# On the Mixed Chinese Postman Problem

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**Abstract**

The mixed Chinese postman problem is a version of the well-known Chinese postman problem in which the underlying graph consists of both directed and undirected edges. We give an integer linear programming formulation for this problem and then show that the extreme points of its linear relaxation polyhedron are all half-integral.

**keywords:** Chinese postman problem, half-integrality, graph, flow

## 1 Introduction

The Chinese postman problem is a well-known combinatorial optimization problem which has many practical applications. For instance, it is the problem faced by a postman delivering mail along a network of roads, or the problem of a trash hauler plotting an efficient route to pick up trash at a set of residences. The problem consists of finding a shortest tour of a strongly connected graph under the restriction that each edge and arc must be traversed at least once. For an arbitrary mixed graph with nonnegative costs, it can be checked in polynomial time whether the graph has a tour in which each edge and arc is traversed exactly once [3, 8]. The special case in which the underlying graph is completely undirected (i.e. consists entirely of two-way streets) has been examined in detail and can be solved efficiently [6, 1]. The special case in which the network consists entirely of directed edges can also be solved in polynomial time [2].

The mixed Chinese postman problem, in which the graph consists of both directed and undirected edges is known to be NP-complete [8]. In [5], Kappauf and Koehler formulated the mixed Chinese postman problem as an integer linear program and

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showed that the polyhedron corresponding to the linear relaxation of this program has half-integral extreme points. In this paper, we present a similar integer linear programming formulation of the mixed Chinese postman problem. We then show that a related flow problem first formulated by Frederickson [4] can be used to provide a more concise and intuitive proof of the half-integrality property than was previously known.

# 2 Definitions

An *undirected graph* $G = (V, E)$ is a set of vertices $V$ along with a multiset of edges $E$ that connect the vertices in $V$. An *edge* $\langle u, v\rangle$ is an unordered pair of vertices that are connected in $G$. Notice that by this definition there can be more than one edge between a single pair of vertices. A *directed graph* or *digraph* $G = (N, A)$ is a set of nodes $N$ along with a multiset of arcs $A$ that connect the nodes in $N$. An *arc* or *directed edge* $(u, v)$ is an ordered pair of nodes that are connected in $G$. If $(u, v)$ is an arc in $A$, then $u$ is called its *tail* and $v$ is called its *head*. The arc is said to go from $u$ to $v$. An *oriented edge* is an edge $\langle u, v\rangle$ that has been given a direction from $u$ to $v$. A *mixed graph* $G = (V, E, A)$ is a graph in which both directed and undirected edges connect a set of vertices $V$. A nonnegative cost function will be defined on $A \cup E$.

The *degree* of a vertex $v$ is the total number of edges and arcs incident to it and is denoted $d(v)$. In this paper, the *indegree* of $v$, denoted $\delta^-(v)$, will be the number of arcs and oriented edges directed into $v$, while the *outdegree*, $\delta^+(v)$, is the number of arcs and oriented edges directed out of $v$. The *deficiency* of $v$ is $\delta^-(v) - \delta^+(v)$.

In a mixed graph $G = (V, E, A)$, a *path* is a sequence of edges and arcs such that if $[u, v]$ is followed by $[w, x]$ in the sequence, then $v = w$. A *tour* is a path such that if $[u, v]$ is the first edge in the path and $[w, x]$ is the last edge, then $u = x$. An *Eulerian tour* is a tour which traverses every edge and arc in the graph exactly once. The *cost of a tour* is the total cost of all edges and arcs traversed by the tour. A graph is strongly connected if for every pair of vertices $u$ and $v$, there is a path from $u$ to $v$ and a path from $v$ to $u$ in $G$.

An *augmented graph* is a graph $G = (V, E, A)$ along with multisets $E'$ and $A'$, referred to as *the augmentation*. In this definition $E'$ and $A'$ represent additional copies of arcs and edges added to the original graph to allow it to satisfy some desired property. An augmented graph is said to satisfy such a property if and only if the graph $G' = (V, E \cup E', A \cup A')$ satisfies the property. The *cost of an augmentation* is the total cost of all edge and arc copies in $E'$ and $A'$.

In a mixed graph $G = (V, E, A)$, $\hat{E}$ and $\check{E}$ represent multisets of oriented edges, one set containing an oriented copy of each edge in $E$ oriented in one (arbitrary) direction, and the other set containing an oppositely oriented copy of each edge. If $e$ is an oriented edge copy from either $\hat{E}$ or $\check{E}$, then $\tilde{e}$ is the corresponding, oppositely oriented edge copy. We refer to $e$ and $\tilde{e}$ as an *edge pair*.

# 3 Problem description and formulation

Let $G = (V, E, A)$ be a strongly connected mixed graph with vertex set $V$, arc set $A$, and edge set $E$. Our objective is to find a shortest tour of this graph that contains every edge and arc at least once. We can think of this problem as that of finding the least-cost augmentation of the original network that yields an Eulerian tour in the augmented graph. To do so, we want to find an augmentation and then assign orientations to all of the edges in such a way that the deficiency at every vertex in the augmented graph is zero. If we do this in a minimal fashion, then we can use this augmentation to find a shortest tour of the graph satisfying the given conditions. This tour can be found in polynomial time given the augmented graph. We can hence formulate the mixed postman problem as shown in Figure 1.

Min

$$\sum_{s \in A \cup \hat{E} \cup \check{E}} c_s x_s \tag{1}$$

Subject to

$$\begin{aligned}
y'_e + y'_{\tilde{e}} &\geq 1 \qquad \forall e \in E && (2)\\
x_s &= y'_s + y_s \qquad \forall s \in A \cup \hat{E} \cup \check{E} && (3)\\
\sum_{s \in \delta^+(v)} x_s - \sum_{s \in \delta^-(v)} x_s &= 0 \qquad \forall v \in V && (4)\\
y'_a &= 1 \qquad \forall a \in A && (5)\\
y'_e &\in \{0, 1\} \qquad \forall e \in \hat{E} \cup \check{E} && (6)\\
y_s &\geq 0,\ integer \qquad \forall s \in A \cup \hat{E} \cup \check{E} && (7)
\end{aligned}$$

Figure 1: Integer linear programming formulation for the mixed postman problem

In this formulation, we think of each feasible solution as specifying a particular augmented graph. For each arc, $y'_a$ represents the "original" copy of that arc (the one contained in $A$) and $y_a$ represents the number of additional copies included in the augmentation. Hence, (5) reflects the necessity of including at least one copy of each arc in the augmented graph. Since the variables $y'_a$ are constants, they are only included for clarity. Two sets of variables correspond to each edge. One set represents copies of the edge oriented in one direction and the other set represents copies of the same edge oriented in the opposite direction. The variables $y'_e$ and $y'_{\tilde{e}}$ represent the first oriented copies of an edge taken in each direction, while $y_e$ and $y_{\tilde{e}}$ represent additional oriented copies of the same edge included in the augmentation. Then (2) represents the requirement that there be at least one copy of each edge in the augmented graph. For both edges and arcs, $x_s$ represents the total number of copies of the arc or oriented edge in the augmented graph. Therefore, (4) is the requirement that in the augmented graph, the deficiency of each vertex be zero. The $\delta^+(v)$ and $\delta^-(v)$ in (4) refer to the indegree and outdegree of nodes in the augmented graph and hence this is a slight abuse of notation, but it should not cause any confusion.

# 4 Proof of the half-integrality property

Now we want to look at the linear relaxation of the integer linear program (1)-(7). Since we know that at least one oriented copy of each edge and arc will be included in the optimal augmented graph, we could consider the cost of the first copy of each edge and arc as a constant in the objective function. In so doing, we then consider only the cost of the augmentation when solving the program. This makes sense since the augmentation cost is the "true" cost of any feasible solution. In order to do this, however, we need the following proposition.

**Proposition 1** *In any basic solution to the linear relaxation of (1)-(7),* $y'_e + y'_{\tilde{e}} = 1 \ \forall e \in E$*.*

*Proof.* Suppose $y'_e + y'_{\tilde{e}} > 1$ for some $e \in E$. Then we could set

$$\begin{array}{lcl} y'_e & \leftarrow & y'_e - \epsilon \\ y'_{\tilde{e}} & \leftarrow & y'_{\tilde{e}} - \epsilon \end{array} \quad \text{or} \quad \begin{array}{lcl} y'_e & \leftarrow & y'_e + \epsilon \\ y'_{\tilde{e}} & \leftarrow & y'_{\tilde{e}} + \epsilon \end{array}$$

for some $\epsilon > 0$ and still maintain feasibility. This implies that the original solution was the linear combination of two other feasible solutions, and hence contradicts the assumption that the original solution was basic. □

Proposition 1 implies that in any basic solution to the linear relaxation, $y'_a = 1 \ \forall a \in A$ and $y'_e + y'_{\tilde{e}} = 1 \ \forall e \in E$ and so we can rewrite the linear relaxation equivalently as in Figure 2. Notice that in the modified objective function $\sum_{s \in A \cup E} c_s$ is now a constant and hence we only consider the variables $y_s$ in our cost. Also, $b_v$ is just the deficiency at each vertex in the original graph. This serves to eliminate the excess variables $y'_a$ from the program. Now we look at a further relaxation of the formulation in Figure 2 that we claim has an integer optimum. Consider the following proposition.

**Proposition 2** *The program (8) and (10)-(15) has an integer optimum.*

*Proof.* Once we eliminate (9), the remaining constraints are just those of a capacitated transshipment problem. We construct the graph $G' = (N', A')$ as follows. For every $v \in V$, place a node $v'$ in $V'$ with demand $b_{v'} := b_v$. For every arc $a \in A$, place an arc $a'$ in $A'$ with $l_{a'} := 0$, $u_{a'} := \infty$, and $c_{a'} := c_a$. For each oriented edge copy $e \in \hat{E}$ from $u$ to $v$, place two arcs $e'$ and $e''$ from $u'$ to $v'$ in $A'$ with the following costs and capacities ($l$ is the lower bound on the capacity and $u$ is the upper bound).

Min

$$\sum_{s\in A\cup\hat{E}\cup\check{E}} c_s y_s + \sum_{s\in A\cup E} c_s \tag{8}$$

Subject to

$$\begin{aligned}
y'_e + y'_{\tilde{e}} &= 1 \qquad \forall e \in E & (9)\\
x_e &= y'_e + y_e \qquad \forall e \in \hat{E}\cup\check{E} & (10)\\
x_a &= y_a \qquad \forall a \in A & (11)\\
\sum_{s\in\delta^+(v)} x_s - \sum_{s\in\delta^-(v)} x_s &= b_v \qquad \forall v \in V & (12)\\
y'_e &\le 1 \qquad \forall e \in \hat{E}\cup\check{E} & (13)\\
y'_e &\ge 0 \qquad \forall e \in \hat{E}\cup\check{E} & (14)\\
y_s &\ge 0 \qquad \forall s \in A\cup\hat{E}\cup\check{E} & (15)
\end{aligned}$$

Figure 2: Linear relaxation of formulation from Figure 1

$$\begin{aligned}
l_{e'} &:= 0\\
u_{e'} &:= 1\\
c_{e'} &:= 0\\
l_{e''} &:= 0\\
u_{e''} &:= \infty\\
c_{e''} &:= c_e
\end{aligned}$$

We do the same for each oriented edge copy in $\check{E}$. Now solving a capacitated transshipment problem on $G'$ is equivalent to solving (8) and (10)-(15) We know that such a problem has an integer optimum. Hence (8) and (10)-(15) does also. □

It is interesting to note that in [4], Frederickson used the solution of this flow problem as part of a heuristic solution procedure for solving the mixed Chinese postman problem. See [4] for more details. Now for the primary results of this paper.

**Proposition 3** *Every integral solution of (8) and (10)-(15) can be mapped into a solution of (8)-(15) which is half-integral and which has the same objective function value.*

*Proof.* There is a very simple mapping to obtain this result. For each edge $e$ for which $y'_e - y'_{\tilde{e}} = 0$ in the solution to (8) and (10)-(15), set $y'_e = y'_{\tilde{e}} = \frac{1}{2}$. In this new solution, (9) is satisfied for all edges. Furthermore, we have not changed the values of any variable $y_s$ and hence the objective function value is still the same. □

**Theorem 1** *Every extreme point of the polyhedron (9)-(15) is half-integral.*

*Proof.* The program (8) and (10)-(15) is a relaxation of the program (8)-(15). But Proposition 3 shows that there exists a half-integral solution of (8)-(15) with the same objective function value as any integral solution of (8) and (10)-(15). More specifically, this is true of any integer optimum of (8) and (10)-(15). Since by Proposition 2, such an integer optimum always exists, there always exists a half-integral optimum to (8)-(15). Finally, since this is true for any set of costs, this implies directly that every extreme point of the polyhedron (9)-(15) is half-integral. □

It should be noted that a proof of the half-integrality property for the windy postman polyhedron (the asymmetric version of the problem) appeared in a doctoral thesis by Zaw Win [9]. He used a somewhat different approach based on the solution of a minimum cost flow problem. This paper was completed independently of Zaw Win's work. We did not learn of his thesis until after this paper had been submitted for publication.

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