inverse of $P(\tau)$:

$$P(\tau)^{-1} = \left(I + \tau a a^\top\right)^{-1} = I - \frac{\tau}{1+\tau} a a^\top. \tag{3.3}$$

As expected from Lemma 3.3, the inverse does not exist if $\tau = -1$. This case is discussed in §3.2.2.

Now, let us evaluate $p(\tau)^\top P(\tau)^{-1} p(\tau) - \rho(\tau)$. Using (3.3) we have:

$$\begin{aligned} p(\tau)^\top P(\tau)^{-1} p(\tau) - \rho(\tau) &= \frac{\tau^2 (\alpha_1 + \alpha_2)^2}{4} a^\top \left(I - \frac{\tau}{1+\tau} a a^\top \right) a - (\tau \alpha_1 \alpha_2 - 1) \\ &= \frac{4\tau^2 \left(\frac{(\alpha_1+\alpha_2)^2}{4} - \alpha_1\alpha_2 \right) + 4\tau(1 - \alpha_1\alpha_2) + 4}{4(1+\tau)} \\ &= \frac{\tau^2 \frac{(\alpha_1-\alpha_2)^2}{4} + \tau(1-\alpha_1\alpha_2) + 1}{(1+\tau)}. \end{aligned} \tag{3.4}$$

Since $\tau \neq -1$, then the denominator in (3.4) is non-zero. Hence, we need to focus only on the roots of the numerator in (3.4). Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function whose value is

$$f(\tau) = \tau^2 \frac{(\alpha_1 - \alpha_2)^2}{4} + \tau(1 - \alpha_1\alpha_2) + 1, \forall \tau \in \mathbb{R},$$

which is a quadratic function of τ. Let $\bar{\tau}_1$ and $\bar{\tau}_2$ be the roots of f.

The discriminant of f is:

$$(1 - \alpha_1\alpha_2)^2 - 4\left(\frac{(\alpha_1 - \alpha_2)^2}{4} \right) = (1 - \alpha_1^2)(1 - \alpha_2^2). \tag{3.5}$$

Therefore, if $(1 - \alpha_1^2) \geq 0$ and $(1 - \alpha_1^2) \geq 0$, then f has real roots. Thus, from Lemma 2.2 we can conclude that f has real roots when $\mathcal{Q} \cap \mathcal{A}^= \neq \emptyset$ and $\mathcal{Q} \cap \mathcal{B}^= \neq \emptyset$.

Now, since the two hyperplanes are distinct, the coefficient of τ^2 in f is positive. For the coefficient of τ in $f(\tau)$ we have $1 - \alpha_1\alpha_2 \geq 0$, where the inequality is implied by the assumption that $\mathcal{Q} \cap \mathcal{A}^= \neq \emptyset$ and $\mathcal{Q} \cap \mathcal{B}^= \neq \emptyset$. This shows that all three coefficients in $f(\tau)$ are non-negative. Hence, we have $\bar{\tau}_1 < 0$ and $\bar{\tau}_2 < 0$.

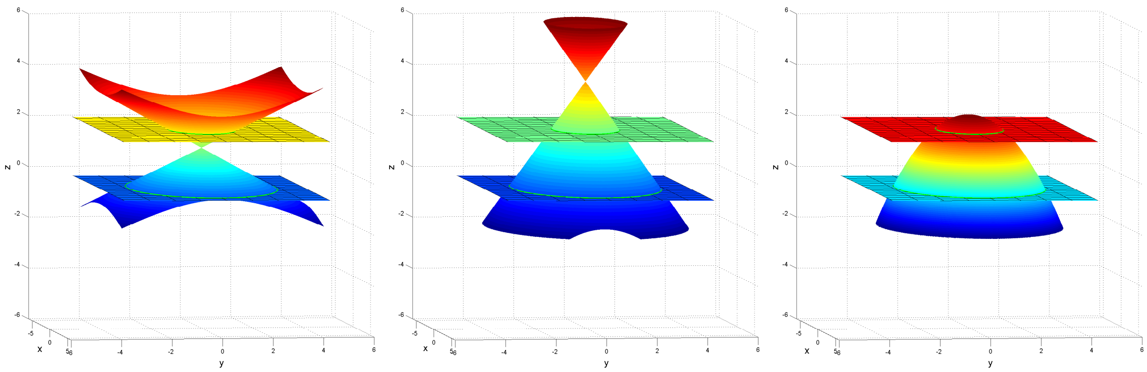
Let us see how the two roots of f compare to -1, at which value $P(\tau)$ becomes singular. We have

$$f(-1) = \frac{(\alpha_1 - \alpha_2)^2}{4} - (1 - \alpha_1\alpha_2) + 1 = \frac{(\alpha_1 + \alpha_2)^2}{4} \geq 0 \tag{3.6}$$

and -1 is not between the two roots of f. Next, we check the derivative of f to decide on which branch of f the value -1 lies. We have

$$f'(-1) = -\frac{(\alpha_1 - \alpha_2)^2}{2} + 1 - \alpha_1\alpha_2 = 1 - \frac{\alpha_1^2 + \alpha_2^2}{2} \geq 0,$$

where the inequality follows from the assumption that $\mathcal{Q} \cap \mathcal{A}^= \neq \emptyset$ and $\mathcal{Q} \cap \mathcal{B}^= \neq \emptyset$. This shows that $\bar{\tau}_1 < -1$ and $\bar{\tau}_2 < -1$. As a result, $\mathcal{F}(\bar{\tau}_1)$ and $\mathcal{F}(\bar{\tau}_1)$ are both cones.

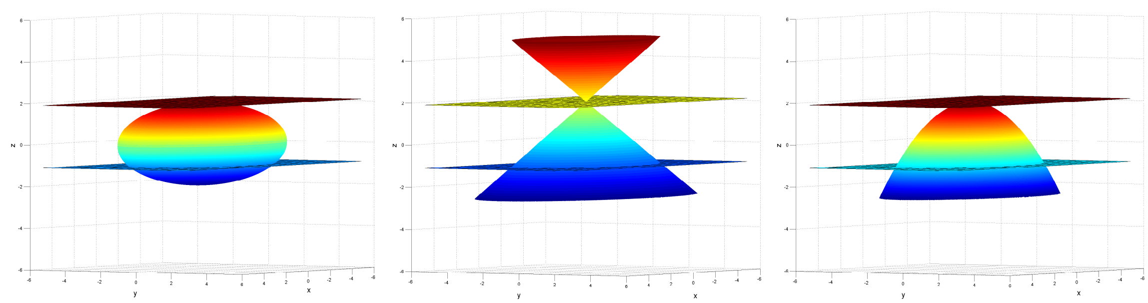


(a) $\tau = \bar{\tau}_2$ (b) $\tau = \bar{\tau}_2$ (c) $\tau = \hat{\tau}$

Figure 1: $f(\tau)$ has two distinct roots which do not coincide with $\hat{\tau}$.

Summary of Shapes According to the values of the discriminant (3.5) we can classify the shapes of $\mathcal{F}(\tau)$ at the roots of f. Recall that $\tau \neq -1$, and $\bar{\tau}_1 \neq -1$, $\bar{\tau}_2 \neq -1$. We may further assume w.l.o.g. that $\bar{\tau}_1 \leq \bar{\tau}_2$. We have the following cases:

- If the discriminant (3.5) is not equal to zero, then $-1 > \bar{\tau}_2 > \bar{\tau}_1$, and there are two different cones at $\tau = \bar{\tau}_1$ and $\tau = \bar{\tau}_2$ in the family $\mathcal{F}(\tau)$. For illustrations see Figure 1.
- If the discriminant(3.5) is equal to zero, then $-1 > \bar{\tau}_2 = \bar{\tau}_1$, and there is a unique cone in the family $\mathcal{F}(\tau)$ at $\tau = \bar{\tau}_1 = \bar{\tau}_2$. Observe that in this case, it follows by Lemma 2.2 that one of the hyperplanes is tangent to the ellipsoid. See Figure 2.

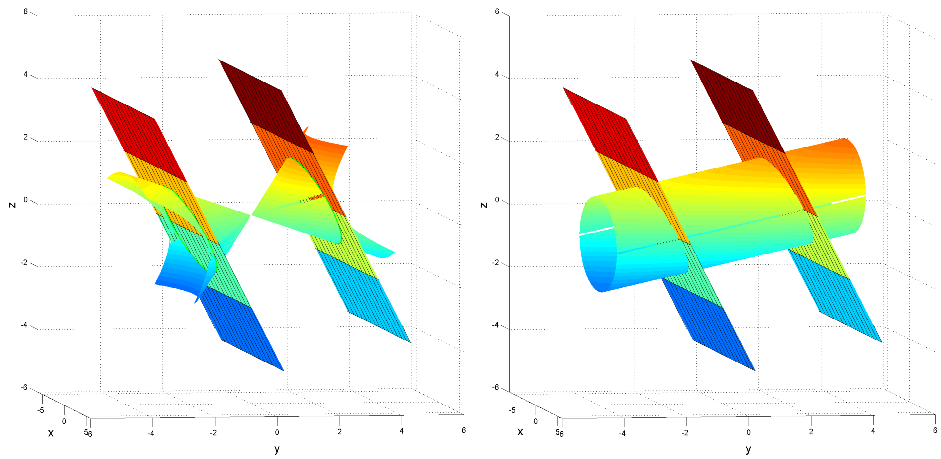


(a) One hyperplane is tangent to the ellipsoid. (b) $\tau = \bar{\tau}_1 = \bar{\tau}_2$ (c) $\tau = -1$

Figure 2: The two roots of $f(\tau)$ coincide, but are different from $\hat{\tau}$.

3.2.2 $P(\tau)$ is Singular

It follows from Lemma 3.3 that $P(\tau)$ is singular when $\tau = -1$. In this case we have that $P(-1) \succeq 0$ but not $P(-1) \succ 0$. Thus, from §2.1 we have that $\mathcal{F}(-1)$ is either a line, a cylinder, or a paraboloid. The shape of $\mathcal{F}(-1)$ can be decided by verifying if $p(-1)$ is in the range of $P(-1)$. This is equivalent



(a) $\tau = \bar{\tau}_1$

(b) $\tau = \hat{\tau}$

Figure 3: $f(\tau)$ has two distinct roots, but the larger root coincides with $\hat{\tau}$.

to deciding if $p(-1)$ is orthogonal to the eigenvector corresponding to the zero eigenvalue of $P(-1)$. One can verify easily that a is such an eigenvector of $P(-1)$, thus we need to check if $p(-1)^\top a = 0$. We have

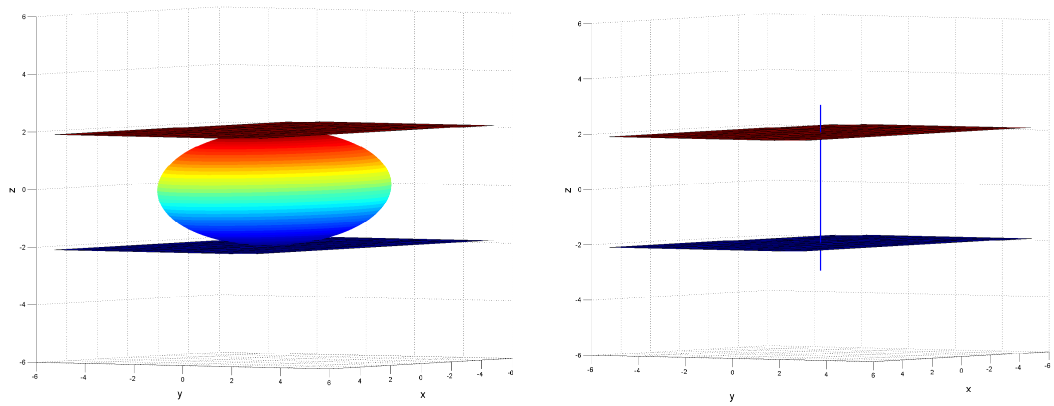
$$p(-1)^\top a = \left(\frac{\alpha_1 + \alpha_2}{2} a\right)^\top a = \frac{\alpha_1 + \alpha_2}{2}. \tag{3.7}$$

Hence, $p(-1)^\top a$ is zero if and only if $\alpha_1 = -\alpha_2$, i.e., the two hyperplanes $\mathcal{A}^=$ and $\mathcal{B}^=$ are symmetric about the center of the hypersphere $\mathcal{Q}$. Therefore, if $\alpha_1 = -\alpha_2$ any vector $x_c = \eta a$, $\eta \in \mathbb{R}$, satisfies the condition $P(-1)x_c = p(-1)$ of **Case 1** in §2.1. To the contrary, if $\alpha_1 \neq -\alpha_2$, then $p(-1)$ is not orthogonal to a, and there is no x_c such that $P(-1)x_c = -p(-1)$. Recall that this is true because a is an eigenvector corresponding to the zero eigenvalue of $P(-1)$. Then, from **Case 2** in §2.1 we conclude that $\mathcal{F}(-1)$ is a paraboloid. These are the cases discussed in §3.2.1. For illustrations, see Figures 1(c) and 2(c).

Summary of Shapes Based on equation (3.7) and the values of the discriminant (3.5) we can classify the shapes of $\mathcal{F}(\tau)$ at -1, $\bar{\tau}_1$, $\bar{\tau}_2$ when $p(-1)^\top a = 0$. We may assume w.l.o.g. that $\bar{\tau}_1 \leq \bar{\tau}_2$. We have the following cases:

- If the discriminant (3.5) is not equal to zero and $-1 = \bar{\tau}_2 > \bar{\tau}_1$, then for the vector $x_c = 0$ we obtain from (3.7) that $x_{\rm c}^\top P(-1)x_{\rm c} - \rho(-1) = (1 - \alpha_1^2) > 0$, and from **Case 1** in §2.1 we have that $\mathcal{F}(-1)$ is a cylinder. Additionally, $\mathcal{F}(\bar{\tau}_1)$ is a cone. For illustrations see Figure 3.

- If the discriminant (3.5) is zero and $-1 = \bar{\tau}_2 = \bar{\tau}_1$, then for the vector $x_c = 0$ from (3.7) we obtain that $x_{\rm c}^\top P(-1)x_{\rm c} - \rho(-1) = (1 - \alpha_1^2) = 0$, and from **Case 1** in §2.1 we have that $\mathcal{F}(-1)$ is a line. For illustrations see Figure 4.



(a) The two hyperplanes are tangent to the ellipsoid.

(b) $\tau = \hat{\tau}$

Figure 4: The two roots of $f(\tau)$ coincide with $\hat{\tau}$.

3.2.3 Summarizing the Shapes of $\mathcal{F}(\tau)$

We can summarize the shapes of the quadrics in the family $\{\mathcal{F}(\tau) \mid \tau \in \mathbb{R}\}$ using $\bar{\tau}_1$, and $\bar{\tau}_2$ in the following theorem. We assume w.l.o.g. that $\bar{\tau}_1 \leq \bar{\tau}_2$.

Theorem 3.4. *The following cases may occur for the shape of $\mathcal{F}(\tau)$:*

- $\bar{\tau}_1 < \bar{\tau}_2 < -1$: *$\mathcal{F}(\hat{\tau})$ is a paraboloid, and $\mathcal{F}(\bar{\tau}_1)$, $\mathcal{F}(\bar{\tau}_2)$ are two cones.*
- $\bar{\tau}_1 = \bar{\tau}_2 < -1$: *$\mathcal{F}(\hat{\tau})$ is a paraboloid and $\mathcal{F}(\bar{\tau}_1)$ is a cone.*
- $\bar{\tau}_1 < \bar{\tau}_1 = -1$: *$\mathcal{F}(\hat{\tau})$ is a cylinder and $\mathcal{F}(\bar{\tau}_1)$ is cone.*
- $\bar{\tau}_1 = \bar{\tau}_2 = -1$: *$\mathcal{F}(\hat{\tau})$ is a line.*

This completes the description of the family $\{\mathcal{F}(\tau) \mid \tau \in \mathbb{R}\}$ of quadrics when $P \succ 0$ and $\mathcal{A}^=$ and $\mathcal{B}^=$ are parallel.

4 Intersections with Nonparallel Hyperplanes

In this section, we investigate the intersection of an ellipsoid with two non-parallel hyperplanes. Consider a quadric $\mathcal{Q}$ represented by (P, p, ρ) for some matrix $P \in \mathbb{R}^{n \times n}$, a vector $p \in \mathbb{R}^n$, and $\rho \in \mathbb{R}$. Let $\mathcal{A}^=$ and $\mathcal{B}^=$ be two non-parallel hyperplanes represented by (a_1, α_1) and (a_2, α_2) respectively, for some $a_1, a_2 \in \mathbb{R}^n$ and $\alpha_1, \alpha_2 \in \mathbb{R}$, with $\|a_1\| = \|a_2\| = 1$, and $a_1^\top a_2 \neq \pm 1$. Additionally, assume that the intersections $\mathcal{Q} \cap \mathcal{A}^=$ and $\mathcal{Q} \cap \mathcal{B}^=$ are nonempty. We first present a generalization of Theorem 3.2 to the case of non-parallel hyperplanes. Then, we analyze the behavior of the new family of quadrics when $P \succ 0$, to show that there is always a quadric that satisfies the definition of either a cone or a cylinder given in §2.1.

4.1 The Family of Quadrics with Fixed Planar Sections

Theorem 4.1. *Consider a quadric $\mathcal{Q}$ represented by (P, p, ρ) and two non-parallel hyperplanes $\mathcal{A}^=$, $\mathcal{B}^=$ represented by (a_1, α_1) and (a_2, α_2) respectively. The uni-parametric family of quadrics parametrized by $\tau \in \mathbb{R}$ and having the same intersection with $\mathcal{A}^=$ and $\mathcal{B}^=$ as the quadric $\mathcal{Q}$ is defined by the pencil of quadrics $\{\mathcal{F}(\tau) \mid \tau \in \mathbb{R}\}$, where $\mathcal{F}(\tau)$ is represented by $P(\tau) = P + \tau\tilde{P}$, $p(\tau) = p + \tau\tilde{p}$, and $\rho(\tau) = \rho + \tau\tilde{\rho}$, with*

$$\tilde{P} = \frac{a_1 a_2^\top + a_2 a_1^\top}{2}, \qquad \tilde{p} = -\frac{\alpha_2 a_1 + \alpha_1 a_2}{2}, \qquad \tilde{\rho} = \alpha_1\alpha_2.$$

Proof. Consider the set $\mathcal{A}^= \cup \mathcal{B}^=$, which can be described as

$$\{x \in \mathbb{R}^l \mid (a_1^\top x - \alpha_1)(a_2^\top x - \alpha_2) = 0\},$$

and observe that

$$\begin{aligned}(a_1^\top x - \alpha_1)(a_2^\top x - \alpha_2) &= x^\top a_1 a_2^\top x - (\alpha_1 a_2^\top + \alpha_2 a_1^\top)x + \alpha_1\alpha_2 \\ &= x^\top \left(\frac{a_1 a_2^\top + a_2 a_1^\top}{2}\right) x - (\alpha_1 a_2^\top + \alpha_2 a_1^\top)x + \alpha_1\alpha_2 = 0.\end{aligned}$$

Now, let

$$\tilde{P} = \frac{a_1 a_2^\top + a_2 a_1^\top}{2}, \qquad \tilde{p} = -\frac{(\alpha_1 a_2 + \alpha_2 a_1)}{2}, \qquad \tilde{\rho} = \alpha_1\alpha_2.$$

Then, the set of solutions of the equation (3.1) can be described by the quadric surface $\tilde{\mathcal{Q}} = (\tilde{Q}, \tilde{q}, \tilde{\rho})$. Now, consider the pencil $\{\mathcal{F}(\tau) \mid \tau \in \mathbb{R}\}$, where $\mathcal{F}(\tau)$ is represented by $(P, p, \rho) + \tau(\tilde{P}, \tilde{p}, \tilde{\rho}) = 0$. Let $\bar{x}$ be a given vector satisfying $\bar{x}^\top \tilde{P}\bar{x} + 2\tilde{p}^\top\bar{x} + \tilde{\rho} = 0$. Then, for $\tau \in R$ we have $\bar{x} \in \mathcal{F}(\tau)$ if and only if

$$\bar{x}^\top(P + \tau\tilde{P})\bar{x} + 2(p - \tau\tilde{p})^\top\bar{x} + (\rho + \tau\tilde{\rho}) = \bar{x}^\top P\bar{x} + 2p^\top\bar{x} + \rho \leq 0.$$

Hence, we have $\bar{x} \in \mathcal{F}(\tau) \cap (\mathcal{A}^= \cup \mathcal{B}^=)$ if and only if $\bar{x} \in \mathcal{Q} \cap (\mathcal{A}^= \cup \mathcal{B}^=)$ for $\tau \in \mathbb{R}$. $\square$

Remark 4.2. *In particular, if $a_1 = a_2 = a$, this simplifies to the result of Theorem 3.2.*

4.2 Classification of the Family $\{\mathcal{F}(\tau) \mid \tau \in \mathbb{R}\}$

In what follows, we assume that the quadric $\mathcal{Q}$ is an ellipsoid, i.e., $P \succ 0$. If not said otherwise, we assume that the quadric $\mathcal{Q} = (P, p, \rho)$ is not a single point, i.e., $(\|u_p\|^2 - \rho) > 0$. Recall the assumption from the affine transformation (2.6) that $\mathcal{Q}$ is a unit hypersphere centered at the origin. In this case we have the following simplified representation of $\mathcal{F}(\tau)$, defined by

$$P(\tau) = I + \tau\frac{a_1 a_2^\top + a_2 a_1^\top}{2}, \quad p(\tau) = -\tau\frac{\alpha_2 a_1 + \alpha_1 a_2}{2}, \quad \rho(\tau) = -1 + \tau\alpha_1\alpha_2. \tag{4.1}$$

We characterize the behavior of the family $\{\mathcal{F}(\tau) \mid \tau \in \mathbb{R}\}$ in (4.1) as a function of parameter τ. First, we discuss the inertia of $P(\tau)$. Then, we analyze the cases: (1) when the matrix $P(\tau)$ is non-singular and (2) when the matrix $P(\tau)$ is singular. Finally, we present a summary of the shapes of the family $\{\mathcal{F}(\tau) \mid \tau \in \mathbb{R}\}$ in Theorem 4.5.

4.2.1 The Eigenvalues of $P(\tau)$

The most important factor in deciding the shape of $\mathcal{F}(\tau)$ is the number of negative or zero eigenvalues of $P(\tau)$. Since P is modified with a rank-2 matrix in (4.1), $P(\tau)$ may possibly have two negative eigenvalues. The following lemma shows that this can not happen when $P \succ 0$.

Lemma 4.3. *If $P \succ 0$, then $P(\tau)$ can have at most one non-positive eigenvalue.*

Proof. The eigenvalues of

$$P(\tau) = \left(I + \frac{\tau}{2}\left(a_1 a_2^\top + a_2 a_1^\top\right)\right), \tag{4.2}$$

are as follows:

- 1 is an eigenvalue with multiplicity $n-2$, the corresponding eigenvectors are orthogonal to a_1 and a_2;
- $1 + \frac{\tau}{2}\left(a_1^\top a_2 + 1\right)$, with the eigenvector $(a_1 + a_2)$;
- $1 + \frac{\tau}{2}\left(a_1^\top a_2 - 1\right)$, with the eigenvector $(a_2 - a_1)$;

Let

$$\hat{\tau}_1 = \frac{-2}{a_1^\top a_2 + 1} \tag{4.3a}$$

$$\hat{\tau}_2 = \frac{-2}{a_1^\top a_2 - 1}, \tag{4.3b}$$

then using the Cauchy-Schwartz inequality

$$\left|a_1^\top a_2\right| \leq \|a_1\| \, \|a_2\|, \tag{4.4}$$

we can see that $\hat{\tau}_1 < 0 < \hat{\tau}_2$. This implies that $\hat{P}(\tau)$ is positive definite if $\tau \in (\hat{\tau}_1, \hat{\tau}_2)$. It has a zero eigenvalue if $\tau = \hat{\tau}_1$ or $\tau = \hat{\tau}_2$, and it is indefinite with exactly one negative eigenvalue otherwise. ☐

From Lemma 4.3 we have that the possible shapes for $\mathcal{F}(\tau)$ are still only those given in §2.1. We distinguish two cases: $P(\tau)$ is non-singular, and $P(\tau)$ is singular. In the following sections, we analyze these two cases separately.

4.2.2 $P(\tau)$ is Non-singular

If $\tau \neq \hat{\tau}_1, \hat{\tau}_2$, then it follows from Lemma 4.3 that $P(\tau)$ is non-singular, which restricts the quadrics to the shapes in Table 2.1. Hence, to verify the existence of a cone in the family $\mathcal{F}(\tau)$, it is necessary to identify a τ for which $p(\tau)^\top P(\tau)^{-1} p(\tau) - \rho(\tau) = 0$.

We use the Sherman-Morrison-Woodbury formula [18] to compute the inverse of $P(\tau)$:

$$\begin{aligned} P^{-1}(\tau) &= \left(I + [a_1, a_2] \begin{bmatrix} 0 & \frac{\tau}{2} \\ \frac{\tau}{2} & 0 \end{bmatrix} [a_1, a_2]^\top \right)^{-1} \\ &= I - \frac{[a_1, a_2] \begin{bmatrix} \tau^2 & -2\tau - \tau^2 a_1^\top a_2 \\ -2\tau - \tau^2 a_1^\top a_2 & \tau^2 \end{bmatrix} [a_1, a_2]^\top}{\tau^2 \left(1 - (a_1^\top a_2)^2\right) - 4a_1^\top a_2 \tau - 4} \\ &= I - \frac{(a_1 a_1^\top + a_2 a_2^\top)\tau^2 - (a_1^\top a_2 \tau^2 + 2\tau)(a_2 a_1^\top + a_1 a_2^\top)}{\tau^2 \left(1 - (a_1^\top a_2)^2\right) - 4a_1^\top a_2 \tau - 4}. \end{aligned} \tag{4.5}$$

Note that the roots of denominator of the second term in (4.5) are $\hat{\tau}_1$ and $\hat{\tau}_2$. These are the values for which $P(\tau)$ is not invertible, as was expected from Lemma 4.3.

Now, we evaluate $p(\tau)^\top P^{-1}(\tau)p(\tau) - \rho(\tau)$, and substituting $p(\tau)$, $P^{-1}(\tau)$, and $\rho(\tau)$ from (4.1) we obtain

$$\begin{aligned} &p(\tau)^\top P^{-1}(\tau)p(\tau) - \rho(\tau) \\ &= \frac{\left((1-\alpha_1^2)(1-\alpha_2^2) - (\alpha_1\alpha_2 - a_1^\top a_2)^2\right)\tau^2 + 4(\alpha_1\alpha_2 - a_1^\top a_2)\tau - 4}{\tau^2 \left(1 - (a_1^\top a_2)^2\right) - 4a_1^\top a_2 \tau - 4}. \end{aligned} \tag{4.6}$$

Recall that the denominator of (4.6) is non-zero if $\tau \neq \hat{\tau}_1, \hat{\tau}_2$, then we need to focus only on its numerator. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function whose value is

$$f(\tau) = \left((1-\alpha_1^2)(1-\alpha_2^2) - (\alpha_1\alpha_2 - a_1^\top a_2)^2\right)\tau^2 + 4(\alpha_1\alpha_2 - a_1^\top a_2)\tau - 4, \forall \tau \in \mathbb{R},$$

which is a quadratic function of τ. The discriminant of f is

$$16(1-\alpha_1^2)(1-\alpha_2^2). \tag{4.7}$$

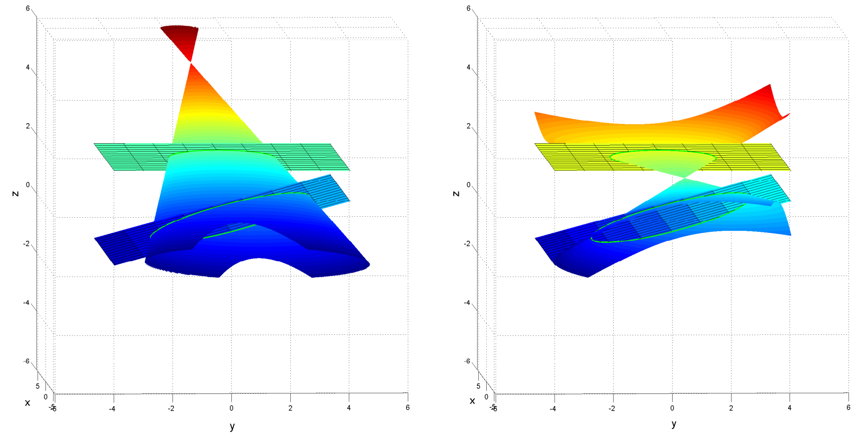
Thus, from Lemma 2.2 we know that f has real roots if $\mathcal{Q} \cap \mathcal{A}^= \neq \emptyset$ and $\mathcal{Q} \cap \mathcal{B}^= \neq \emptyset$. Let the roots of f be denoted by $\bar{\tau}_1$ and $\bar{\tau}_2$. We assume w.l.o.g. that $\bar{\tau}_1 \leq \bar{\tau}_2$.

Summary of Shapes We need to compare the roots of f to $\hat{\tau}$ and $\hat{\tau}_2$ to characterize the shapes of $\mathcal{F}(\tau)$. Recall that $\tau \neq \hat{\tau}_1, \hat{\tau}_2$. We first analyze the case $\hat{\tau}_1 < \bar{\tau}_i < \hat{\tau}_2$, for some $i = 1, 2$. In such case it follows from Lemma 4.3 that $P(\bar{\tau}_i) \succ 0$. Now, since

$$p(\bar{\tau}_i)^\top P^{-1}(\bar{\tau}_i)\bar{p}(\bar{\tau}_i) - \rho(\bar{\tau}_i) = 0,$$

from Table 2.1 in §2.1 we know that $\mathcal{F}(\bar{\tau}_i)$ is a point. This is possible only if $\mathcal{Q}$ is a point, because $\mathcal{A}^=$ and $\mathcal{B}^=$ are non-parallel and $\mathcal{Q} \cap (\mathcal{A}^= \cup \mathcal{B}^=) = \mathcal{F}(\tau) \cap (\mathcal{A}^= \cup \mathcal{B}^=)$. This implies that $p^\top P^{-1} p - \rho = 0$ and $\alpha_1 = \alpha_2 = 0$. Hence, $p(\tau) = 0$ and $\rho(\tau) = 0$ for any $\tau \in \mathbb{R}$, which simplifies the characterization of all the shapes of $\mathcal{F}(\tau)$ for $\tau \in \mathbb{R}$. First, for any $\hat{\tau}_1 < \tau < \hat{\tau}_2$ the quadric $\mathcal{F}(\tau)$ is a point. Second, the identity $-P(\hat{\tau}_i)0 = p(\hat{\tau}_i)$ holds for $\hat{\tau}_1$ and $\hat{\tau}_2$. Thus, it follows from **Case 1** in §2.1 that the quadrics $\bar{\mathcal{F}}(\hat{\tau}_i)$, $i = 1, 2$, are two lines. Finally, for $\tau < \hat{\tau}_1$ and $\tau > \hat{\tau}_2$, the quadrics $\mathcal{F}(\hat{\tau})$ are cones.

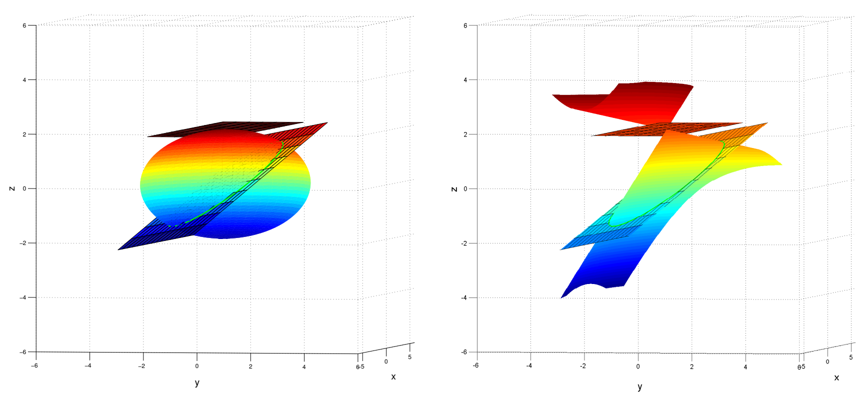
Now, if $\bar{\tau}_i \notin (\hat{\tau}_1, \hat{\tau}_2)$, $i = 1, 2$, the shapes of $\mathcal{F}(\tau)$ depend on the value of the discriminant (4.7). We have the following cases:



(a) $\tau = \bar{\tau}_1$ (b) $\tau = \bar{\tau}_2$

Figure 5: f has two distinct roots $\bar{\tau}_1$, $\bar{\tau}_1$ which are not coinciding with $\hat{\tau}_1$ and $\hat{\tau}_2$.

- If the discriminant (4.7) of f is not equal to zero, then $\hat{\tau}_2 < \bar{\tau}_1 < \bar{\tau}_2$, or $\bar{\tau}_1 < \bar{\tau}_2 < \hat{\tau}_1$, or $\bar{\tau}_1 < \hat{\tau}_1 < \hat{\tau}_2 < \bar{\tau}_2$. In these cases we have that $\mathcal{F}(\hat{\tau}_1)$ and $\mathcal{F}(\hat{\tau}_2)$ are two paraboloids, and $\bar{\mathcal{F}}(\bar{\tau}_1)$ and $\mathcal{F}(\bar{\tau}_2)$ are two different cones. For illustrations see Figure 5.
- If the discriminant (4.7) of f is zero, then $\bar{\tau}_1 = \bar{\tau}_2 < \hat{\tau}_1$ or $\hat{\tau}_2 < \bar{\tau}_1 = \bar{\tau}_2$. In these cases $\mathcal{F}(\hat{\tau}_1)$ and $\mathcal{F}(\hat{\tau}_2)$ are two paraboloids, and there is a unique cone $\mathcal{F}(\bar{\tau}_1) = \mathcal{F}(\bar{\tau}_2)$. Observe that in these cases one of the hyperplanes must be tangent to the hypersphere $\mathcal{Q}$. For illustrations see Figure 6.



(a) One hyperplane tangent to the ellipsoid (b) $\tau = \bar{\tau}_1 = \bar{\tau}_2$

Figure 6: The two roots of f coincide, but are different from $\hat{\tau}_1$ and $\hat{\tau}_2$.

4.2.3 $P(\tau)$ is Singular

If $\tau = \hat{\tau}_1$ or $\tau = \hat{\tau}_2$, then it follows from Lemma 4.3 that $p(\hat{\tau}_i)$, $i = 1, 2$ is singular. In this case we have $p(\hat{\tau}_i) \succeq 0$ but not $p(\hat{\tau}_i) \succ 0$. Thus, from §2.1 we have that $\mathcal{F}(\hat{\tau}_i)$ is either a line, a cylinder, or a paraboloid. The shape of $\mathcal{F}(\tau_i)$ can be decided by verifying if $p(\hat{\tau}_i)$ is in the range of $p(\hat{\tau}_i)$. This happens exactly when $p(\hat{\tau}_i)$ is orthogonal to the eigenvector corresponding to the zero eigenvalue of $p(\hat{\tau}_i)$. Lemma 4.4 provides the zero eigenvectors of $\mathcal{F}(\tau_i)$, $i = 1, 2$.

Lemma 4.4. *The eigenvector for the zero eigenvalue of $p(\hat{\tau}_1)$ is $(a_2 + a_1)$, and for the zero eigenvalue of $P(\hat{\tau}_2)$ is $(a_2 - a_1)$.*

Proof. For $p(\hat{\tau}_1)$, direct computation yields

$$\begin{aligned} p(\hat{\tau}_1)\,(a_2 + a_1) &= \left(I - \frac{a_1 a_2^\top + a_2 a_1^\top}{a_1^\top a_2 + 1}\right)(a_2 + a_1) \\ &= (a_2 + a_1) - \frac{(a_2 + a_1)\left(a_1^\top a_2 + 1\right)}{a_1^\top a_2 + 1} = 0, \end{aligned}$$

and similarly for $p(\hat{\tau}_2)$, we obtain

$$\begin{aligned} p(\hat{\tau}_2)\,(a_2 - a_1) &= \left(I - \frac{a_1 a_2^\top + a_2 a_1^\top}{a_1^\top a_2 - 1}\right)(a_2 - a_1) \\ &= (a_2 - a_1) - \frac{(a_2 - a_1)\left(a_1^\top a_2 - 1\right)}{a_1^\top a_2 - 1} = 0. \end{aligned}$$

Recall that a_1 and a_2 are linearly independent, thus the two eigenvectors are different from the zero vector. This completes the proof. $\square$

Now we can compute the inner product of these eigenvectors with $p(\hat{\tau}_1)$ and $p(\hat{\tau}_2)$. Consider first $p(\hat{\tau}_1)$, then we obtain:

$$p(\hat{\tau}_1)^\top (a_1 + a_2) = \frac{(\alpha_1 a_2^\top + \alpha_2 a_1^\top)\,(a_1 + a_2)}{a_1^\top a_2 + 1} = \alpha_1 + \alpha_2. \tag{4.8}$$

For the case $p(\hat{\tau}_2)$ we obtain:

$$p(\hat{\tau}_2)^\top (a_2 - a_1) = \frac{(\alpha_1 a_2^\top + \alpha_2 a_1^\top)\,(a_2 - a_1)}{a_1^\top a_2 - 1} = \alpha_2 - \alpha_1. \tag{4.9}$$

Recall that if (4.8) or (4.9) is not zero, then we have that either $-p(\hat{\tau}_1)$ is not in the range of $p(\hat{\tau}_1)$ or $-p(\hat{\tau}_2)$ is not in the range of $p(\hat{\tau}_2)$, or we have both cases. Hence, from **Case 2** in §2.1 either $\mathcal{F}(\hat{\tau}_1)$ or $\mathcal{F}(\hat{\tau}_2)$ is a paraboloid, or both are paraboloids.

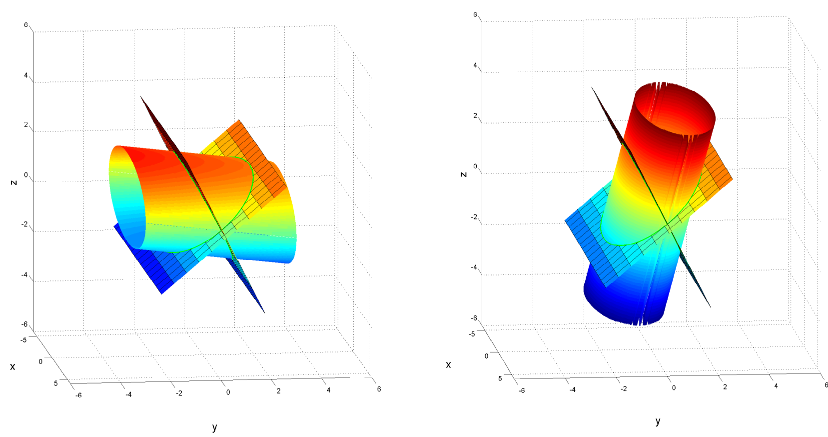
Summary of Shapes We use the discriminant of f in (4.7) to classify the remaining cases of $\mathcal{F}(\tau)$ at $\hat{\tau}_1$, $\hat{\tau}_2$, $\bar{\tau}_1$, and $\bar{\tau}_2$. Recall that $\bar{\tau}_1 \leq \bar{\tau}_2$ and $\hat{\tau}_1 \leq \hat{\tau}_2$. Then, we have the following cases:

- If the discriminant (4.7) of f is not equal to zero, then we need to consider two possibilities:

◇ $\hat{\tau}_1 = \bar{\tau}_1$ and $\hat{\tau}_2 = \bar{\tau}_2$, which is illustrated in Figure 7. If $f(\hat{\tau}_1) = f(\hat{\tau}_2) = 0$, then

$$(\alpha_1 + \alpha_2)^2 = 0 \tag{4.10}$$
$$(\alpha_1 - \alpha_2)^2 = 0, \tag{4.11}$$

which implies that $\alpha_1 = \alpha_2 = 0$, i.e., both hyperplanes intersect at the origin, which is the center of $\mathcal{Q}$. Hence, for the vector $x_c = 0$ the identity $-p(\hat{\tau}_i)x_c = \bar{p}(\hat{\tau}_i)$ holds for $\hat{\tau}_1$ and $\hat{\tau}_2$. Furthermore, since $p(\hat{\tau}_i) = 0$ and $\bar{\rho}(\hat{\tau}_i) = -1$, then $p(\hat{\tau}_i)^\top p(\hat{\tau}_i)p(\hat{\tau}_i) - \rho(\hat{\tau}_i) = 1 > 0$ for $i = 1, 2$. Thus, it follows from **Case 1** in §2.1 that the quadrics $\mathcal{F}(\hat{\tau}_i)$, $i = 1, 2$, are two cylinders.



(a) $\tau = \bar{\tau}_1 = \hat{\tau}_1$ (b) $\tau = \bar{\tau}_2 = \hat{\tau}_2$

Figure 7: $\bar{\tau}_1 \neq \bar{\tau}_2$, and $\bar{\tau}_1 = \hat{\tau}_1$, $\bar{\tau}_2 = \hat{\tau}_2$.

◇ Exactly one of the roots $\bar{\tau}_1, \bar{\tau}_2$ is equal to either $\hat{\tau}_1$ or $\hat{\tau}_2$, which is illustrated in Figure 8. Recall that if the discriminant is not equal to zero, then $|\alpha_1| < 1$ and $|\alpha_2| < 1$, i.e., neither of the hyperplanes $\mathcal{A}^=$ or $\mathcal{B}^=$ are tangent to $\mathcal{Q}$. Assume that one of the roots is equal to $\hat{\tau}_1$. It follows from equations (4.10) and (4.8) that $\alpha_1 = -\alpha_2$, and that $(a_1 + a_2)$ is orthogonal to $p(\hat{\tau}_1)$. Now, let

$$x_c = \frac{\alpha_2}{2}(a_2 - a_1). \tag{4.12}$$

Then, we have

$$\begin{aligned} p(\hat{\tau}_1)x_c &= \left(I - \frac{(a_1 a_2^\top + a_2 a_1^\top)}{(a_1^\top a_2 + 1)}\right)\left(\frac{\alpha_2(a_2 - a_1)}{2}\right) \\ &= -\frac{\alpha_2(a_1 - a_2)}{(a_1^\top a_2 + 1)} = -p(\hat{\tau}_1). \end{aligned} \tag{4.13}$$

Additionally, for the choice of x_c in (4.12) we have that

$$x_c^\top p(\hat{\tau}_1)x_c - \rho(\hat{\tau}_1) = \frac{\alpha_2^2(a_2 - a_1)^\top(a_2 - a_1)}{2(a_1^\top a_2 + 1)} - \rho(\hat{\tau}_1) = 1 - \alpha_2^2 > 0, \tag{4.14}$$

where the strict inequality follows since $\mathcal{B}^{=}$ is non tangent to $\mathcal{Q}$. As a result, from **Case 1** in §2.1 we obtain that the quadric $\mathcal{F}(\hat{\tau}_1)$ is a cylinder.

Similarly, when one of the roots equals $\hat{\tau}_2$, we can choose

$$x_c = \frac{\alpha_2}{2}\left(a_2 + a_1\right). \tag{4.15}$$

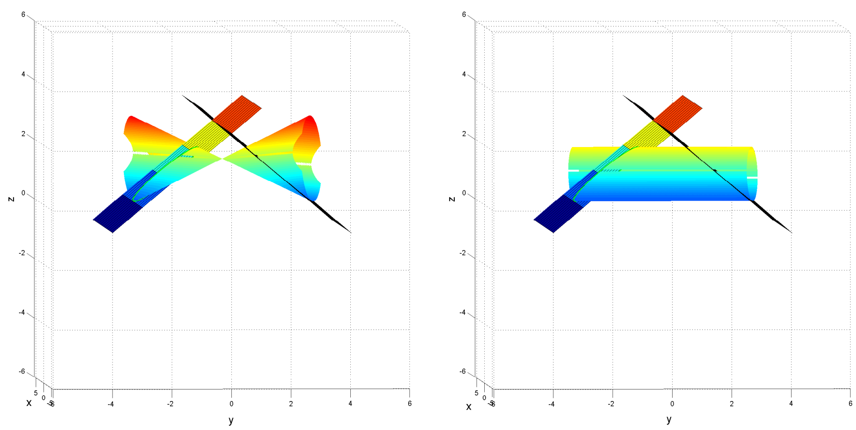
In this case, it follows from equations (4.11) and (4.9) that $\alpha_1 = \alpha_2$, and that $(a_2 - a_1)$ is orthogonal to $p(\hat{\tau}_2)$. Additionally, we have that

$$\begin{aligned} p(\hat{\tau}_2)x_c &= \left(I - \frac{(a_1 a_2^\top + a_2 a_1^\top)}{(a_1^\top a_2 - 1)}\right)\left(\frac{\alpha_2(a_2 + a_1)}{2}\right) \\ &= -\frac{\alpha_2(a_2 + a_1)}{(a_1^\top a_2 - 1)} = -p(\hat{\tau}_2), \end{aligned} \tag{4.16}$$

and for the choice of x_c in (4.15) we have that

$$x_c^\top p(\hat{\tau}_2)x_c - \rho(\hat{\tau}_2) = -\frac{\alpha_2^2(a_2 + a_1)^\top(a_2 + a_1)}{2(a_1^\top a_2 - 1)} - \rho(\hat{\tau}_2) = 1 - \alpha_2^2 > 0. \tag{4.17}$$

As a result, from **Case 1** in §2.1 we obtain that the quadric $\mathcal{F}(\hat{\tau}_2)$ is a cylinder as well.



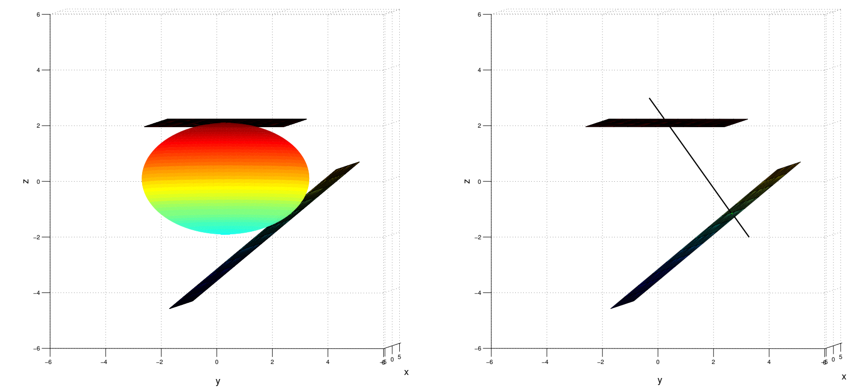
(a) $\tau = \bar{\tau}_1$ (b) $\tau = \bar{\tau}_2 = \hat{\tau}_1$

Figure 8: $f(\tau)$ has two distinct roots, but one of the roots coincides with either $\hat{\tau}_1$ or $\hat{\tau}_2$.

- If the discriminant (4.7) of f in (4.7) is zero, then the two roots of f are equal, i.e., $\bar{\tau} = \bar{\tau}_1 = \bar{\tau}_2$. Let $\bar{\tau} = \hat{\tau}_1$, then from equation (4.10) we obtain the identity $\alpha_1 = -\alpha_2$. Now, since the discriminant of f is zero, we have

$$\alpha_1^2 = \alpha_2^2 = 1, \tag{4.18}$$

and it follows from Lemma 2.2 that the hyperplanes $\mathcal{A}^{=}$ and $\mathcal{B}^{=}$ are tangent to the ellipsoid $\mathcal{Q}$. Recall that for x_c in (4.12) we have from Equation (4.13) that $p(\hat{\tau}_1)x_c = -p(\hat{\tau}_1)$. Furthermore,



(a) Two hyperplanes tangent to the ellipsoid

(b) $\tau = \bar{\tau}_1 = \bar{\tau}_2 = \hat{\tau}_1$

Figure 9: The two roots of $f(\tau)$ coincide with either $\hat{\tau}_1$ or $\hat{\tau}_2$.

from (4.14) and (4.18) we have that $x_c^\top p(\hat{\tau}_1)x_c - \rho(\hat{\tau}_1) = 0$. Hence, the quadric $\mathcal{F}(\hat{\tau}_1)$ is a line.

Similarly, if $\bar{\tau} = \hat{\tau}_2$, then from equation (4.11) we obtain $\alpha_2 = \alpha_1$, and the identity (4.18) still holds. Then, for x_c in (4.15) we have from (4.16) that $\bar{P}(\hat{\tau}_2)x_c = -p(\hat{\tau}_2)$ and from (4.17) we have that $x_c^\top p(\hat{\tau}_2)x_c - \rho(\hat{\tau}_2) = 0$. Then, the quadric $\mathcal{F}(\hat{\tau}_1)$ is a line in this case as well. For illustrations of these cases see Figure 9.

4.2.4 Summarizing the Shapes of $\mathcal{F}(\tau)$

We can now summarize the possible shapes of the quadrics in the family $\{\mathcal{F}(\tau) \mid \tau \in \mathbb{R}\}$ at $\hat{\tau}_1$, $\hat{\tau}_2$, and at $\bar{\tau}_1$, and $\bar{\tau}_2$, where $\hat{\tau}_1 < \hat{\tau}_2$ and $\bar{\tau}_1 < \bar{\tau}_2$.

Theorem 4.5. *The following cases may occur for the shape of $\mathcal{F}(\tau)$:*

- $\hat{\tau}_2 < \bar{\tau}_1 < \bar{\tau}_2$*, or* $\bar{\tau}_1 < \bar{\tau}_2 < \hat{\tau}_1$*, or* $\bar{\tau}_1 < \hat{\tau}_1 < \hat{\tau}_2 < \bar{\tau}_2$*:* $\mathcal{F}(\hat{\tau}_1)$*,* $\mathcal{F}(\hat{\tau}_2)$ *are two paraboloids, and* $\mathcal{F}(\bar{\tau}_1)$*,* $\mathcal{F}(\bar{\tau}_2)$ *are two cones.*

- $\bar{\tau}_1 = \bar{\tau}_2 < \hat{\tau}_1$ *or* $\hat{\tau}_2 < \bar{\tau}_1 = \bar{\tau}_2$*:* $\mathcal{F}(\hat{\tau}_1)$*,* $\mathcal{F}(\hat{\tau}_2)$ *are two paraboloids, and* $\mathcal{F}(\bar{\tau}_1) = \mathcal{F}(\bar{\tau}_2)$ *is a cone.*

- $\bar{\tau}_1 = \hat{\tau}_1$ *and* $\hat{\tau}_2 = \bar{\tau}_2$*:* $\hat{\mathcal{F}}(\hat{\tau}_1) = \mathcal{F}(\bar{\tau}_1)$*,* $\mathcal{F}(\hat{\tau}_2) = \hat{\mathcal{F}}(\bar{\tau}_2)$ *are two cylinders.*

- $\bar{\tau}_1 \neq \hat{\tau}_2$ *and exactly one of* $\bar{\tau}_1$ *or* $\bar{\tau}_2$*, is equal to either* $\hat{\tau}_1$ *or* $\hat{\tau}_2$*: either* $\mathcal{F}(\hat{\tau}_1)$ *is a cylinder and* $\mathcal{F}(\hat{\tau}_2)$ *is a paraboloid, or* $\mathcal{F}(\hat{\tau}_2)$ *is a cylinder and* $\hat{\mathcal{F}}(\hat{\tau}_1)$ *is a paraboloid. In both cases exactly one of* $\hat{\mathcal{F}}(\bar{\tau}_1)$ *or* $\mathcal{F}(\bar{\tau}_2)$ *is a cone.*

- $\bar{\tau}_1 = \bar{\tau}_2 = \hat{\tau}_1$ *or* $\hat{\tau}_2 = \bar{\tau}_1 = \bar{\tau}_2$*: either* $\mathcal{F}(\hat{\tau}_1)$ *is a line and* $\hat{\mathcal{F}}(\hat{\tau}_2)$ *is a paraboloid or* $\mathcal{F}(\hat{\tau}_2)$ *is a line and* $\mathcal{F}(\hat{\tau}_1)$ *is a paraboloid, correspondingly.*

This completes the description of the family $\{\mathcal{F}(\tau) \mid \tau \in \mathbb{R}\}$ of quadratic when $P \succ 0$ and $\mathcal{A}^=$ and $\mathcal{B}^=$ are non-parallel.

5 Generalization

It is important to highlight that the results presented in Theorems 3.2 and 4.1 apply for general quadrics. No assumption is needed about the matrix P. On the other hand, in §3.2 and §4.2 we assume that the initial quadric is an ellipsoid. This assumption indeed facilitates the analysis of the family. However, the results obtained in §3.2 and §4.2 cover the cases where *$\mathcal{Q}$ has an ID1 matrix P and the intersection with the hyperplanes are bounded.* These cases are important because they may occur as the feasible set of problem (MISOCO). We formalize this result, which follows directly from Theorem 4.1 and Lemma 4.3, as the following corollary.

Corollary 5.1. *If there exists a $\tilde{\tau} \in \mathbb{R}$ such that $P(\tilde{\tau}) \succ 0$, then $P(\tau)$ can have at most one non-positive eigenvalue.*

Observe that in this case the analysis reduces to taking the base case at the value $\tilde{\tau}$, where $\mathcal{F}(\tilde{\tau})$ is an ellipsoid.

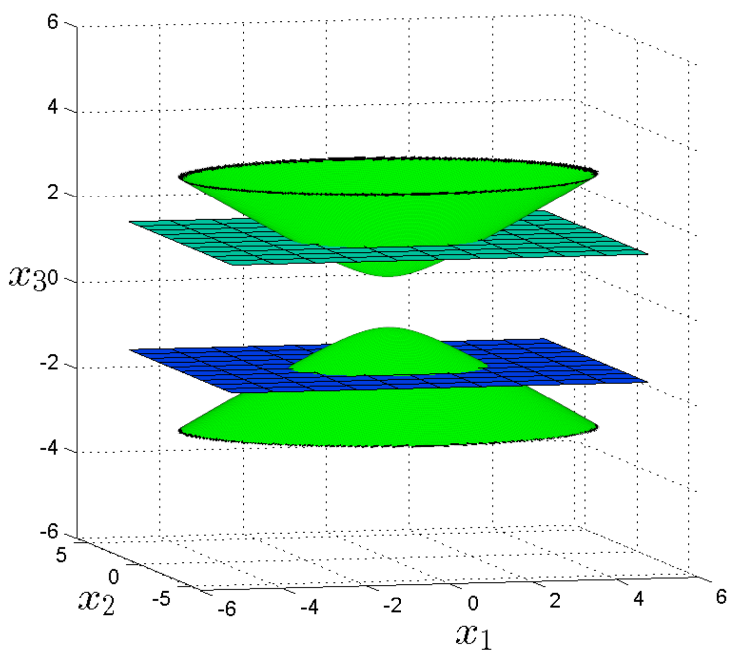


Figure 10: A generalization of our problem: the intersection of a quadric with one negative eigenvalue.

Recall that the original motivation of this work is the analysis of a conic convex optimization problem. However, the results presented here also cover cases as the one illustrated in Figure 10. One can notice that in such cases there is a disjunction over a non-convex set.

6 Conclusions and Future Work

In this paper, we gave a complete characterization of the quadric surfaces that maintain a fixed intersection with two hyperplanes. Such surfaces can be parametrized with only one parameter.

We also analyzed the properties of these families and showed that they consist of quadrics with at most one non-positive eigenvalue if P is PD. The interested reader can find a video illustration of this family of quadrics showing its evolution for different values of τ at `https://coral.ie.lehigh.edu/projects/ciclops`.

For our motivating application, the most important members of these families are the cones, which can be used to obtain the convex hull of a disjunction in mixed integer second order conic optimization (MISOCO). We showed that such cones always exist. The properties of the cones in the context of MISOCO are being investigated now. One limitation of the theory presented in §3.2 and §4.2 is the assumption that P is positive definite. Geometrically, this assumption assures that $\mathcal{Q}$ is an ellipsoid, or that there is a member of the family $\{\mathcal{F}(\tau) \mid \tau \in \mathbb{R}\}$ of quadrics that is an ellipsoid. Thus the intersections with the two hyperplanes are also (lower dimensional) ellipsoids. This was also a crucial assumption in proving that $P(\tau)$ has at most one non-positive eigenvalue, which simplified the description of the family $\{\mathcal{F}(\tau) \mid \tau \in \mathbb{R}\}$.

If P was indefinite, then the intersection of $\mathcal{Q}$ with one hyperplane may be an ellipsoid, while the intersection with the other hyperplane can be a hyperboloid. This situation complicates the description of the quadratic families. Nevertheless, if P has at most one non-positive eigenvalue, the intersections of $\mathcal{Q}$ with the two hyperplanes do not need to be bounded. In this case, the analysis in §3.2 and §4.2 can be extended. In such cases one can repeat the same analysis based on the inertia of the matrix P. The details of this extension are the subject of further research, which is needed for MISOCO problems, our targeted application.

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