A Complete Characterization of Disjunctive Conic Cuts for Mixed Integer Second Order Cone Optimization

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Abstract

We study the convex hull of the intersection of a disjunctive set defined by parallel hyperplanes and the feasible set of a mixed integer second order cone optimization (MISOCO) problem. We extend our prior work on disjunctive conic cuts (DCCs), which has thus far been restricted to the case in which the intersection of the hyperplanes and the feasible set is bounded. Using a similar technique, we show that one can extend our previous results to the case in which that intersection is unbounded. We provide a complete characterization in closed form of the conic inequalities required to describe the convex hull when the hyperplanes defining the disjunction are parallel.

1 Introduction

The use of valid inequalities in mixed integer linear optimization (MILO) was essential for the development of successful optimization solvers [Cornuéjols, 2008]. Particularly, the disjunctive inequalities proposed originally by Balas have proven to be especially powerful [Balas, 1979]. Here, we present an extension of the disjunctive approach of Balas to mixed integer second order cone optimization (MISOCO). The contribution of this paper lies in the consolidation and completion of the characterizations of disjunctive conic cuts (DCCs) for MISOCO problems proposed in Belotti et al. [2013, 2015].

A MISOCO problem is that of minimizing a linear function over the intersection of an affine subspace with the Cartesian product of k Lorentz cones. A Lorentz cone is defined as $\mathbb{L}^p = \{x^i \in \mathbb{R}^p | x^i_1 \geq \|x^i_{2:p}\|\}$, where the notation $x_{2:p}$ is used to denote a vector formed by the components 2 to

p of a vector $x \in \mathbb{R}^p$. Formally, the standard form of a MISOCO problem is then as follows:

$$\begin{aligned} \text{minimize: } & c^\top x \\ \text{subject to: } & Ax = b \\ & x \in \mathcal{K} \\ & x \in \mathbb{Z}^d \times \mathbb{R}^{n-d}, \end{aligned} \tag{MISOCO}$$

where $A \in \mathbb{Q}^{m\times n}$, $c \in \mathbb{Q}^n$, $b \in \mathbb{Q}^m$, $\mathcal{K} = \mathbb{L}_1^{n_1} \times \ldots \times \mathbb{L}_k^{n_k}$, and $\sum_{i=1}^k n_i = n$. We assume the rows of A are linearly independent. It is well-known that the formulation of a MISOCO problem is similar to the formulation of a MILO problem, except for the requirement that the feasible region be contained in the cone $\mathcal{K}$. Given the similarities between MISOCO and MILO problems, it stands to reason that the application of disjunctive techniques to MISOCO problems may also become a powerful tool to improve the performance of MISOCO solvers.

For simplicity of presentation, we focus in what follows on the case in which $\mathcal{K} = \mathbb{L}^n$ ($k = 1$). Our main objective is then to obtain a description of the convex hull of the set

$$\mathcal{F}^D = \mathcal{F} \cap (\mathcal{U} \cup \mathcal{V}), \tag{1}$$

where $\mathcal{F} = \{x \in \mathbb{R}^n \mid Ax = b,\ x \in \mathbb{L}^n\}$, $\mathcal{U} = \{x \in \mathbb{R}^n \mid u^\top x \geq \varphi\}$, $\mathcal{V} = \{x \in \mathbb{R}^n \mid u^\top x \leq \varpi\}$, and $\varpi \leq \varphi$. Throughout the paper we use the superscript $^=$ to denote the hyperplanes forming the boundary of a half-space, as for example $\mathcal{U}^= = \{x \in \mathbb{R}^n \mid u^\top x = \varphi\}$. To achieve our goal we use the result in Belotti et al. [2013], which shows that the set $\mathcal{F}$ may be described in terms of a quadric

$$\mathcal{Q} = \{w \in \mathbb{R}^\ell \mid w^\top P w + 2p^\top w + \rho \leq 0\},$$

where $P \in \mathbb{R}^{\ell\times\ell}$, $p \in \mathbb{R}^\ell$, and ρ is a scalar. Our characterization of $\text{conv}(\mathcal{F}^{\mathcal{D}})$ is obtained analyzing the geometry of the intersections of $\mathcal{Q}$ with the sets $\mathcal{U}^=$ and $\mathcal{V}^=$.

Several generalizations of Balas' disjunctive approach have been investigated in the literature. Namely, Stubbs and Mehrotra generalized Balas' approach for $0-1$ mixed integer convex optimization problems [Stubbs and Mehrotra, 1999]. Later, Çezik and Iyengar proposed an extension of *lift-and-project* techniques for 0-1 mixed integer conic optimization (MICO) [Çezik and Iyengar, 2005]. In particular, they showed how to generate linear and convex quadratic valid inequalities using the relaxation obtained by a projection procedure. That work also extended the Chvátal-Gomory procedure [Gomory, 1958] for generating linear cuts to MICO problems. Further, Drewes narrowed the analysis in Drewes [2009], applying the extension studied in Çezik and Iyengar [2005] and Stubbs and Mehrotra [1999] to MISOCO problems. More recently, close forms for disjunctive cuts derived by extending Balas' approach to MISOCO problems were also derived in Andersen and Jensen [2013], Belotti et al. [2013, 2015], Dadush et al. [2011], Kılınç-Karzan and Yıldız [2014], Modaresi et al. [2016, 2015].

In this paper we build on the theory of DCCs previously developed in Belotti et al. [2013, 2015], in which boundedness of the intersection of $\mathcal{F}$ with the sets $\mathcal{U}^=$ and $\mathcal{V}^=$ was assumed and a closed form description of the DCCs was provided. Here, we consider the case where the boundedness of these intersections is no longer assumed. This requires an additional analysis of the behaviour of the family of quadrics described in Belotti et al. [2013]. In particular, with unboundedness it is not possible to have a unified result such as Theorem 3.4 in Belotti et al. [2013]. These results were originally described in Goez's dissertation Góez [2013], which provided a complete characterization of conv($\mathcal{F}^D$). A different approach for the derivation of these cuts was independently proposed in Modaresi et al. [2016], with two major differences. First, the construction presented here uses an algebraic analysis of quadrics for the derivation of the cuts, while the approach in Modaresi et al. [2016] uses an interpolation technique. Second, the derivation proposed here generalizes to all quadrics that are needed to describe the geometry of the continuous relaxation of (MISOCO) problems. This is not the case in Modaresi et al. [2016], since with that approach the characterization of hyperboloids was not possible due to the involved formulas required in the analysis.

We organize our presentation in this paper as follows. We begin in Section 2 by characterizing the shapes of the feasible set of a MISOCO problem with a single cone. That characterization provides the basis for obtaining the closed form for the DCCs to be introduced. In Section 3, we describe a detailed general procedure to derive DCCs for MISOCO problems. That description recalls the results previously provided in Belotti et al. [2013, 2015] and discusses the new extensions to the unbounded case supported by the results of Góez's Ph.D. thesis [Góez, 2013]. Finally, we close the paper with some conclusions and directions for future research in Section 4.

2 Quadrics and the feasible set of a MISOCO problem

Before discussing the derivation of the DCCs, we need to establish the relationship between quadrics and the feasible set of (MISOCO). To simplify the analysis, we restrict ourselves to the set $\mathcal{F}$ introduced in (1). The goal of this section is twofold. First, we show that the set $\mathcal{F}$ can be described in terms of a quadric $\mathcal{Q}$. Second, we show how that quadric $\mathcal{Q}$ may be used to obtain a conic inequality valid for $\mathcal{F}^D$. Finally, we provide a characterization of the shapes of the quadrics that can be used to represent $\mathcal{F}$. That characterization is the basis for the analysis in Section 3.

We first consider the set $\{x \in \mathbb{R}^n \mid Ax = b\}$ and a vector $x^0 \in \mathcal{F}$. Let $H_{n\times\ell}$ be a matrix whose columns form an orthonormal basis for the null space of A, where $\ell = n - m$. Then, we have that

$$\{x \in \mathbb{R}^n \mid Ax = b\} = \{x \in \mathbb{R}^n \mid \exists w \in \mathbb{R}^\ell,\ x = x^0 + Hw\}. \tag{2}$$

We may use (2) to rewrite the set $\mathcal{F}$ in terms of a quadric as follows. First, let $J \in \mathbb{R}^{n\times n}$ be a diagonal matrix defined as

$$J = \begin{bmatrix} -1 & 0 \\ 0 & I \end{bmatrix}.$$

Then, we have that

$$\mathbb{L}^n = \{x \in \mathbb{R}^n \mid x^\top J x \le 0, x_1 \ge 0\}.$$

Substituting $x = x^0 + Hw$ in the constraint involving J, it becomes

$$(x^0 + Hw)^\top J(x^0 + Hw) = w^\top H^\top JHw + 2(x^0)^\top JHw + (x^0)^\top Jx^0 \le 0. \tag{3}$$

Defining $P = H^\top JH$, $p = H^\top Jx^0$, and $\rho = (x^0)^\top Jx^0$, we can re-write (3) as the quadric

$$\mathcal{Q} = \{w \in \mathbb{R}^\ell \mid w^\top Pw + 2p^\top w + \rho \le 0\}. \tag{4}$$

In what follows, it will be convenient to define the set

$$\mathcal{F}^{\mathcal{Q}} = \{w \in \mathbb{R}^\ell \mid w \in \mathcal{Q},\ x_1^0 + H_{1:}^\top w \ge 0\}, \tag{5}$$

where $H_{1:}$ is the first row of H. Using $\mathcal{F}^{\mathcal{Q}}$, we can express $\mathcal{F}$ can in terms of $\mathcal{Q}$ as follows:

$$\mathcal{F} = \{x \in \mathbb{R}^n \mid \exists w \in \mathcal{F}^{\mathcal{Q}},\ x = x^0 + Hw\}. \tag{6}$$

The condition $x_1 = x_1^0 + H_{1:}^\top w \geq 0$ in (5) is necessary because there are cases in which $\mathcal{Q}$ is non-convex (see Figure 1). In such a case, substituting $\mathcal{Q}$ for $\mathcal{F}^{\mathcal{Q}}$ in (6) would admit solutions for which $x_1 < 0$ that do not correspond to solutions to the original problem.

Recall the set $\mathcal{F}^D$ defined in (1) that we wish to convexify and the sets $\mathcal{U}$ and $\mathcal{V}$ defined earlier. Notice that the sets $\mathcal{U}$ and $\mathcal{V}$ can be reformulated as

$$\mathcal{U} = \{x \in \mathbb{R}^n \mid \exists w \in \mathbb{R}^\ell,\ x = x^0 + Hw,\ u^\top Hw \geq \varphi - u^\top x^0\},$$

and

$$\mathcal{V} = \{x \in \mathbb{R}^n \mid \exists w \in \mathbb{R}^\ell,\ x = x^0 + Hw,\ u^\top Hw \leq \varpi - u^\top x^0\}.$$

To express $\mathcal{U}$ and $\mathcal{V}$ in a form that can be used with the reformulation we have obtained, define $a = u^\top H$, $\alpha = \varphi - u^\top x^0$ and $\beta = \varpi - u^\top x^0$. Now, let $\mathcal{A} = \{w \in \mathbb{R}^\ell \mid a^\top w \geq \alpha\}$ and $\mathcal{B} = \{w \in \mathbb{R}^\ell \mid a^\top w \leq \beta\}$. One can then equivalently define $\mathcal{F}^D$ from (1) in terms of $\mathcal{Q}$ and the set $\mathcal{A} \cup \mathcal{B}$ as

$$\mathcal{F}^D = \{x \in \mathbb{R}^n \mid \exists w \in \mathcal{F}^{\mathcal{Q}} \cap (\mathcal{A} \cup \mathcal{B}), x = x^0 + Hw\}. \tag{7}$$

We now derive a Lemma that shows that separating a point $\hat{x} \in \mathcal{F}$ from $\mathcal{F}^D$ is equivalent to separating a certain related point $\hat{w} \in \mathcal{F}^{\mathcal{Q}}$ from $\mathcal{F}^{\mathcal{Q}} \cap (\mathcal{A} \cup \mathcal{B})$.

Lemma 1. *Consider a vector $\hat{x} \in \mathcal{F}$ and a vector $\hat{w} \in \mathcal{F}^{\mathcal{Q}}$ such that $\hat{x} = x^0 + H\hat{w}$. Then $\hat{x} \notin \mathcal{F}^D$ if and only if $\hat{w} \notin \mathcal{F}^{\mathcal{Q}} \cap (\mathcal{A} \cup \mathcal{B})$.*

Proof. Recall the alternative representation of $\mathcal{F}$ given in (6). Note that any $x \in \mathcal{F}$ is a linear combination of x^0 and the columns of H. Additionally, recall that the columns of H are linearly independent. Then, the vector $\hat{w}$ defining $\hat{x}$ is unique. The result follows. □

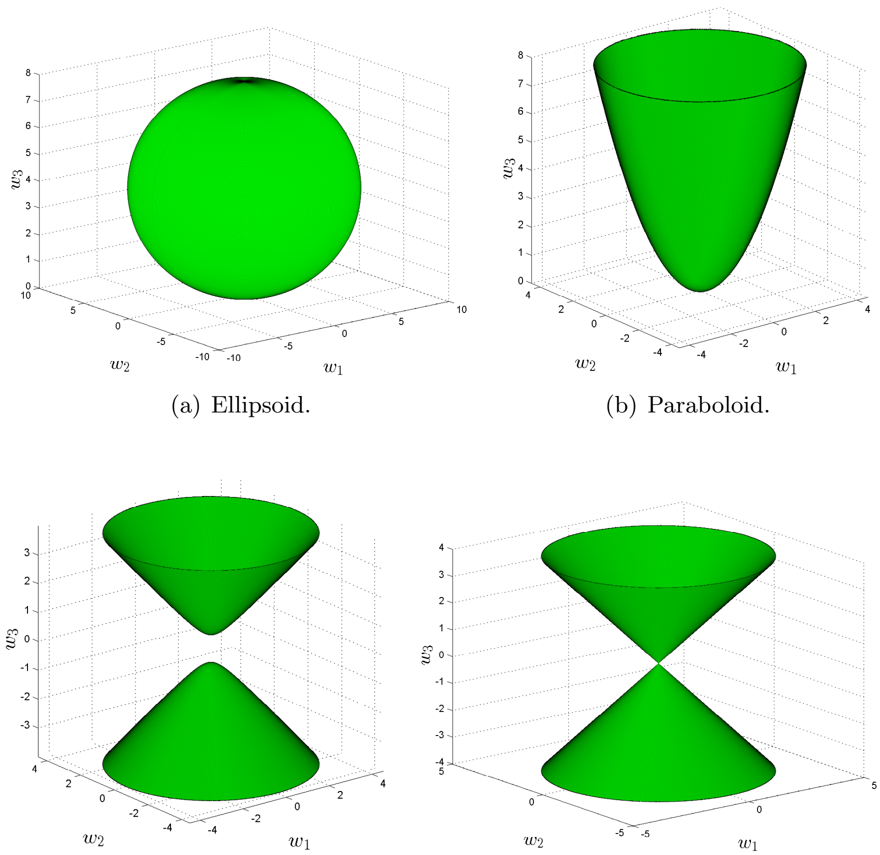
Thus, from Lemma 1 we obtain that convexifying the set $\mathcal{F}^{\mathcal{Q}} \cap (\mathcal{A} \cup \mathcal{B})$ one can derive disjunctive cuts for the feasible set of (MISOCO).

Before moving forward with the derivation of the DCCs for MISOCO problems in Section 3, we need to analyze the shapes of the quadric $\mathcal{Q}$. The inertia of the matrix P in the representation of $\mathcal{Q}$ is the key element defining its shape. Hence, we recall the following result about the inertia of P.

Lemma 2 (Belotti et al. [2013])**.** *The matrix P in the definition of the quadric $\mathcal{Q}$ has at most one non-positive eigenvalue, and at least $\ell - 1$ positive eigenvalues.*

Using Lemma 2 it may be shown that for the analysis of Section 3 we only need to account for the following possible shapes of $\mathcal{Q}$:

1. if $P \succ 0$, then $\mathcal{Q}$ is an ellipsoid, see Figure 1(a) for an illustration;
2. if $P \succeq 0$ and it is singular, then $\mathcal{Q}$ is:
   (a) a paraboloid if there is no vector $w^{\mathrm{c}} \in \mathbb{R}^\ell$ such that $Pw^{\mathrm{c}} = -p$, see Figure 1(b) for an illustration;



(a) Ellipsoid. (b) Paraboloid.

(c) Hyperboloid. (d) Cone.

Figure 1: Illustration of the shapes of $\mathcal{Q}$.

(b) a line if there is a vector $w^{\text{c}} \in \mathbb{R}^{\ell}$ such that $Pw^{\text{c}} = -p$;

3. if P is indefinite with one negative eigenvalue (ID1), then $\mathcal{Q}$ is:

   (a) a hyperboloid of two sheets if $p^{\top}P^{-1}p - \rho < 0$, see Figure 1(c) for an illustration;

   (b) a quadratic cone if $p^{\top}P^{-1}p - \rho = 0$, see Figure 1(d) for an illustration.

The proof that we only need to consider the cases above is mainly technical and is presented in A. Note that the cases 3a and 3b are the two cases metioned earlier in which the quadric $\mathcal{Q}$ is a non-convex set. Finally, notice that by construction we have that the set $\mathcal{F}^{\mathcal{Q}}$ is either equal to $\mathcal{Q}$ when it is a convex quadric or to one of the branches of $\mathcal{Q}$ when it is a non-convex quadric. Hence, the boundedness or unboundedness of the intersections of $\mathcal{F}^{\mathcal{Q}}$ with $\mathcal{A}^{=}$ and $\mathcal{B}^{=}$ will be determined by the intersections of those hyperplanes with $\mathcal{Q}$.

Example 1. As an example illustrating the cases, consider the following MISOCO problem:

$$\begin{aligned} \text{minimize: } & c_1x_1 + c_1x_2 + c_1x_3 + c_1x_4 \\ \text{subject to: } & -\sigma x_3 + x_4 = 2 \\ & (x_4, x_1, x_2, x_3)^\top \in \mathbb{L}^4 \\ & x_1, x_2, x_3 \in \mathbb{Z}. \end{aligned} \tag{8}$$

In this example we have that $\ell = 3$ and we assume $\sigma \geq 0$, obtaining the following cases: if $\sigma < 1$, then $\mathcal{Q}$ would be an ellipsoid; if $\sigma = 1$, then $\mathcal{Q}$ would be a paraboloid; if $\sigma > 1$, then $\mathcal{Q}$ would be a hyperboloid.

3 Derivation of DCCs for MISOCO problems

We now present the derivation of DCCs for MISOCO problems using the reformulation introduced in Section 2. The quadric $\mathcal{Q}$ and sets $\mathcal{F}^\mathcal{Q}$, $\mathcal{A}$, and $\mathcal{B}$ are as defined earlier in Section 2.

To facilitate the algebra, we may assume w.l.o.g. that $\beta < \alpha$ and that the quadric $\mathcal{Q}$ has been normalized. In this context, a normalized quadric $\mathcal{Q}$ implies three things: first, that the matrix P is a diagonal matrix with all its entries taking values in $\{-1, 0, 1\}$; second, that the scalar ρ takes a value in $\{-1, 0, 1\}$; and third, that $p = 0$ for the case of ellipsoids, hyperboloids, or cones. This normalization may be achieved using one of the linear transformations L described in B.

We provide in this section a full list of the DCCs that can be derived by an explicit characterization of the set $\text{conv}(\mathcal{F}^\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B}))$. From Section 2, using the setup described above, we know that the scope of our derivation can be limited to the following cases for $\mathcal{Q}$: an ellipsoid, a paraboloid, a branch of a hyperboloid of two sheets, and a Lorentz cone. Note that if $\mathcal{A} \cap \mathcal{F}^\mathcal{Q} = \emptyset$, then $\text{conv}(\mathcal{F}^\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{F}^\mathcal{Q} \cap \mathcal{B}$. Similarly, if $\mathcal{B} \cap \mathcal{F}^\mathcal{Q} = \emptyset$, then $\text{conv}(\mathcal{F}^\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{F}^\mathcal{Q} \cap \mathcal{A}$. Therefore, for the rest of this section we assume that $\mathcal{A} \cap \mathcal{F}^\mathcal{Q} \neq \emptyset$ and $\mathcal{B} \cap \mathcal{F}^\mathcal{Q} \neq \emptyset$. Additionally, given that $\alpha \neq \beta$, we have that $\mathcal{A} \cap \mathcal{B} = \emptyset$. In order to keep the paper self-contained, we divide our derivation as follows. In Section 3.1, we recall the definitions of DCCs and disjunctive cylindrical cuts (DCyCs) introduced in Belotti et al. [2015]. In Section 3.2, we revisit some results from Belotti et al. [2013] about quadrics. Section 3.3 reviews the derivation when $\mathcal{A}^= \cap \mathcal{F}^\mathcal{Q}$ and $\mathcal{B}^= \cap \mathcal{F}^\mathcal{Q}$ are bounded [Belotti et al., 2015]. Finally, in Section 3.4 we complete the derivation of possible DCCs considering the case when $\mathcal{A}^= \cap \mathcal{F}^\mathcal{Q}$ and $\mathcal{B}^= \cap \mathcal{F}^\mathcal{Q}$ are unbounded.

3.1 Disjunctive conic cuts

We recall in this section the main definitions and results of Belotti et al. [2015]. We begin with the formal definition of a DCC.

Definition 1 (Belotti et al. [2015])**.** A full-dimensional closed convex cone $\mathcal{K} \subset \mathbb{R}^\ell$ is called a *disjunctive conic cut* (DCC) for the set $\mathcal{F}^\mathcal{Q}$ and the disjunctive set $\mathcal{A} \cup \mathcal{B}$ if

$$\text{conv}\left(\mathcal{F}^\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})\right) = \mathcal{F}^\mathcal{Q} \cap \mathcal{K}.$$

In what follows, we may need to consider translated cones, i.e., a cone whose vertex is not at the origin. Since it is always possible to use a translation to reposition the vertex of a given cone at the origin, we may assume in our proofs that the vertex of a cone is at the origin unless a translated cone is strictly required, in which case we explicitly specify it. The following proposition provides a sufficient condition for a convex cone $\mathcal{K}$ to be a disjunctive conic cut for the set $\mathcal{F}^{\mathcal{Q}}$ and the disjunctive set $\mathcal{A} \cup \mathcal{B}$ when the sets $\mathcal{F}^{\mathcal{Q}} \cap \mathcal{A}^{=}$ and $\mathcal{F}^{\mathcal{Q}} \cap \mathcal{B}^{=}$ are bounded.

Proposition 1 (Belotti et al. [2015])**.** *Let $\mathcal{F}^{\mathcal{Q}}$ and the disjunctive set $\mathcal{A} \cup \mathcal{B}$ be such that $(\mathcal{A} \cap \mathcal{B}) \cap \mathcal{F}^{\mathcal{Q}}$ is empty, and both $\mathcal{F}^{\mathcal{Q}} \cap A^{=}$ and $\mathcal{F}^{\mathcal{Q}} \cap \mathcal{B}^{=}$ are nonempty and bounded. A full-dimensional convex cone $\mathcal{K} \subset \mathbb{R}^{\ell}$ is the unique DCC for $\mathcal{F}^{\mathcal{Q}}$ and the disjunctive set $\mathcal{A} \cup \mathcal{B}$ if*

$$\mathcal{K} \cap \mathcal{A}^{=} = \mathcal{F}^{\mathcal{Q}} \cap \mathcal{A}^{=} \quad \textit{and} \quad \mathcal{K} \cap \mathcal{B}^{=} = \mathcal{F}^{\mathcal{Q}} \cap \mathcal{B}^{=}. \tag{9}$$

We must also recall the formal definition of a DCyC. For this definition we need to introduce first the definition of a cylinder that is used here.

Definition 2 (Belotti et al. [2015])**.** Let $\mathcal{D} \subseteq \mathbb{R}^n$ be a convex set and $d_0 \in \mathbb{R}^n$ a vector. Then, the set $\mathcal{C} = \{x \in \mathbb{R}^n | x = d + \sigma d_0, d \in \mathcal{D}, \sigma \in \mathbb{R}\}$ is called a convex cylinder in $\mathbb{R}^n$.

Notice that using this definition a line is a one-dimensional cylinder. We may define a DCyC as follows.

Definition 3 (Belotti et al. [2015])**.** A closed convex cylinder $\mathcal{C} \subset \mathbb{R}^{\ell}$ is a *disjunctive cylindrical cut (DCyC)* for the set $\mathcal{F}^{\mathcal{Q}}$ and the disjunctive set $\mathcal{A} \cup \mathcal{B}$ if

$$\text{conv}(\mathcal{F}^{\mathcal{Q}} \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{C} \cap \mathcal{F}^{\mathcal{Q}}.$$

The following proposition provides a sufficient condition for a convex cylinder $\mathcal{C}$ to be a disjunctive cylindrical cut for the set $\mathcal{F}^{\mathcal{Q}}$.

Proposition 2. *Let $\mathcal{F}^{\mathcal{Q}}$ and the disjunctive set $\mathcal{A} \cup \mathcal{B}$ be such that $(\mathcal{A} \cap \mathcal{B}) \cap \mathcal{F}^{\mathcal{Q}}$ is empty, and both $\mathcal{F}^{\mathcal{Q}} \cap A^{=}$ and $\mathcal{F}^{\mathcal{Q}} \cap \mathcal{B}^{=}$ are nonempty. A convex cylinder $\mathcal{C} \in \mathbb{R}^{\ell}$ with a unique direction $d^0 \in \mathbb{R}^{\ell}$, such that $a^\top d^0 \neq 0$, is the unique DCyC for $\mathcal{F}^{\mathcal{Q}}$ and the disjunctive set $\mathcal{A} \cup \mathcal{B}$ if*

$$\mathcal{C} \cap \mathcal{A}^{=} = \mathcal{F}^{\mathcal{Q}} \cap \mathcal{A}^{=} \quad \textit{and} \quad \mathcal{C} \cap \mathcal{B}^{=} = \mathcal{F}^{\mathcal{Q}} \cap \mathcal{B}^{=}. \tag{10}$$

Notice that this version of Proposition 2 requires that the sets $\mathcal{F}^{\mathcal{Q}} \cap \mathcal{A}^{=}$ and $\mathcal{F}^{\mathcal{Q}} \cap \mathcal{B}^{=}$ be nonempty, but not bounded, in contrast to the corresponding result in Belotti et al. [2015]. Since the proof of this version of Proposition 2 does not differ significantly from the proof in Belotti et al. [2015], we omit the proof here. For the interested reader, it can be found in Góez [2013].

3.2 Family of quadrics

We now recall the key elements from Belotti et al. [2013] with regard to the derivation of the DCCs and DCyCs. We start with the following result about quadrics.

Theorem 1 (Belotti et al. [2013])**.** *The uni-parametric family $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$ of quadrics having the same intersection with $\mathcal{A}^{=}$ and $\mathcal{B}^{=}$ as the quadric $\mathcal{Q}$ is defined as $\mathcal{Q}(\tau) = \{w \in \mathbb{R}^{\ell} \mid w^\top P(\tau) w + 2p(\tau)^\top w + \rho(\tau) \leq 0\}$, where*

$$P(\tau) = P + \tau a a^\top, \qquad p(\tau) = p - \tau \frac{(\alpha + \beta)}{2} a, \qquad \rho(\tau) = \rho + \tau \alpha \beta.$$

We may now use Theorem 1 to ensure that for any $\tau \in \mathbb{R}$ the quadric $\mathcal{Q}(\tau)$ satisfies the conditions $\mathcal{Q}(\tau) \cap \mathcal{A}^= = \mathcal{Q} \cap \mathcal{A}^=$ and $\mathcal{Q}(\tau) \cap \mathcal{B}^= = \mathcal{Q} \cap \mathcal{B}^=$. It also allows us to characterize the shapes of the quadrics in the family $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$ as follows. The matrix $P(\tau)$ is in general non-singular, except for some specific values of τ, which are discussed in Sections 3.3 and 3.4. For the case where $P(\tau)$ is non-singular, one can rewrite the definition of the quadric $\mathcal{Q}(\tau)$ as

$$(w + P(\tau)^{-1}p(\tau))^\top P(\tau)(w + P(\tau)^{-1}p(\tau)) \leq p(\tau)^\top P(\tau)^{-1}p(\tau) - \rho(\tau). \tag{11}$$

Hence, using the same approach of Section 2, the shapes of the quadrics in the family $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$ can be classified using the inertia of $P(\tau)$ and the term $p(\tau)^\top P(\tau)^{-1}p(\tau) - \rho(\tau)$. We use this result in Sections 3.3 and 3.4 to proceed with the derivation of the closed forms of the DCCs and DCyCs.

3.3 Derivation of DCCs when $\mathcal{F}^\mathcal{Q} \cap \mathcal{A}^=$ and $\mathcal{F}^\mathcal{Q} \cap \mathcal{B}^=$ are bounded

We begin our derivation considering the case when the intersections of $\mathcal{F}^\mathcal{Q}$ with the hyperplanes $\mathcal{A}^=$ and $\mathcal{B}^=$ are bounded. Recall the possible shapes of $\mathcal{Q}$ obtained at the end of Section 2. In Belotti et al. [2013] the properties of the family of quadrics $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$ of Theorem 1 were analyzed taking an ellipsoid as the initial quadric $\mathcal{Q}$. This choice has two advantages: a) the sets $\mathcal{Q} \cap \mathcal{A}^=$ and $\mathcal{Q} \cap \mathcal{B}^=$ are always bounded; and b) it simplifies the algebra. For the sake of consistency with Belotti et al. [2013, 2015], we assume here that $\mathcal{Q}$ is an ellipsoid. As a consequence, we have in this case that $\mathcal{F}^\mathcal{Q} = \mathcal{Q}$. However, we highlight that this analysis is also valid when $\mathcal{Q}$ is a paraboloid, one branch of a hyperboloid of two sheets, or a quadratic cone, provided that the sets $\mathcal{Q} \cap \mathcal{A}^=$ and $\mathcal{Q} \cap \mathcal{B}^=$ are bounded. In fact, the following result is a direct consequence of Lemmas 3.2 and 3.5 in Góez [2013].

Corollary 1. *Let $\mathcal{Q}$ be an ellipsoid, a paraboloid, a hyperboloid of two sheets, or a quadratic cone, and let $\mathcal{A}^=$ and $\mathcal{B}^=$ to parallel hyperplanes such that $\mathcal{A}^= \cap \mathcal{Q} \neq \emptyset$ and $\mathcal{B}^= \cap \mathcal{Q} \neq \emptyset$. If $\mathcal{Q} \cap \mathcal{A}^=$ and $\mathcal{Q} \cap \mathcal{B}^=$ are bounded, then there exists an ellipsoid in the family $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$ of Theorem 1.*

Hence, the close forms for the cases of a paraboloid, one branch of a hyperboloid, or a quadratic cone, may be obtained with the analysis presented here by taking one of the ellipsoids in the family as the initial quadric $\mathcal{Q}$.

In Belotti et al. [2013], it is shown that the term $p(\tau)^\top P(\tau)^{-1}p(\tau) - \rho(\tau)$ can be written as the ratio

$$p(\tau)^\top P(\tau)^{-1}p(\tau) - \rho(\tau) = \frac{\tau^2 \frac{(\alpha_1 - \alpha_2)^2}{4} + \tau(1 - \alpha_1\alpha_2) + 1}{(1 + \tau)}. \tag{12}$$

Notice that for $\tau = -1$ the denominator in (12) is zero, this is also the only value where $P(\tau)$ becomes singular. Let $\bar{\tau}_1 \leq \bar{\tau}_2$ be the roots of the numerator of (12). A full characterization of the family $\mathcal{Q}(\tau)$ for $\tau \in \mathbb{R}$, depending on the geometry of $\mathcal{Q}$ and the hyperplanes $\mathcal{A}^=$ and $\mathcal{B}^=$, is presented in [Belotti et al., 2013, Theorem 3.4], which we recall here.

Theorem 2. *The following cases may occur for the shape of $\mathcal{Q}(\tau)$:*

1. $\bar{\tau}_1 < \bar{\tau}_2 < -1$: *$\mathcal{Q}(-1)$ is a paraboloid, and $\mathcal{Q}(\bar{\tau}_1)$, $\mathcal{Q}(\bar{\tau}_2)$ are two cones.*
2. $\bar{\tau}_1 = \bar{\tau}_2 < -1$: *$\mathcal{Q}(-1)$ is a paraboloid and $\mathcal{Q}(\bar{\tau}_2)$ is a cone.*
3. $\bar{\tau}_1 < \bar{\tau}_2 = -1$: *$\mathcal{Q}(\bar{\tau}_1)$ is cone and $\mathcal{Q}(\bar{\tau}_2)$ is a cylinder.*

4. $\bar{\tau}_1 = \bar{\tau}_2 = -1$*:* $\mathcal{Q}(\bar{\tau}_2)$ *is a line.*

We may now used this characterization to identify a DCC or DCyC in the family of Theorem 1, which convexifies the intersection of $\mathcal{Q}$ with a parallel disjunction. Our strategy in the next sections works as follows. First, we identify a convex cone $\mathcal{K}$ or a convex cylinder $\mathcal{C}$ in families of quadrics of Theorem 2. Second, we use the results of Propositions 1 and 2 to show that $\mathcal{K} \cap \mathcal{Q}$ or $\mathcal{C} \cap \mathcal{Q}$ characterizes the convex hull for $\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})$. To simplify the analysis, we separate the cases of cylinders and cones in the following two sections.

3.3.1 Cylinders

Consider the families $\{\mathcal{Q}(\tau),\, \tau \in \mathbb{R}\}$, described in the third and fourth cases in Theorem 2, where $\mathcal{Q}(\bar{\tau}_2)$ is a cylinder. Recall that

$$\mathcal{Q}(\bar{\tau}_2) = \left\{ w \in \mathbb{R}^\ell \;\middle|\; w^\top P(\bar{\tau}_2) w + 2p(\bar{\tau}_2)^\top w + \rho(\bar{\tau}_2) \le 0 \right\}, \tag{13}$$

where $P(\bar{\tau}_2)$ is a positive semidefinite matrix. Hence, the cylinder $\mathcal{Q}(\bar{\tau}_2)$ is convex, and from Proposition 2 we obtain that $\mathcal{F}^{\mathcal{Q}} \cap \mathcal{Q}(\bar{\tau}_2) = \mathcal{F}^{\mathcal{Q}} \cap (\mathcal{A} \cup \mathcal{B})$. In other words, we have that $\mathcal{Q}(\bar{\tau}_2)$ is a DCyC.

Example 2. To illustrate this result, consider the case when $\sigma = \sqrt{5} - 2$ in Example (8). Using this value we obtain that

$$\mathcal{F}^{\mathcal{Q}} = \{(x_1, x_2, x_3)^\top \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + (4\sqrt{5} - 8)x_3^2 + (8 - 4\sqrt{5})x_3 - 4 \le 0\},$$

which is a non-normalized ellipsoid centered at $(0, 0, 0.5)^\top$. Now, for the objective function we use $(c_1, c_2, c_3, c_4) = (0, 1, 0, 0)$, the optimal solution for the continuous relaxation with this set up is $x^* = (0, \sqrt{2 + \sqrt{5}}, 0.5, 1 + \frac{\sqrt{5}}{2})$. Given the integrality constraint over x_3, we may use the disjunction $x_3 \le 0 \vee x_3 \ge 1$ to derive a DCyC to cut off this solution. From B we obtain that the normalized quadric $\mathcal{Q}$ is the unit ball and that $\alpha = -\beta = \frac{\sqrt{\sqrt{5}-2}}{\sqrt{6-\sqrt{5}}}$. Hence, from (12) we have

$$p(\tau)^\top P(\tau)^{-1} p(\tau) - \rho(\tau) = \frac{\alpha^2\tau^2 + \tau(1 + \alpha^2) + 1}{1 + \tau},$$

and the roots of the numerator are $\bar{\tau}_1 = -\frac{1}{2\alpha^2}$ and $\bar{\tau}_2 = -1$. Henceforth, we know from Theorem 2 that $\mathcal{Q}(\bar{\tau}_2)$ is a cylinder. We highlight that the normalization has been used for the analysis of the quadrics, but it is not necessary to compute the cut. Specifically, in this case we have in the original space of $\mathcal{F}^{\mathcal{Q}}$ that

$$\tilde{\tau}_2 = \frac{\bar{\tau}_2}{\left\| P^{-\frac{1}{2}} a \right\|^2} = 8 - 4\sqrt{5},$$

where $P^{-\frac{1}{2}}$ is the inverse of the matrix square root of P. Hence, using $\tilde{\tau}_2$ in Theorem 1 we obtain the following DCyC,

$$\tilde{\mathcal{Q}}(\tilde{\tau}_2) = \{(x_1, x_2, x_3)^\top \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - 4 \le 0\}.$$

Figure 2 illustrates Example 2, where one can appreciate how the DCyC cuts off the relaxed solution. In this particular example, after adding the DCyC, we obtain that the solutions $(0, 2, 1, 1 + \frac{\sqrt{5}}{2})$ and $(0, 2, 0, 1 + \frac{\sqrt{5}}{2})$ are optimal for the continuous relaxation.

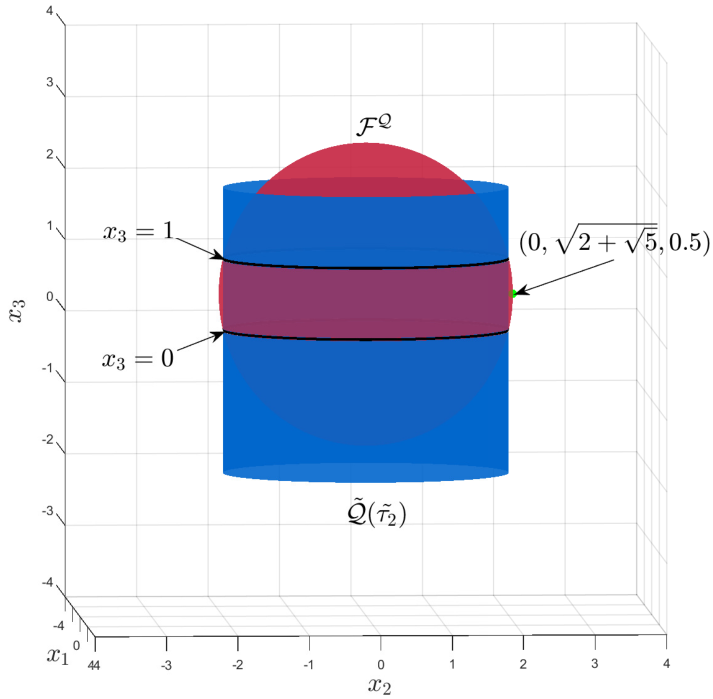


Figure 2: Illustration of Example 2: a DCyC when both $\mathcal{Q} \cap \mathcal{A}^=$ and $\mathcal{Q} \cap \mathcal{B}^=$ are bounded.

3.3.2 Cones

Consider now the cones described in the first and second cases of Theorem 2. The following results provide a criterion to identify which of the cones $\mathcal{Q}(\bar{\tau}_1)$ and $\mathcal{Q}(\bar{\tau}_2)$ provides a DCC in those cases.

Lemma 3 (Belotti et al. [2015]). *The quadric $\mathcal{Q}(\bar{\tau}_2)$ in the families $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$ of the cases 1 and 2 of Theorem 2 contains a cone that satisfies Definition 1.*

The proof of Lemma 3 is provided in Belotti et al. [2015] and is omitted here for the sake of space. However, we still need to discuss how to obtain a DCC from $\mathcal{Q}(\bar{\tau}_2)$, which is a non-convex cone. For this purpose, we first show that $\mathcal{Q}(\bar{\tau}_2)$ is the union of two convex cones. Then, we give a criterion to decide which cone is a DCC.

Recall that $\bar{\tau}_2 < -1$, hence $P(\bar{\tau}_2)$ is a symmetric and non-singular matrix and it has exactly one negative eigenvalue. Then, $P(\bar{\tau}_2)$ can be diagonalized as $P(\bar{\tau}_2) = VDV^\top$, where $V \in \mathbb{R}^{\ell \times \ell}$ is an orthogonal matrix and $D \in \mathbb{R}^{\ell \times \ell}$ is a diagonal matrix having the eigenvalues of $P(\bar{\tau}_2)$ in its diagonal. We may assume w.l.o.g. that $D_{1,1} < 0$, and let $w^{\mathrm{c}} = -P(\bar{\tau}_2)^{-1}p(\tau_2)$, and $B = V\tilde{D}^{1/2}$, where $\tilde{D}_{l,k} = |D_{l,k}|$. Thus, we can write $\mathcal{Q}(\bar{\tau}_2)$ in terms of B as follows:

$$\mathcal{Q}(\bar{\tau}_2) = \left\{ w \in \mathbb{R}^\ell \,\middle|\, (w - w^{\mathrm{c}})^\top B_{2:\ell} B_{2:\ell}^\top (x - w^{\mathrm{c}}) \leq \left(B_1^\top (w - w^{\mathrm{c}}) \right)^2 \right\},$$

where $B_{2:\ell} \in \mathbb{R}^{\ell \times (\ell-1)}$, is formed by the columns 2 to ℓ of B. Now, define the sets $\mathcal{Q}(\bar{\tau}_2)^+$, $\mathcal{Q}(\bar{\tau}_2)^-$

as follows

$$\mathcal{Q}(\bar{\tau}_2)^+ \equiv \left\{ w \in \mathbb{R}^\ell \;\middle|\; \left\| B_{2:\ell}^\top (w - w^{\mathrm{c}}) \right\| \le B_1^\top (w - w^{\mathrm{c}}) \right\}, \tag{14}$$

$$\mathcal{Q}(\bar{\tau}_2)^- \equiv \left\{ w \in \mathbb{R}^\ell \;\middle|\; \left\| B_{2:\ell}^\top (w - w^{\mathrm{c}}) \right\| \le -B_1^\top (w - w^{\mathrm{c}}) \right\}, \tag{15}$$

which are two second order cones, i.e., two convex cones. It is easy to verify that $\mathcal{Q}(\bar{\tau}_2) = \mathcal{Q}(\bar{\tau}_2)^+ \cup \mathcal{Q}(\bar{\tau}_2)^-$.

The final step is to decide which of the cones $\mathcal{Q}(\bar{\tau}_2)^+$ and $\mathcal{Q}(\bar{\tau}_2)^-$ is a DCC. This is decided using the sign of $B_1^\top(-P^{-1}p - w^{\mathrm{c}})$. Thus, using Proposition 1 we obtain that:

- if $B_1^\top(-P^{-1}p - w^{\mathrm{c}}) > 0$, then $\mathcal{Q}(\bar{\tau}_2)^+$ is a DCC for $\mathcal{F}^{\mathcal{Q}}$ and the disjunctive set $\mathcal{A} \cup \mathcal{B}$;
- if $B_1^\top(-P^{-1}p - w^{\mathrm{c}}) < 0$, then $\mathcal{Q}(\bar{\tau}_2)^-$ is a DCC for $\mathcal{F}^{\mathcal{Q}}$ and the disjunctive set $\mathcal{A} \cup \mathcal{B}$.

Notice that $B_1^\top(-P^{-1}p - w^{\mathrm{c}}) = 0$ implies that the center of the ellipsoid $\mathcal{Q}$ coincides with the vertex of the selected cone. In this case the set $\mathcal{F}^{\mathcal{Q}}$ is a single point, which is a trivial case that does not allow the generation of cuts. This completes the procedure.

Example 3. To illustrate this result, let us consider the case when $\sigma = \sqrt{2} - 1$ in Example (8). Using this value we obtain the quadric

$$\mathcal{F}^{\mathcal{Q}} = \{(x_1, x_2, x_3)^\top \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + 2(\sqrt{2} - 1)x_3^2 + 4(1 - \sqrt{2})x_3 - 4 \le 0\},$$

which is a non-normalized ellipsoid centered at $(0, 0, 1)^\top$. Now, for the objective function we use $(c_1, c_2, c_3, c_4) = (0, \sqrt{10 + 6\sqrt{2}}, 2(1-\sqrt{2}), 0)$, and the optimal solution for the continuous relaxation of the problem with this set up is $x^* = (0, \sqrt{\frac{5+3\sqrt{2}}{2}}, 0.5, \frac{3+\sqrt{2}}{2})$. We may use again the disjunction $x_3 \le 0 \vee x_3 \ge 1$ to derive a DCC to cut off this solution. Here, we have again that the normalize quadric $\mathcal{Q}$ is the unit ball with $\alpha = 0$ and $\beta = -\sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}}$. Hence, from (12) we have

$$p(\tau)^\top P(\tau)^{-1} p(\tau) - \rho(\tau) = \frac{\frac{\sqrt{2}-1}{4(\sqrt{2}+1)}\tau^2 + \tau + 1}{1 + \tau},$$

and from the roots of the numerator we obtain

$$\bar{\tau}_2 = \frac{2\left(\sqrt{2(\sqrt{2}+1)} - \sqrt{2} - 1\right)}{\sqrt{2} - 1}.$$

Henceforth, we know from Theorem 2 that $\mathcal{Q}(\bar{\tau}_2)$ is a cone. In this case, we have in the original space of $\mathcal{F}^{\mathcal{Q}}$ that

$$\tilde{\tau}_2 = \frac{\bar{\tau}_2}{\left\| P^{-\frac{1}{2}} a \right\|^2} = 4\left(\sqrt{2(\sqrt{2}+1)} - \sqrt{2} - 1\right).$$

Hence, using $\tilde{\tau}_2$ in Theorem 1 we obtain the following quadric,

$$\tilde{\mathcal{Q}}(\tilde{\tau}_2) = \{(x_1, x_2, x_3)^\top \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \le 0.0390\,(x_3 + 10.1333)^2\},$$

where the results are given with a precision of up to 4 digits. Now, we have that $-w_c = (0,0,10.1333)^\top$, $-P^{-1}p = (0,0,1)^\top$, and $B_1 = (0,0,0.1974)^\top$. Hence, we obtain that $B_1^\top\left(-P^{-1}p - w_c\right) \geq 0$, and our DCC is

$$\tilde{\mathcal{Q}}(\tilde{\tau}_2)^+ = \{(x_1,x_2,x_3)^\top \in \mathbb{R}^3 \mid \left\|(x_1,x_2)^\top\right\| \leq 0.1974\,(x_3 + 10.1333)\}.$$

Figure 3 illustrates Example 3, where one can appreciate how the DCC cuts off the relaxed solution. Notice that the DCC is a translated cone with the vertex at $(0,0,10.1333)^\top$.

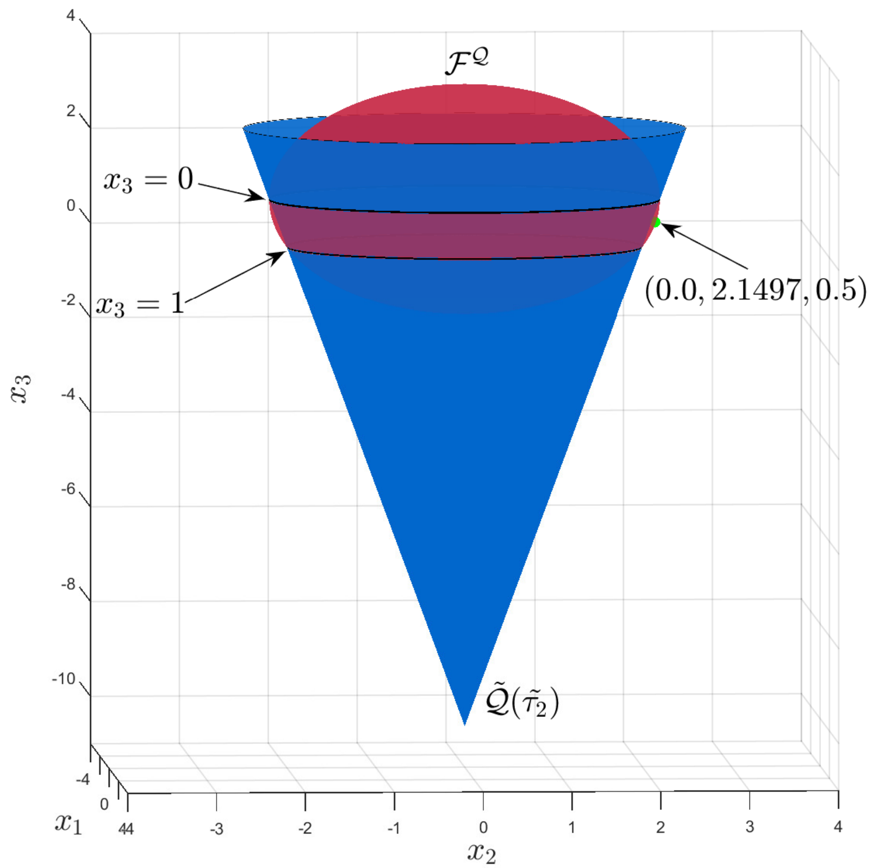


Figure 3: Illustration of Example 3: a DCC when both $\mathcal{Q} \cap \mathcal{A}^=$ and $\mathcal{Q} \cap \mathcal{B}^=$ are bounded.

Up to this point we have shown that for all the cases in Theorem 2, one can find a cone $\mathcal{K}$ or a cylinder $\mathcal{C}$ that satisfies Definitions 1 or 3 respectively. Hence, by combining Theorem 2 with Propositions 1 and 2 one obtains a procedure to find the convex hull of $\mathcal{F}^{\mathcal{Q}} \cap (\mathcal{A} \cup \mathcal{B})$, when the disjunctive set $\mathcal{A} \cup \mathcal{B}$ is such that both $\mathcal{F}^{\mathcal{Q}} \cap \mathcal{A}^=$ and $\mathcal{F}^{\mathcal{Q}} \cap \mathcal{B}^=$ are bounded.

3.4 Derivation of DCCs when both $\mathcal{Q} \cap \mathcal{A}^=$ and $\mathcal{Q} \cap \mathcal{B}^=$ are unbounded

To complete the derivation of all the DCCs and DCyCs for MISOCO problems we need to consider the case when both intersections $\mathcal{Q} \cap \mathcal{A}^=$ and $\mathcal{Q} \cap \mathcal{B}^=$ are unbounded. Notice that in this case the quadric $\mathcal{Q}$ cannot be an ellipsoid. Hence, for the rest of this section we focus on the cases when $\mathcal{Q}$ is a paraboloid, a hyperboloid of two sheets, or a non-convex cone.

Our strategy in the next sections works as follows. First, in Section 3.4.1 we analyze the case of cylinders. In that case we have the additional challenge of having quadrics associated with the DCyCs which may not be convex cylinders. We show that it is possible to find a convex cylinder $\mathcal{C}$ such that $\mathcal{C}\cap(\mathcal{A}\cup\mathcal{B}) = \mathcal{F}^{\mathcal{Q}}\cap(\mathcal{A}\cup\mathcal{B})$. Then, we use Proposition 2 to prove that $\mathcal{F}^{\mathcal{Q}}\cap\mathcal{C}$ characterizes the convex hull for $\mathcal{F}^{\mathcal{Q}} \cap (\mathcal{A} \cup \mathcal{B})$. Second, in Section 3.4.2 we analyze the case of cones. In that case the challenge we face is that a cone sharing the intersections $\mathcal{F}^{\mathcal{Q}} \cap (\mathcal{A} \cup \mathcal{B})$ with $\mathcal{F}^{\mathcal{Q}}$ may not be unique, as it is shown in Belotti et al. [2015]. We prove that it is possible to find a convex cone $\mathcal{K}$ in the family of quadrics of Theorem 1 such that $\mathcal{F}^{\mathcal{Q}} \cap \mathcal{K}$ characterizes the convex hull for $\mathcal{F}^{\mathcal{Q}} \cap (\mathcal{A} \cup \mathcal{B})$. Notice that the results in C and D are used in the proofs of this section.

3.4.1 Cylinders

We divide the derivation of the DCyCs in this section in two parts. First, we analyze the case when $\mathcal{Q}$ is a paraboloid. Second, we analyze the case when $\mathcal{Q}$ is a hyperboloid of two sheets or a non-convex cone.

Paraboloids Let us assume that $\mathcal{Q}$ is a paraboloid, which implies that the system $Px = -p$ has no solution, and also we have that $\mathcal{F}^{\mathcal{Q}} = \mathcal{Q}$. Recall our assumption that $\mathcal{Q}$ is normalized using the procedure given in B. From that normalization we have that if $a_1 \neq 0$, then the intersections $\mathcal{Q} \cap \mathcal{A}^=$ and $\mathcal{Q} \cap \mathcal{B}^=$ would be bounded. Hence, for the analysis of this case we may assume that $a_1 = 0$. As the first step in our strategy we have the following result for the family of quadrics in Theorem 1.

Lemma 4 (Góez [2013])**.** *If $\mathcal{Q}$ is a paraboloid and $a_1 = 0$, then the quadric $\mathcal{Q}(-1)$ in the family $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$ is a convex cylinder.*

Proof. In this case the characteristic polynomial (37) for $P(\tau)$ simplifies to

$$(1-\lambda)^{n-2}(\lambda^2 - \lambda(1+\tau\,\|a\|^2) + \tau a_1^2) = (1-\lambda)^{n-2}(\lambda^2 - \lambda(1+\tau) + \tau a_1^2) = 0.$$

Thus, 1 is an eigenvalue of P with multiplicity $\ell - 2$. The other two eigenvalues are given by the roots of $\lambda^2 - \lambda(1+\tau) + \tau a_1^2 = 0$, which are

$$\frac{(1+\tau) \pm \sqrt{(1+\tau)^2 - 4\tau a_1^2}}{2}. \tag{16}$$

Hence, we have that zero is an eigenvalue of the matrix $P(-1)$, with multiplicity 2. We perform the proof in three steps. First, we find a basis for the null space of $P(-1)$. Then, we find a direction in that space that is orthogonal to $p(-1)$. Finally, we show that $\mathcal{Q}(-1)$ is a convex cylinder in that direction.

Recall that for $\tau \in \mathbb{R}$ the first row and first column of $P(-1)$ are zero vectors. Since $\|a\| = 1$, and $a_1 = 0$, we have that

$$P(-1)a = \left(\tilde{J} - aa^\top\right) a = a - a = 0.$$

Thus, a and $(1, 0^\top)^\top$ are eigenvectors of $P(-1)$ associated with the 0 eigenvalue, and form a basis for the null space of $P(-1)$. Hence, any vector of the form $(\gamma, a_{2:\ell}^\top)^\top$, for all $\gamma \in \mathbb{R}$, belongs to the null space of $P(-1)$. Define $\tilde{\gamma}$ as

$$\tilde{\gamma} = \frac{-p_{2:\ell}^\top a_{2:\ell} - \frac{\alpha+\beta}{2}}{p_1},$$

and recall that $\mathcal{Q}$ being a paraboloid implies that $p_1 \neq 0$. The vector $[\tilde{\gamma}, a_{2:\ell}^\top]^\top$ is orthogonal to $p(-1)$, since

$$\begin{aligned} p(-1)^\top \begin{bmatrix} \tilde{\gamma} \\ a_{2:\ell} \end{bmatrix} &= \left(p^\top + \frac{\alpha+\beta}{2} a\right) \begin{bmatrix} \frac{-p_{2:\ell}^\top a_{2:\ell} - \frac{\alpha+\beta}{2}}{p_1} \\ a_{2:\ell} \end{bmatrix} \\ &= -p_{2:\ell}^\top a_{2:\ell} - \frac{\alpha+\beta}{2} + \frac{\alpha+\beta}{2} + p_{2:\ell}^\top a_{2:\ell} = 0. \end{aligned}$$

Let $\tilde{w} \in \mathbb{R}^\ell$ be a vector such that $\tilde{w} \in \mathcal{Q}(-1) \cap (\mathcal{A}^= \cup \mathcal{B}^=)$, then we have that

$$\tilde{w}^\top P(-1)\tilde{w} + 2p(-1)^\top \tilde{w} + \rho(-1) \leq 0.$$

Now, let $\tilde{u}^\top = \tilde{w}^\top + \theta[\tilde{\gamma}, a_{2:\ell}^\top]^\top$ for some $\theta \in \mathbb{R}$, then we have that

$$\begin{aligned} &\tilde{u}^\top P(-1)\tilde{u} + 2p(-1)^\top \tilde{u} + \rho(-1) \\ &\quad = \tilde{w}^\top P(-1)\tilde{w} + \theta(\tilde{\gamma}, a_{2:\ell}^\top)P(-1)\tilde{w} + \theta^2(\tilde{\gamma}, a_{2:\ell}^\top)P(-1) \begin{bmatrix} \tilde{\gamma} \\ a_{2:\ell} \end{bmatrix} \\ &\qquad + 2p(-1)^\top \tilde{u} + \rho(-1) \\ &\quad = \tilde{w}^\top P(-1)\tilde{w} + 2p(-1)^\top \tilde{w}^\top + 2\theta p(-1)^\top \begin{bmatrix} \tilde{\gamma} \\ a_{2:\ell} \end{bmatrix} + \rho(-1) \\ &\quad = \tilde{w}^\top P(-1)\tilde{w} + 2p(-1)^\top \tilde{w}^\top + \rho(-1) \leq 0, \end{aligned}$$

where the last inequality follows from the assumption $\tilde{w} \in \mathcal{Q}(-1) \cap (\mathcal{A}^= \cup \mathcal{B}^=)$. Hence, the distance of any vector $\tilde{u}$ to the boundary of $\mathcal{Q}(-1)$ is constant for any $\theta \in \mathbb{R}$. Finally, we need to show that any cross section of $\mathcal{Q}(-1)$ in the direction $[\tilde{\gamma}, a_{2:\ell}^\top]^\top$ is a convex set. Consider the hyperplane $[\tilde{\gamma}, a_{2:\ell}^\top]w = \varrho$, where $\varrho \in \mathbb{R}$, and let $\tilde{P}(-1)$ be the lower right $\ell - 1 \times \ell - 1$ sub-matrix of $P(-1)$. Then, for a fixed ϱ we obtain a quadric $\tilde{\mathcal{Q}}(-1) \in \mathbb{R}^{\ell-1\times\ell-1}$ defined by the inequality

$$w_{2:n}^\top \tilde{P}(-1)w_{2:n} + 2\left(p(-1)_{2:n} - \frac{p_1}{\tilde{\gamma}} a_{2:n}\right) w_{2:n} + 2p_1\varrho + \rho(-1) \leq 0.$$

Note that $\tilde{P}(-1)$ is a positive semi-definite matrix, thus $\tilde{\mathcal{Q}}(-1)$ is a convex set. Therefore, $Q(-1)$ is a convex cylinder in the direction $(\tilde{\gamma}, a_{2:\ell}^\top)^\top$. □

Now, following our strategy, we use Proposition 2 to show that the quadric $\mathcal{Q}(-1)$ of Lemma 4 is a DCyC. Recall from the proof of Lemma 4 that the direction of the cylinder $\mathcal{Q}(-1)$ is given by a vector $[\gamma\ a_{2:\ell}^\top]^\top$, for some $\gamma \in \mathbb{R}$. Hence, we have that the product of the normal vector a of the hyperplanes $\mathcal{A}^=$ and $\mathcal{B}^=$ with the direction of the cylinder is different from 0. Hence, from Proposition 2 and Lemma 1 we obtain that $\mathcal{Q}(-1)$ is a DCyC.

Non-convex cones and hyperboloids of two sheets We analyze now the cases when $\mathcal{Q} \in \mathbb{R}^\ell$ is a non-convex cone or a hyperboloid of two sheets. Hence, using the procedure described in Section 3.3.2, one can show that there are two convex sets $\mathcal{Q}^+$ and $\mathcal{Q}^-$ such that $\mathcal{Q}^+ \cup \mathcal{Q}^- = \mathcal{Q}$. In this case we have either $\mathcal{F}^\mathcal{Q} = \mathcal{Q}^+$ or $\mathcal{F}^\mathcal{Q} = \mathcal{Q}^-$, we may assume w.l.o.g. that $\mathcal{F}^\mathcal{Q} = \mathcal{Q}^+$. Finally, given the assumption that $\mathcal{Q}$ is normalized, from B we know that the vector p in (4) is the zero vector, and $P_{1,1} = -1$.

Recall the first step of our strategy. We need to identify a cylinder in the family $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$ of Theorem 1. For that, we need the inertia of the matrix $P(\tau)$, and the explicit expression of $p(\tau)P(\tau)^{-1}p(\tau) - \rho(\tau)$ in terms of P, p, and ρ. The inertia of $P(\tau)$ is characterized in the following result.

Lemma 5. *The eigenvalues of $P(\tau)$ are*

$$\frac{\tau \pm \sqrt{\tau^2 + 4 + 4\tau(1 - 2a_1^2)}}{2}$$

and 1 *with multiplicity* $\ell - 2$.

Proof. Using (37) we obtain that the characteristic polynomial of $P(\tau)$ simplifies to

$$(1-\lambda)^{\ell-2}\left(\lambda^2 - \lambda\tau \|a\|^2 + (\tau a_1^2 - \tau \sum_{i=2}^{\ell} a_1^2 - 1)\right)$$
$$= (1-\lambda)^{\ell-2}(\lambda^2 - \lambda\tau + (2\tau a_1^2 - \tau - 1)) = 0.$$

Thus, 1 is an eigenvalue of P with multiplicity $\ell - 2$. The other two eigenvalues are given by the roots of $\lambda^2 - \lambda\tau + (2\tau a_1^2 - \tau - 1) = 0$, which are

$$\frac{\tau \pm \sqrt{\tau^2 + 4 + 4\tau(1 - 2a_1^2)}}{2}.$$

□

Now, assuming that $P(\tau)$ is non-singular we have that

$$\begin{aligned} p(\tau)^\top P(\tau)^{-1} p(\tau) - \rho(\tau) \\ &= \left(-\tau\frac{\alpha+\beta}{2}a\right)^\top \left(\tilde{J} + \tau a a^\top\right)^{-1} \left(-\tau\frac{\alpha+\beta}{2}a\right) - (\rho + \tau\alpha\beta). \\ &= \frac{\tau^2(1-2a_1^2)\frac{(\alpha-\beta)^2}{4} - \tau(\rho(1-2a_1^2) + \alpha\beta) - \rho}{1 + \tau(1-2a_1^2)}. \end{aligned} \tag{17}$$

Notice that for $\hat{\tau} = -\frac{1}{(1-2a_1^2)}$ the denominator of (17) becomes 0. In fact, $\hat{\tau}$ is such that $P(\hat{\tau})$ is singular, and $\mathcal{Q}(\hat{\tau})$ is a cylinder. Given that (17) depends on ρ and a_1^2, we must analyze non-convex cones and hyperboloids of two sheets separately in order to formalize this result.

Let us begin with the case when $\mathcal{Q}$ is a non-convex cone. From the normalization in B we know that $\rho = 0$, which allows us to simplify (17) to:

$$p(\tau)^\top P(\tau)^{-1} p(\tau) - \rho(\tau) = \frac{\tau^2(1-2a_1^2)\frac{(\alpha-\beta)^2}{4} - \tau\alpha\beta}{1 + \tau(1-2a_1^2)}. \tag{18}$$

Hence, we have that $p(\tau)^\top P(\tau)^{-1}p(\tau) = 0$ when the numerator of (18) is zero, and we obtain the values

$$\bar{\tau}_1 = 2\left(\frac{\alpha\beta - |\alpha\beta|}{(1-2a_1^2)(\alpha-\beta)^2}\right) \quad \text{and} \quad \bar{\tau}_2 = 2\left(\frac{\alpha\beta + |\alpha\beta|}{(1-2a_1^2)(\alpha-\beta)^2}\right). \tag{19}$$

Now, we need to analyze the dependency of (17) on a_1^2. First, notice that if $a_1^2 > \frac{1}{2}$, then from Lemma 5 we have that there is always a $\tau > 0$ such that all the eigenvalues of $P(\tau)$ are positive. In other words, there will be an ellipsoid in the family of quadrics, and this case falls within the analysis of Section 3.3. Second, if $a_1^2 = \frac{1}{2}$, then from Lemma 5 we have that $P(\tau)$ has always one negative eigenvalue and $\ell - 1$ positive eigenvalues. That implies that there is no cylinder in this case. Even more, if $a_1^2 = \frac{1}{2}$, then a is parallel to an extreme ray of $\mathcal{Q}^+$, which results in $\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}) = \operatorname{conv}(\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}))$. In that case the original cones is a DCC. As a result, we need to focus our analysis on the range $0 \le a_1^2 < \frac{1}{2}$. Let $\bar{\tau}$ be the non-zero root in (19), then based on the value of $\bar{\tau}$ we have the following result.

Theorem 3 (Classification for a non-convex cone [Góez, 2013])**.** *If $\mathcal{Q}$ is a non-convex cone, and $a_1^2 < \frac{1}{2}$, then the shape of the quadric $\mathcal{Q}(\bar{\tau})$ in the family $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$ is:*

1. *a cone if $\bar{\tau} > \hat{\tau}$,*
2. *a hyperbolic cylinder of two sheets if $\bar{\tau} = \hat{\tau}$.*

Proof. To facilitate the flow of the discussion we moved the proof of this result to D.1. □

Consider now the case when $\mathcal{Q}$ is a hyperboloid of two sheets. From the normalization in B we know that $\rho = 1$, which simplifies (17) to:

$$p(\tau)^\top P(\tau)^{-1}p(\tau) - \rho(\tau) = \frac{\tau^2(1-2a_1^2)\frac{(\alpha-\beta)^2}{4} - \tau((1-2a_1^2)+\alpha\beta) - 1}{1+\tau(1-2a_1^2)}. \tag{20}$$

In this case, similarly to the non-convex cone, if $a_1^2 > \frac{1}{2}$ we also obtain an ellipsoid in the family of quadrics, which takes us to the analysis of Section 3.3. Additionally, if $a_1^2 = \frac{1}{2}$ one may show that there is no cylinder in the family $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$. That case is part of the analysis of Section 3.4.2, where it becomes relevant. As a result, we need to focus our analysis on the range $0 \le a_1^2 < \frac{1}{2}$. Let $\bar{\tau}_1 \le \bar{\tau}_2$ be the roots of the numerator of (20), then based on this roots we have the following result.

Theorem 4 (Classification for a hyperboloid of two sheets [Góez, 2013])**.** *If $\mathcal{Q}$ is a hyperboloid of two sheets, and $a_1^2 < \frac{1}{2}$, then the shapes of the quadrics $\mathcal{Q}(\bar{\tau}_1)$ and $\mathcal{Q}(\bar{\tau}_2)$ may be as follows:*

1. *if $\beta \neq -\alpha$, then both quadrics are cones,*
2. *if $\beta = -\alpha$, then $\bar{\tau}_1 = \hat{\tau}$, $\mathcal{Q}(\hat{\tau})$ is a hyperbolic cylinder of two sheets, and $\mathcal{Q}(\bar{\tau}_2)$ is a cone.*

Proof. To facilitate the flow of the discussion we moved the proof of this result to D.2. □

We need to analyze further the cylinders in the second cases of Theorems 4 and 3. In these cases the quadric $\mathcal{Q}(\hat{\tau})$ is a hyperbolic cylinder of two sheets, which is a non-convex quadric. For these reasons, we must show that the feasible set of a MISOCO will be contained in one of the branches of $\mathcal{Q}(\hat{\tau})$. From the proofs of Theorems 4 and 3 we know that in both cases $\alpha = -\beta$.

Then, we have that $p(\hat{\tau}) = 0$, $\rho(\hat{\tau}) > 0$, and $\mathcal{Q}(\hat{\tau}) = \{w \in \mathbb{R}^\ell \mid w^\top P(\hat{\tau})w \le -\rho(\hat{\tau})\}$. Consider the eigenvalue decomposition $P(\hat{\tau}) = VDV^\top$, where $D \in \mathbb{R}^{\ell\times\ell}$ is a diagonal matrix, and $V \in \mathbb{R}^{\ell\times\ell}$ is non-singular. We may assume w.l.o.g. that $D_{1,1} = -1$, $D_{2,2} = 0$, and $D_{i,i} > 0$, $i \in \{3,\dots,\ell\}$. Now, let $B = V\bar{D}^{\frac{1}{2}}$, where $\bar{D}$ is a diagonal matrix such that $\bar{D}_{i,i} = |D_{i,i}|$. Let $B_{3:\ell}$ be the matrix that has the last $\ell-2$ columns of B, and B_1 be the first column of B. Then,

$$\mathcal{Q}(\hat{\tau}) = \left\{ w \in \mathbb{R}^\ell \;\middle|\; \left\|B_{3:\ell}^\top w\right\|^2 \le -\rho(\hat{\tau}) + \left(B_1^\top w\right)^2 \right\}.$$

Let us define the following two sets

$$\mathcal{Q}^+(\hat{\tau}) = \left\{ x \in \mathbb{R}^\ell \;\middle|\; \left\|B_{3:\ell}^\top x\right\| \le \xi,\ \left\|\begin{bmatrix}\xi & \sqrt{\rho(\hat{\tau})}\end{bmatrix}^\top\right\| \le B_1^\top x \right\},$$
$$\mathcal{Q}^-(\hat{\tau}) = \left\{ x \in \mathbb{R}^\ell \;\middle|\; \left\|B_{3:\ell}^\top x\right\| \le \xi,\ \left\|\begin{bmatrix}\xi & \sqrt{\rho(\hat{\tau})}\end{bmatrix}^\top\right\| \le -B_1^\top x \right\},$$

where $\begin{bmatrix}\xi & \sqrt{\rho(\hat{\tau})}\end{bmatrix}^\top \in \mathbb{R}^2$. Thus, $\mathcal{Q}(\hat{\tau}) = \mathcal{Q}^+(\hat{\tau}) \cup \mathcal{Q}^-(\hat{\tau})$, and each of these branches of $\mathcal{Q}(\hat{\tau})$ are convex cylinders in the direction V_2, which is the second column of V. Also, note that $\mathcal{Q}^+(\hat{\tau}) \cap \mathcal{Q}^-(\hat{\tau}) = \emptyset$. Then we obtain the following result.

Lemma 6 (Góez [2013])**.** *In the second case in Theorem 3 and the second case in Theorem 4 the set $(\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q}^+$ is a subset of a single branch of $\mathcal{Q}(\hat{\tau})$.*

Proof. The proof is by contradiction. We show that if the set $(\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q}^+$ is not a subset of a single branch of $\mathcal{Q}(\hat{\tau})$, then $(\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q} \neq (\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q}(\hat{\tau})$, which is a contradiction. Let $u, v \in (A^= \cup \mathcal{B}^=) \cap \mathcal{Q}^+$ be two vectors such that $u \in \mathcal{Q}^+(\hat{\tau})$ and $v \in \mathcal{Q}^-(\hat{\tau})$.

First, consider the case when $u, v \in \mathcal{A}^=$ or $u, v \in \mathcal{B}^=$. Hence, we either have $a^\top u = \alpha$ and $a^\top v = \alpha$, or $a^\top u = \beta$ and $a^\top v = \beta$, and then there must exist a $0 \le \tilde{\lambda} \le 1$ such that $w = \tilde{\lambda} v + (1-\tilde{\lambda})u \in (\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q}^+$ but $w \notin (\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q}(\hat{\tau})$. This statement is true because $\mathcal{Q}^+$, $\mathcal{Q}^+(\hat{\tau})$, and $\mathcal{Q}^-(\hat{\tau})$ are convex sets, and $\mathcal{Q}^+(\hat{\tau}) \cap \mathcal{Q}^-(\hat{\tau}) = \emptyset$. This contradicts $(\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q} = (\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q}(\hat{\tau})$.

Second, consider the case when $u \in \mathcal{A}^=$ and $v \in \mathcal{B}^=$. Therefore, $a^\top u = \alpha$, $a^\top v = \beta$, and let $\tilde{a} = [-a_1\ a_{2:\ell}^\top]^\top$. From Theorem 1 we obtain that $P(\hat{\tau}) = \tilde{J} - \frac{aa^\top}{(1-2a_1^2)}$, and for any $\theta \in \mathbb{R}$ we have that

$$(v + \theta\tilde{a})^\top P(\hat{\tau})(v + \theta\tilde{a}) + \rho(\hat{\tau}) = v^\top P(\hat{\tau})v + \rho(\hat{\tau}) \le 0.$$

Similarly, we have for any $\theta \in \mathbb{R}$ that

$$(u + \theta\tilde{a})^\top P(\hat{\tau})(u + \theta\tilde{a}) + \rho(\hat{\tau}) = u^\top P(\hat{\tau})u + \rho(\hat{\tau}) \le 0,$$

Additionally, since $a^\top\tilde{a} \neq 0$, then $\exists\hat{\theta}$ such that $u + \hat{\theta}\tilde{a} \in \mathcal{Q}^+(\hat{\tau})$ and $a^\top(u + \hat{\theta}\tilde{a}) = \beta$, which shows that $\mathcal{Q}^+(\hat{\tau}) \cap \mathcal{B}^= \neq \emptyset$. Similarly, $\exists\tilde{\theta}$ such that $v + \tilde{\theta}\tilde{a} \in \mathcal{Q}^-(\hat{\tau})$ and $a^\top(v + \tilde{\theta}, \tilde{a}) = \alpha$, which shows that $\mathcal{Q}^-(\hat{\tau}) \cap \mathcal{A}^= \neq \emptyset$.

Now, we show that $\mathcal{Q}^+(\hat{\tau}) \cap \mathcal{B}^= \cap \mathcal{Q}^+ = \emptyset$ and $\mathcal{Q}^-(\hat{\tau}) \cap \mathcal{A}^= \cap \mathcal{Q}^+ = \emptyset$. Assume to the contrary that $\mathcal{Q}^+(\hat{\tau}) \cap \mathcal{B}^= \cap \mathcal{Q}^+ \neq \emptyset$. Then, for any $s \in \mathcal{Q}^+(\hat{\tau}) \cap \mathcal{B}^= \cap \mathcal{Q}^+$ there must exist a $0 \le \tilde{\lambda} \le 1$ such that $w = \tilde{\lambda}s + (1-\tilde{\lambda})v \in \mathcal{B}^= \cap \mathcal{Q}^+$ but $w \notin \mathcal{B}^= \cap \mathcal{Q}(\hat{\tau})$. This is true because $\mathcal{Q}^+$ is convex, $\mathcal{Q}^+(\hat{\tau}) \cap \mathcal{Q}^-(\hat{\tau}) = \emptyset$, and $v \in \mathcal{Q}^-(\hat{\tau}) \cap \mathcal{B}^=$. A similar contradiction would be obtained if $\mathcal{Q}^-(\hat{\tau}) \cap \mathcal{A}^= \cap \mathcal{Q}^+ \neq \emptyset$.

Now, since $\mathcal{Q}^+(\hat{\tau}) \cap \mathcal{B}^= \neq \emptyset$ and $\mathcal{Q}^-(\hat{\tau}) \cap \mathcal{A}^= \neq \emptyset$, and also $\mathcal{Q}^+(\hat{\tau}) \cap \mathcal{B}^= \cap \mathcal{Q}^+ = \emptyset$ and $\mathcal{Q}^-(\hat{\tau}) \cap \mathcal{A}^= \cap \mathcal{Q}^+ = \emptyset$, we have that $\mathcal{Q}^+(\hat{\tau}) \cap \mathcal{B}^= \cap \mathcal{Q}^- \neq \emptyset$ and $\mathcal{Q}^-(\hat{\tau}) \cap \mathcal{A}^= \cap \mathcal{Q}^- \neq \emptyset$, because $(\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q} = (\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q}(\hat{\tau})$.

Let $w \in \mathcal{Q}^+(\hat{\tau}) \cap \mathcal{B}^= \cap \mathcal{Q}^-$. Then, $\lambda w + (1-\lambda)u \in \mathcal{Q}^+(\hat{\tau})$ for $0 \leq \lambda \leq 1$, since $\mathcal{Q}^+(\hat{\tau})$ is convex. Now, if $\mathcal{Q}$ is a hyperboloid, then there exists a $0 \leq \tilde{\lambda} \leq 1$ such that $\tilde{\lambda} w + (1-\tilde{\lambda})u \notin \mathcal{Q}$, because $u \in \mathcal{Q}^+$ and $w \in \mathcal{Q}^-$. Hence, we obtain that $(\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q} \neq (\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q}(\hat{\tau})$, which is a contradiction. On the other hand, if $\mathcal{Q}$ is a cone, there must exist a $\tilde{\lambda}$ such that either $\tilde{\lambda} w + (1-\tilde{\lambda})u \notin \mathcal{Q}$ or $\tilde{\lambda} w + (1-\tilde{\lambda})u$ is the zero vector. In the first case, we find a contradiction to $(\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q} = (\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q}(\hat{\tau})$ again. In the second case, let us consider a vector $s \in \mathcal{Q}^-(\hat{\tau}) \cap \mathcal{A}^= \cap \mathcal{Q}^-$. Then, $\lambda s + (1-\lambda)v \in \mathcal{Q}^-(\hat{\tau})$ for $0 \leq \lambda \leq 1$, since $\mathcal{Q}^-(\hat{\tau})$ is convex. In this case, there must exist a $\bar{\lambda}$ such that $\bar{\lambda} s + (1-\bar{\lambda})v \notin \mathcal{Q}$. The last statement is true because $v \in \mathcal{Q}^+$ and $s \in \mathcal{Q}^-$, the zero vector is in $\mathcal{Q}^+(\hat{\tau})$ and $\mathcal{Q}^-(\hat{\tau}) \cap \mathcal{Q}^+(\hat{\tau}) = \emptyset$. Hence, we obtain that $(\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q} \neq (\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q}(\hat{\tau})$, which is again a contradiction. This completes the proof. □

We can now complete the derivation of the DCyCs of this section. From Proposition 2 and Lemma 6, we know that the branch of $\mathcal{Q}(\hat{\tau})$ containing the set $(\mathcal{A}^= \cup \mathcal{B}^=) \cap \mathcal{Q}^+$ is a DCyC for Problem (MISOCO). Henceforth, to complete the derivation we need to define a criteria to identify the branch of $\mathcal{Q}(\hat{\tau})$ that defines the DCyC. First, consider the case when $\mathcal{Q}^+ = \{x \in \mathbb{R}^\ell \mid x \in \mathcal{Q}, x_1 \geq 0\}$, then the DCyC is given by $\mathcal{Q}^+(\hat{\tau})$. Second, consider the case when $\mathcal{Q}^+ = \{x \in \mathbb{R}^\ell \mid x \in \mathcal{Q}, x_1 \leq 0\}$, then the DCyC is given by $\mathcal{Q}^-(\hat{\tau})$. This completes the derivation of all the possible DCyCs for MISOCO problems.

Example 4. To illustrate these results, let us consider the case when $\sigma = 1$ in Example (8). Using this value we obtain the set

$$\mathcal{F}^{\mathcal{Q}} = \{(x_1, x_2, x_3)^\top \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - 4x_3 - 4 \leq 0\},$$

which is a non-normalized paraboloid. Now, for the objective function we use $(c_1, c_2, c_3, c_4) = (0, 1, -4, 0)$, the optimal solution for the continuous relaxation with this setup is $x^* = (0, 0.5, -\frac{15}{8}, \frac{17}{16})$. Given the integrality constraint over x_2, we may use the disjunction $x_2 \leq 0 \vee x_2 \geq 1$ to derive a DCyC to cut off this solution. From B we obtain that the normalized quadric $\mathcal{Q}$ is a paraboloid and that $\alpha = \frac{1}{2}$ and $\beta = 0$. Henceforth, we know from Lemma 4 that $\mathcal{Q}(-1)$ is a parabolic cylinder. For paraboloids, we have from the normalization process that in the original space of $\mathcal{F}^{\mathcal{Q}}$ we can use the same value found for the normalized quadric, i.e., $\tilde{\tau} = -1$. Hence, using $\tilde{\tau}$ in Lemma 4 we obtain the following DCyC:

$$\tilde{\mathcal{Q}}(-1) = \{(x_1, x_2, x_3)^\top \in \mathbb{R}^3 \mid x_1^2 + x_2 - 4x_3 - 4 \leq 0\},$$

which is a parabolic cylinder. Figure 4 illustrates Example 4, where one can appreciate how the DCyC cuts off the relaxed solution. In Example 4, after adding the DCyC, we obtain that the solutions $(0, 0, -1, 1)$ and $(0, 1, -\frac{3}{4}, \frac{5}{4})$ are optimal for the continuous relaxation.

3.4.2 Cones

The final part of our analysis is to complete the derivation of DCCs. For this there are three remaining cases we must consider: the family $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$ associated with the first cases of Theorems 3 and 4, and the case when $\mathcal{Q}$ is a hyperboloid of two sheets and $a_1^2 = \frac{1}{2}$.

Using our strategy, we must find first a cone in the family $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$ when $\mathcal{Q}$ is a hyperboloid of two sheets and $a_1^2 = \frac{1}{2}$. In this case we have that the numerator of (17) simplifies to

$$-\alpha\beta\tau - \rho. \tag{21}$$

Then, we have the following result.

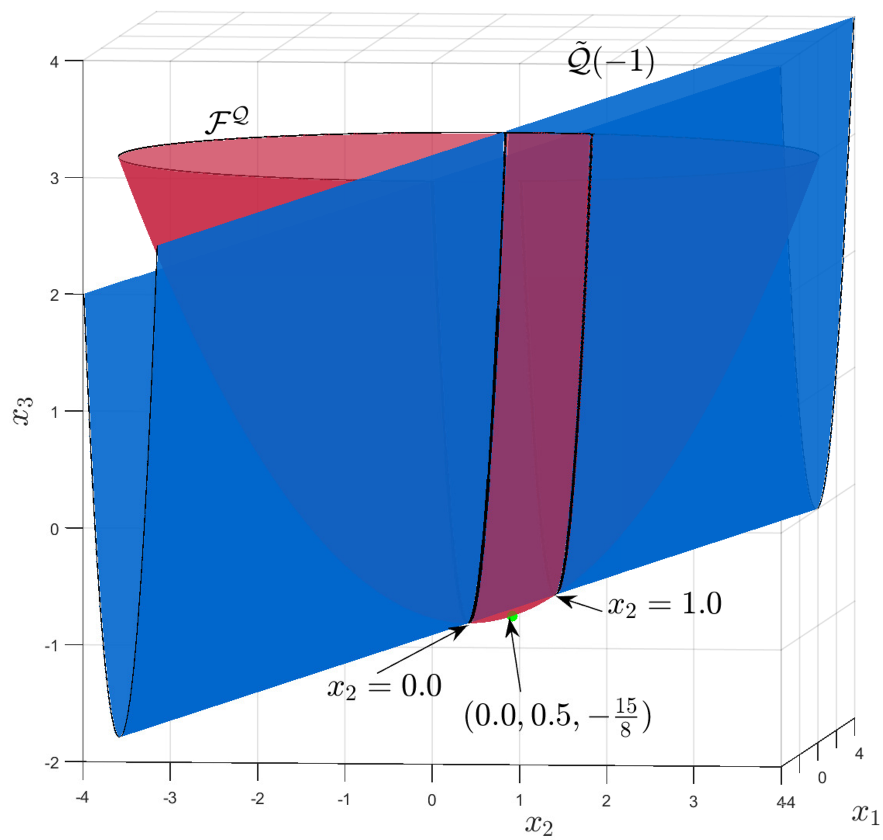


Figure 4: Illustration of Example 4: a DCyC when both $\mathcal{Q} \cap \mathcal{A}^=$ and $\mathcal{Q} \cap \mathcal{B}^=$ are unbounded.

Lemma 7 (Góez [2013]). *If $\mathcal{Q}$ is a hyperboloid, and $a_1^2 = \frac{1}{2}$, then for $\bar{\tau} = -\frac{\rho}{\alpha\beta}$ the quadric $\mathcal{Q}(\bar{\tau})$ in the family $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$ is a cone.*

Proof. In this case we have from Lemma 5 that $P(\tau)$ is always an invertible matrix with one negative eigenvalue. On the other hand, we have from (21) that $p(\bar{\tau})P(\bar{\tau})^{-1}p(\bar{\tau}) - \rho(\bar{\tau}) = 0$ for $\bar{\tau} = -\rho/\alpha\beta$. Hence, we have that the quadric $\mathcal{Q}(\bar{\tau})$ in the family $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$ is a cone. $\square$

For the derivation of the cuts we use the following representation for $\mathcal{Q}$. When $\mathcal{Q}$ is a normalized non-convex cone, we have that $\mathcal{Q} = \{w \in \mathbb{R}^\ell \mid \|w_{2:\ell}\|^2 \leq w_1^2\}$. In this case we may define $\mathcal{Q}^+ = \{w \in \mathbb{R}^\ell \mid \|w_{2:\ell}\| \leq w_1\}$ and $\mathcal{Q}^- = \{w \in \mathbb{R}^n \mid \|w_{2:\ell}\| \leq -w_1\}$, where $\mathcal{Q} = \mathcal{Q}^+ \cup \mathcal{Q}^-$, $\mathcal{Q}^+ \cap \mathcal{Q}^- = 0$, and 0 is the origin. Also, note that $\mathcal{Q}^+$ and $\mathcal{Q}^-$ are two second order cones. Now, when $\mathcal{Q}$ is a normalized hyperboloid, we have that $\mathcal{Q} = \{w \in \mathbb{R}^\ell \mid \|w_{2:\ell}\|^2 \leq w_1^2 - 1\}$. In this case, we may define $\mathcal{Q}^+ = \{w \in \mathbb{R}^\ell \mid \|w_{2:\ell}\|^2 \leq \xi,\ \|(\xi, 1)\| \leq w_1\}$ and $\mathcal{Q}^- = \{w \in \mathbb{R}^\ell \mid \|w_{2:\ell}\|^2 \leq \xi,\ \|(\xi, 1)\| \leq -w_1\}$, then $\mathcal{Q} = \mathcal{Q}^+ \cup \mathcal{Q}^-$ and $\mathcal{Q}^+ \cap \mathcal{Q}^- = \emptyset$. Also, note that $\mathcal{Q}^+$ and $\mathcal{Q}^-$ are two convex sets.

Given that the result for cones and hyperboloids of two sheets are similar, we will use $\mathcal{Q}^+$ and $\mathcal{Q}^-$ indistinctively for cones and hyperboloids. We will specify whether we are referring to a cone or a hyperboloid of two sheets when needed. Notice that in this case neither Proposition 1 or 2 applies. Hence for the derivation of the DCCs we need to prove that the cones in the first cases of Theorems 3 and 4, and of Lemma 7 satisfy Definition 1. To prove this, we use some intermediate results that are omitted here for the ease of the discussion. The interested reader can find the details of these intermediate steps in D.3, in Lemmas 8, and 11.

Theorem 5 (Góez [2013])**.** *Let $\bar{\tau}$ be the smaller root of the numerator of* (17)*. The quadric $\mathcal{Q}(\bar{\tau}) \in \{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$ of the first case of Theorems 3 and 4, and Lemma 7 contains a cone that satisfies Definition 1.*

Proof. We divide the proof into two parts. First, we show that the theorem is true for the first case of Theorem 3 when $0 \in \mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})$. Second, we show that the theorem is true when $\mathcal{Q}$ is a hyperboloid of two sheets, or $\mathcal{Q}$ is a cone and $0 \notin \mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})$.

DCC when $\mathcal{Q}$ is a cone and the vector zero is an element of $\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})$***:*** This occurs when α and β have the same sign. Then, the smallest root of $f(\tau)$ in this case is $\bar{\tau} = 0$. Hence, it is enough to show that $\mathcal{Q}^+ = \text{conv}(\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}))$ in this case. First of all, since $\mathcal{Q}^+$ is a convex set, we have that $\text{conv}(\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B})) \subseteq \mathcal{Q}^+$. Thus, to complete the proof of the first part we need to show that $\mathcal{Q}^+ \subseteq \text{conv}(\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}))$. From the definition of convex hull it is clear that $\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}) \subseteq \text{conv}(\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}))$. Now, let $\hat{x} \in \mathcal{Q}^+$ be such that $\hat{x} \notin \mathcal{A} \cup \mathcal{B}$. Then, we have that $\beta \leq a^\top \hat{x} = \sigma \leq \alpha$. Assume first that $0 \leq \beta \leq \alpha$, and consequently the vector zero is contained in $\mathcal{B}$. Since $\mathcal{Q}^+$ is a cone, then $\gamma \hat{x} \in \mathcal{Q}^+$ for $\gamma \geq 0$. Now, we have that $a^\top(\gamma \hat{x}) = \gamma \sigma$. Then, for $\gamma^1 = \frac{\alpha}{\sigma}$ we obtain $a^\top(\gamma^1 \hat{x}) = \alpha$, and for $\gamma^2 = \frac{\beta}{\sigma}$ we obtain $a^\top(\gamma^2 \hat{x}) = \beta$. Now, consider the convex combination $\lambda(\gamma^1 \hat{x}) + (1-\lambda)(\gamma^2 \hat{x})$, $0 \leq \lambda \leq 1$. For $\hat{\lambda} = \frac{1-\gamma^2}{\gamma^1-\gamma^2}$ we obtain that $0 \leq \hat{\lambda} \leq 1$, and $\lambda(\gamma^1 \hat{x}) + (1-\lambda)(\gamma^2 \hat{x}) = \hat{x}$. Since $\gamma^2 \hat{x} \in \mathcal{Q}^+ \cap \mathcal{B}$ and $\gamma^1 \hat{x} \in \mathcal{Q}^+ \cap \mathcal{A}$, then $\hat{x} \in \text{conv}(\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}))$. Now, if $\beta \leq \alpha \leq 0$, it can be shown with a similar argument that $\hat{x} \in \text{conv}(\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}))$. Hence $\mathcal{Q}^+ \subseteq \text{conv}(\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}))$, and it satisfies Definition 1, i.e., it is a DCC for $\mathcal{Q}^+$ and the disjunctive set $\mathcal{A} \cup \mathcal{B}$.

DCC when $\mathcal{Q}$ is a hyperboloid of two sheets, or $\mathcal{Q}$ is a cone and the vector zero is not an element of $\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})$***:*** In this case we have from Lemma 11 that $\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}) \in \mathcal{Q}^+(\bar{\tau})$ or $\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}) \in \mathcal{Q}^-(\bar{\tau})$. We may assume w.l.o.g. that $\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}) \subseteq \mathcal{Q}^+(\bar{\tau})$. Since $\mathcal{Q}^+(\bar{\tau})$ is a convex set we have that $\text{conv}(\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B})) \subseteq (\mathcal{Q}^+ \cap \mathcal{Q}^+(\bar{\tau})$.

To complete the proof we need to show that $\mathcal{Q}^+ \cap \mathcal{Q}^+(\bar{\tau}) \subseteq \text{conv}((\mathcal{A} \cup \mathcal{B}) \cap \mathcal{Q}^+)$. For this purpose, we prove first that $\mathcal{Q}^+ \cap \mathcal{A}^= = \mathcal{Q}^+(\bar{\tau}) \cap \mathcal{A}^=$ and $\mathcal{Q}^+ \cap \mathcal{B}^= = \mathcal{Q}^+(\bar{\tau}) \cap \mathcal{B}^=$. Observe that $\mathcal{Q}^+ \cap \mathcal{A}^= \subseteq \mathcal{Q}^+(\bar{\tau})$, then $\mathcal{Q}^+ \cap \mathcal{A}^= \subseteq \mathcal{Q}^+(\bar{\tau}) \cap \mathcal{A}^=$. Thus, it is enough to show that $\mathcal{Q}^+(\bar{\tau}) \cap \mathcal{A}^= \subseteq \mathcal{Q}^+ \cap \mathcal{A}^=$. Let $u \in Q^+ \cap \mathcal{A}^=$. Recall that $\mathcal{Q} \cap (\mathcal{A}^= \cup \mathcal{B}^=) = \mathcal{Q}(\bar{\tau}) \cap (\mathcal{A}^= \cup \mathcal{B}^=)$, hence if $\mathcal{Q}^+(\bar{\tau}) \cap \mathcal{A}^= \not\subseteq \mathcal{Q}^+ \cap \mathcal{A}^=$, then there exists a vector $v \in Q^- \cap \mathcal{A}^= \cap \mathcal{Q}^+(\bar{\tau})$. We know that $\mathcal{Q}^+ \cap \mathcal{Q}^- = 0$ if $\mathcal{Q}$ is a cone, and $\mathcal{Q}^+ \cap \mathcal{Q}^- = \emptyset$ if $\mathcal{Q}$ is a hyperboloid of two sheets. Even more, in this case if $\mathcal{Q}$ is a cone, we know that $0 \notin \mathcal{Q} \cap \mathcal{A}^=$. Hence, using the separation theorem we have that there exists a hyperplane $\mathcal{H} = \{x \in \mathbb{R}^\ell \mid h^\top x = \eta\}$ separating $\mathcal{Q}^+$ and $\mathcal{Q}^-$, such that $0 \in \mathcal{H}$. Then, there exists a $0 \leq \lambda \leq 1$ such that $\lambda u + (1-\lambda)v \in \mathcal{Q}^+(\bar{\tau}) \cap \mathcal{A}^=$ and $h^\top(\lambda u + (1-\lambda)v) = \eta$, i.e., $(\lambda u + (1-\lambda)v) \notin \mathcal{Q}$. This contradicts $\mathcal{Q} \cap (\mathcal{A}^= \cup \mathcal{B}^=) = \mathcal{Q}(\bar{\tau}) \cap (\mathcal{A}^= \cup \mathcal{B}^=)$. Hence, $\mathcal{Q}^+(\bar{\tau}) \cap \mathcal{A}^= \subseteq \mathcal{Q}^+ \cap \mathcal{A}^=$. Similarly, we can show that $\mathcal{Q}^+ \cap \mathcal{B}^= = \mathcal{Q}^-(\bar{\tau}) \cap \mathcal{B}^=$.

Now, for any $x \in \mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B})$, we have that $x \in \mathcal{Q}^+ \cap \mathcal{Q}^+(\bar{\tau})$ and $x \in \text{conv}(\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}))$. Next, we need to consider a vector $\tilde{x} \in \mathbb{R}^n$ such that $\tilde{x} \in \mathcal{Q}^+(\bar{\tau}) \cap \mathcal{Q}^+ \cap \bar{\mathcal{A}} \cap \bar{\mathcal{B}}$, where $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ are the complements of $\mathcal{A}$ and $\mathcal{B}$, respectively. From Lemma 8 we have that $x(\bar{\tau}) \in \mathcal{A}$ or $x(\bar{\tau}) \in \mathcal{B}$. We may assume w.l.o.g. that $x(\bar{\tau}) \in \mathcal{B}$. Since $\mathcal{Q}^+(\bar{\tau})$ is a translated cone, then $\{x \in \mathbb{R}^n \mid x = x(\bar{\tau}) + \theta(\tilde{x} - x(\bar{\tau})), \theta \geq 0\} \subseteq \mathcal{Q}^+(\bar{\tau})$. Thus, there exists a scalar $0 < \theta_1 < 1$ such that

$a^\top(x(\bar\tau)+\theta_1(x-x(\bar\tau)))=\beta$ and a scalar $1<\theta_2$ such that $a^\top(x(\bar\tau)+\theta_2(x-x(\bar\tau)))=\alpha$. Let $\lambda=(1-\theta_1)/(\theta_2-\theta_1)$, then $\tilde x=(1-\lambda)(x(\bar\tau)+\theta_1(x-x(\bar\tau)))+\lambda(x(\bar\tau)+\theta_2(x-x(\bar\tau)))$. Therefore, $\tilde x\in\operatorname{conv}((\mathcal{A}\cup\mathcal{B})\cap\mathcal{Q}^+)$. The same conclusion is found if we assume that $x(\bar\tau)\in\mathcal{A}$. This proves that $\mathcal{Q}^+\cap\mathcal{Q}^+(\bar\tau)\subseteq\operatorname{conv}((\mathcal{A}\cup\mathcal{B})\cap\mathcal{Q}^+)$. Thus, the cone $\mathcal{Q}^+(\bar\tau)$ is a DCC for $\mathcal{Q}^+$ and the disjunctive set $\mathcal{A}\cup\mathcal{B}$. Finally, if $\mathcal{Q}^+\cap(\mathcal{A}\cup\mathcal{B})\subseteq\mathcal{Q}^-(\bar\tau)$, then we can use a similar argument to prove that $\mathcal{Q}^-(\bar\tau)$ is a DCC for $\mathcal{Q}^+$ and the disjunctive set $\mathcal{A}\cup\mathcal{B}$. □

To complete the derivation we must define a criterion to identify which branch of $\mathcal{Q}(\bar\tau)$ in Theorem 5 defines a DCC. First, we consider the case when $\mathcal{F}^\mathcal{Q}=\mathcal{Q}^+$, then the conic cut is given by $\mathcal{Q}^+(\bar\tau)$. Second, we consider the case when $\mathcal{F}^\mathcal{Q}=\mathcal{Q}^-$, then the conic cut is given by $\mathcal{Q}^-(\bar\tau)$. This completes the derivation of all the possible DCCs for MISOCO problems.

Example 5. To illustrate this result, let us consider the case when $\sigma=\sqrt{2}+1$ in Example (8). Using this value we obtain the

$$\mathcal{F}^\mathcal{Q}=\{(x_1,x_2,x_3)^\top\in\mathbb{R}^3\mid x_1^2+x_2^2-2(1+\sqrt{2})x_3^2-4(1+\sqrt{2})x_3-4\le 0,\qquad 2+(\sqrt{2}+1)x_3\ge 0\},$$

where the quadric is a non-normalized hyperboloid of two sheets centered at $(0,0,-1)^\top$. Given the constraint $2+(\sqrt{2}+1)x_3\ge 0$, we obtain

$$\mathcal{F}^\mathcal{Q}=\{(x_1,x_2,x_3)^\top\in\mathbb{R}^3\mid \left\|(x_1,x_2)^\top\right\|\le\xi,\ \|\xi,\bar\rho\|\le\sqrt{2(1+\sqrt{2})}(x_3+1)\},$$

where $\bar\rho=\sqrt{2(\sqrt{2}-1)}$. Now, for the objective function we use $(c_1,c_2,c_3,c_4)=(0,0.5,-\sqrt{18+2\sqrt{2}},0)$, and the optimal solution for the continuous relaxation of the problem with this set up is $x^*=(0,0.5,-1+\sqrt{1-\frac{15}{8(1+\sqrt{2})}},\frac{\sqrt{31+15\sqrt{2}}}{2\sqrt{2}})$. We may use again the disjunction $x_2\le 0\vee x_2\ge 1$ to derive a DCC to cut off this solution. Here, we have that the normalized quadric $\mathcal{Q}$ is a hyperboloid of two sheets centered at the origin with $\alpha=\frac{1}{\sqrt{2(1-\sqrt{2})}}$ and $\beta=0$. Hence, from (20) we have

$$p(\tau)^\top P(\tau)^{-1}p(\tau)-\rho(\tau)=\frac{\frac{1}{8(\sqrt{2}-1)}\tau^2+\tau+1}{1+\tau},$$

and from the roots of the numerator we obtain

$$\bar\tau=4(\sqrt{2}-1)-\sqrt{40-24\sqrt{2}}.$$

Henceforth, we know from Theorem 4 that $\mathcal{Q}(\bar\tau)$ is a cone. In this case, we have that $\left\|B^{-\frac{1}{2}}V^\top a\right\|=1$, and in the original space of $\mathcal{F}^\mathcal{Q}$ we obtain

$$\tilde\tau=\frac{\bar\tau}{\left\|B^{-\frac{1}{2}}V^\top a\right\|^2}=\bar\tau.$$

Hence, using $\tilde\tau$ in Theorem 1 we obtain the following quadric:

$$\tilde{\mathcal{Q}}(\tilde\tau)=\{(x_1,x_2,x_3)^\top\in\mathbb{R}^3\mid x_1^2+(1+\tilde\tau)x_2^2-\tilde\tau x_2\le 2(1+\sqrt{2})(x_3^2-2x_3)+4\}.$$

Now, we have that $w^c = (0, -\frac{\tilde{\tau}}{2(1+\tilde{\tau})}, -1)^\top$, and we obtain that our DCC is

$$\tilde{\mathcal{Q}}(\tilde{\tau})^+ = \{(x_1, x_2, x_3)^\top \in \mathbb{R}^3 \mid \left\|(x_1, \sqrt{Q_{2,2}}(x_2 - w^c_2))^\top\right\| \leq \sqrt{Q_{3,3}}\,(x_3 + 1)\},$$

where $Q_{2,2} = 1 + \tilde{\tau}$ and $Q_{3,3} = 2(1 + \sqrt{2})$. Figure 5 illustrates Example 5, where one can appreciate how the DCC cuts off the relaxed solution. Notice that the DCC is a translated cone with the vertex at $(0, -2.0592, -1)^\top$.

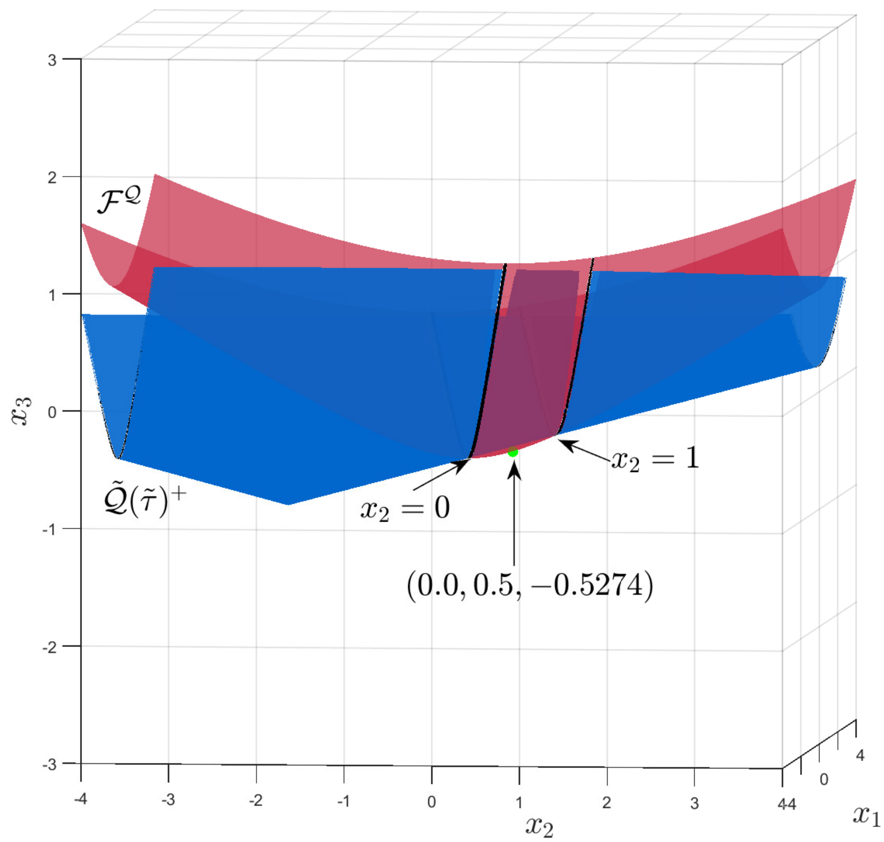


Figure 5: Illustration of Example 5: a DCC when both $\mathcal{Q} \cap \mathcal{A}^=$ and $\mathcal{Q} \cap \mathcal{B}^=$ are unbounded.

4 Conclusions

In this paper, we investigated the derivation of disjunctive conic cuts (DCCs) and Disjunctive Cylindrical Cuts (DCyC) for MISOCO problems. This was achieved by extending the ideas of disjunctive programming that have been applied successfully for obtaining linear cuts for MILO problems. We introduced first the concept of DCCs and DCyCs, which are an extension of the disjunctive cuts that have been well studied for MILO problems. In this analysis we considered disjunctions that are defined by parallel hyperplanes. Under some mild assumptions we were able to show that the intersection of these cuts with a closed convex set, given as the intersection of a SOC and an affine set, is the convex hull of the intersection of the same set with a linear disjunction. Additionally, we provided a full characterization of DCCs and DCyCs for MISOCO problems when the disjunctions are defined by parallel hyperplanes. This analysis provides a procedure for the derivation of DCCs and DCyCs. In this paper we focus our study on the geometric analysis of the sets without any deep discussion on how to solve the separation problem of a given point from the feasible set of a MISOCO problem. This is an important question given its algorithmic implications, but it is outside the scope of this paper. However, this keeps as an open topic for further research in this area.

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A Shapes of quadrics

Here we show that the shapes of the quadric $\mathcal{Q}$ are limited to those described in Section 2. We may assume that $\mathcal{Q}$ is not an empty set, otherwise there is no need for classification. Now, for the analysis of the shapes of $\mathcal{Q}$ we need the following. First, recall that $Ax^0 = b$, then the system $Hw = -x^0$ will have a solution if and only if $b = 0$. Second, recall that $P = H^\top JH$, and let $H_{1:}$ be the first row of H. Then, we have that

$$PH_{1:} = (H^\top JH)H_{1:} = (I - 2H_{1:}H_{1:}^\top)H_{1:} = (1 - 2H_{1:}^\top H_{1:})H_{1:}.$$

As a result, $H_{1:}$ is an eigenvector of P associated with the eigenvalue $(1 - 2H_{1:}^\top H_{1:})$. Third, let us define the set

$$\begin{aligned}\mathcal{F}^r &= \{x \in \mathbb{R}^n \mid Ax = b, x^\top Jx \le 0\} \\ &= \{x \in \mathbb{R}^n \mid x = x^0 + Hw, \text{ with } w \in \mathcal{Q}\}, \quad (22)\end{aligned}$$

which is a relaxation of $\mathcal{F}$. Note that due to the constraint $x^\top Jx \le 0$, if the set $\mathcal{F}^r$ contains a line, then the zero vector is an element of $\mathcal{F}^r$, i.e. $b = 0$. Finally, for the sake of clarity, we present the definition of a cylinder that is used here.

Definition 4 (Convex Cylinder [Belotti et al., 2015])**.** Let $\mathcal{D} \subseteq \mathbb{R}^n$ be a convex set and $d_0 \in \mathbb{R}^n$ a vector. Then, the set $\mathcal{C} = \{x \in \mathbb{R}^n | x = d + \sigma d_0, d \in \mathcal{D}, \sigma \in \mathbb{R}\}$ is a convex cylinder in $\mathbb{R}^n$.

We divide the classification of the shapes of $\mathcal{Q}$ in two cases: P is singular, and P is non-singular.

Let us begin classifying the shapes of $\mathcal{Q}$ when P is singular. First of all, from Lemma 2 we know that if P is singular, then $P \succeq 0$ and $(1 - 2H_{1:}^\top H_{1:}) = 0$. Consequently, $H_{1:}$ is an eigenvector of P associated with its zero eigenvalue. Now, from Section 2 we know that $\mathcal{Q}$ may be a paraboloid or a cylinder. To decide which is the case, one has to verify if the system $Pw = -p$ is solvable. On one hand, if $Pw = -p$ has no solution, then we obtain that $\mathcal{Q}$ is a paraboloid. On the other hand, if the system $Pw = -p$ is solvable, then $\mathcal{Q}$ is a cylinder. We show now that if $Pw = -p$, then given the setup of Section 2, $\mathcal{Q}$ is always a line, i.e., a cylinder whose base is a point.

Let $w^{\mathrm{c}} \in \mathbb{R}^\ell$ be such that $Pw^{\mathrm{c}} = -p$. Hence, $\mathcal{Q}$ is a cylinder, and contains a line. Now, consider the set $\mathcal{L} = \{w \in \mathbb{R}^\ell \mid w = w^{\mathrm{c}} + \sigma H_{1:}, \sigma \in \mathbb{R}\}$. Note that $\mathcal{L} \subseteq \mathcal{Q}$, which follows from the following inequality

$$(w^{\mathrm{c}} + \sigma H_{1:})^\top P(w^{\mathrm{c}} + \sigma H_{1:}) + 2p^\top (w^{\mathrm{c}} + \sigma H_{1:}) + \rho = (w^{\mathrm{c}})^\top P w^{\mathrm{c}} + 2p^\top w^{\mathrm{c}} + \rho \leq 0.$$

The first equality is true because $p^\top H_{1:} = 0$. The last inequality follows from the assumption that $\mathcal{Q}$ is not an empty set. Thus, $H_{1:}$ is the vector defining the direction of the cylinder $\mathcal{Q}$. Let us define the set

$$\mathcal{S} = \{x \in \mathbb{R}^n \mid x = x^0 + H(w^{\mathrm{c}} + \sigma H_{1:}),\ \sigma \in \mathbb{R}\},$$

which is a line in $\mathbb{R}^n$. Hence, since $\mathcal{L} \subseteq \mathcal{Q}$, we obtain from (22) that $\mathcal{S} \subseteq \mathcal{F}^r$. Additionally, recall that $\mathcal{F}^r$ may contain a line if and only if $b = 0$. Hence, it follows from $\mathcal{S} \subseteq \mathcal{F}^r$ that $b = 0$, and the system $Hw = -x^0$ is solvable because $-x^0$ is in the null space of A. Let $w^{\mathrm{c}} = (\hat{w} + \sigma H_{1:})$, where $\sigma \in \mathbb{R}$ and $H\hat{w} = -x^0$. Note that $\hat{w}$ is unique since the columns of H are linearly independent. Then, for $\sigma \in \mathbb{R}$ we have that $Pw^{\mathrm{c}} = H^\top J H \hat{w} = -H^\top J x^0 = -p$, and we obtain that

$$(w^{\mathrm{c}})^\top P w^{\mathrm{c}} + 2p^\top w^{\mathrm{c}} + \rho = -(\hat{w})^\top P \hat{w} + \rho = -(\hat{w})^\top H^\top J H \hat{w} + (x^0)^\top J x^0 = 0.$$

This shows that if the system $Pw = -p$ is solvable, then we have that $\mathcal{Q}$ is a line.

We now classify the shapes of $\mathcal{Q}$ when P is non-singular. In this case we have that (4) is equivalent to

$$\mathcal{Q} = \{w \in \mathbb{R}^\ell \mid \left(w + P^{-1}p\right)^\top P \left(w + P^{-1}p\right) \leq p^\top P^{-1} p - \rho\}. \tag{23}$$

The shape of $\mathcal{Q}$ in this case is determined by the inertia of P and the value of the right hand side of (23) [Belotti et al., 2013]. The first case to consider is when $P \succ 0$, in which case we have that $\mathcal{Q}$ is an ellipsoid. Now, to complete the classification of $\mathcal{Q}$ we need to consider the case when P is an ID1 matrix. We have the following possibilities [Belotti et al., 2013]:

- if $p^\top P^{-1} p - \rho \leq 0$, then $\mathcal{Q}$ is a hyperboloid of two sheets;
- if $p^\top P^{-1} p - \rho = 0$, then $\mathcal{Q}$ is a scaled and translated second order cone;
- if $p^\top P^{-1} p - \rho \geq 0$, then $\mathcal{Q}$ is a hyperboloid of one sheet.

We show here that the setup of Section 2 only allows $p^\top P^{-1} p - \rho \leq 0$. In other words, we need to show that $\mathcal{Q}$ is never a hyperboloid of one sheet.

We know that the vector $-P^{-1}p$ is either the vertex of a scaled second order cone or the intersection point of the asymptotes of a hyperboloid [Belotti et al., 2013]. Now, note that if $\mathcal{Q}$ is a cone or a hyperboloid of one sheet, then $-P^{-1}p \in \mathcal{Q}$. In this case, we need to show that $p^\top P^{-1} p - \rho = 0$ is always true to exclude the possibility of hyperboloid of one sheet. From Lemma 2 we know that if P is ID1, then $(1 - 2H_{1:}^\top H_{1:}) < 0$, and $H_{1:}$ is an eigenvector of P associated with its negative eigenvalue. Recall the set $\mathcal{L}$, then we have the following inequality

$$(-P^{-1}p + \sigma H_{1:} + P^{-1}p)^\top P(-P^{-1}p + \sigma H_{1:} + P^{-1}p) = \sigma^2 H_{1:}^\top P H_{1:} \leq 0,$$

which shows that $\mathcal{L} \subseteq \mathcal{Q}$ when $\mathcal{Q}$ is either a cone or a hyperboloid of one sheet. Define the set

$$\mathcal{T} = \{x \in \mathbb{R}^n \mid x = x^0 + H(-P^{-1}p + \sigma H_{1:}),\ \sigma \in \mathbb{R}\}.$$

Then, $\mathcal{T} \subseteq \mathcal{F}^r$ when $\mathcal{Q}$ is either a cone or a hyperboloid of one sheet. Now, since $\mathcal{T} \subset \mathbb{R}^n$ is a line, from $\mathcal{T} \subseteq \mathcal{F}^r$ follows that $b = 0$, which implies the existence of a unique vector $w^{\mathrm{c}} \in \mathbb{R}^\ell$ such that $Hw^{\mathrm{c}} = -x^0$. Further, we have that

$$\begin{aligned}(w^{\mathrm{c}} + P^{-1}p)^\top P(w^{\mathrm{c}} + P^{-1}p) &= (w^{\mathrm{c}})^\top P w^{\mathrm{c}} + 2p^\top w^{\mathrm{c}} + p^\top P^{-1}p \\ &= (w^{\mathrm{c}})^\top H^\top J H w^{\mathrm{c}} + 2(x^0)^\top J H w^{\mathrm{c}} + p^\top P^{-1}p \\ &= p^\top P^{-1}p + (x^0)^\top J x^0 - 2(x^0)^\top J x^0 \\ &= p^\top P^{-1}p - \rho.\end{aligned}$$

On the other hand, we have that

$$P(w^{\mathrm{c}} + P^{-1}p) = Pw^{\mathrm{c}} + p = H^\top J H w^{\mathrm{c}} + H^\top J x^0 = -H^\top J x^0 + H^\top J x^0 = 0.$$

Henceforth, we have that $p^\top P^{-1}p - \rho = 0$, and the quadric $\mathcal{Q}$ cannot be a hyperboloid of one sheet.

B Normalized quadrics

To facilitate the algebra in Sections 3.3 and 3.4 we use normalized quadrics. For discussing the normalization we need to define

$$\tilde{J} = \begin{bmatrix} \tilde{J}_{1,1} & 0 \\ 0 & I \end{bmatrix}.$$

Also, since P is a real symmetric matrix, recall that P has the following eigenvector decomposition $P = VDV^\top$, where $V \in \mathbb{R}^{\ell\times\ell}$ is an orthonormal matrix and $D \in \mathbb{R}^{\ell\times\ell}$ is a diagonal matrix. Finally, we may assume w.l.o.g. that the diagonal elements of D are arranged from smaller to bigger, where $D_{1,1}$ is the smallest value [Searle, 1982].

Using this framework, we consider the following two cases to obtain a normalized descriptions for a quadric $\mathcal{Q}$.

B.1 P is non-singular

When $\mathcal{Q}$ is a cone, a hyperboloid of two sheets, or an ellipsoid, then P is non-singular and the quadric $\mathcal{Q}$ can be writen as

$$\mathcal{Q} = \left\{ w \in \mathbb{R}^\ell \mid (w + P^{-1}q)^\top P(w + P^{-1}p) \leq pP^{-1}p - \rho \right\}. \tag{24}$$

Equation (24) can be expressed in terms of V and D. First, for $\tilde{J}$ let

$$\tilde{J}_{i,i} = \frac{D_{i,i}}{|D_{i,i}|}, i = 1, \ldots, \ell. \tag{25}$$

Hence, if $D_{1,1} < 0$ we obtain $\tilde{J}_{1,1} = -1$, and $\tilde{J}$ is the identity matrix if $D_{1,1} > 0$. Now, let $\tilde{D} \in \mathbb{R}^{\ell\times\ell}$ be a diagonal matrix defined as $\tilde{D}_{i,i} = |D_{i,i}|$, $i = 1, \ldots, \ell$. Then, we have that $P = (V\tilde{D}^{\frac{1}{2}})\tilde{J}(\tilde{D}^{\frac{1}{2}}V^\top)$ and we obtain

$$\mathcal{Q} = \{w \in \mathbb{R}^\ell \mid \left(w + P^{-1}p\right)^\top (V\tilde{D}^{\frac{1}{2}})\tilde{J}(\tilde{D}^{\frac{1}{2}}V^\top)\left(w + P^{-1}p\right) \leq p^\top P^{-1}p - \rho\}. \tag{26}$$

For this first normalized description we may define the affine transformation $\mathsf{L} : \mathbb{R}^\ell \mapsto \mathbb{R}^\ell$ as follows

$$\mathsf{L}(w) = \tilde{D}^{\frac{1}{2}} V^\top \left(w + P^{-1} p \right). \tag{27}$$

Recall that V is an orthonormal matrix, and that $\tilde{D}$ is non-singular by definition. Hence, the matrix $\tilde{D}^{\frac{1}{2}} V^\top$ is non-singular.

To complete the description of this normalization case we need now to examine the term $p^\top P^{-1} p - \rho$. Consider the case $p^\top P^{-1} p - \rho \neq 0$, and define

$$u = \frac{1}{\sqrt{|p^\top P^{-1} p - \rho|}} \mathsf{L}(w) \quad \text{and} \quad \delta = -\frac{p^\top P^{-1} p - \rho}{|p^\top P^{-1} p - \rho|}. \tag{28}$$

Then, since $\tilde{D}^{\frac{1}{2}} V^\top$ is non-singular, using (28) we obtain a one-to-one mapping between every element of $\mathcal{Q}$ and the set

$$\tilde{\mathcal{Q}} = \left\{ u \in \mathbb{R}^n \mid u^\top \tilde{J} u + \delta \leq 0 \right\}. \tag{29}$$

Now, for the case $p^\top P^{-1} p - \rho = 0$ let

$$u = \mathsf{L}(w) \quad \text{and} \quad \delta = 0. \tag{30}$$

In this case, using (30) we obtain a one-to-one mapping between $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$. The set $\tilde{\mathcal{Q}}$ in (29) defines our normalization when $\mathcal{Q}$ is a cone, a hyperboloid of two sheets or an ellipsoid.

B.2 P is singular

When $\mathcal{Q}$ is a paraboloid P has at most one non-positive eigenvalue, thus its non-positive eigenvalue in this case is 0. Hence, we have that $D_{1,1} = 0$ for the matrix D of the diagonalization of P. Define a diagonal matrix $\tilde{D} \in \mathbb{R}^{\ell \times \ell}$ as $\bar{D}_{i,i} = D_{i,i}$ for $i \in \{2, \ldots, \ell\}$ and $\bar{D}_{1,1} = 1$, and let

$$\tilde{J}_{i,i} = 1, i \in \{2, \ldots, \ell\}, \quad \text{and} \quad \tilde{J}_{1,1} = 0. \tag{31}$$

Thus, we obtain the following equivalent description

$$\mathcal{Q} = \left\{ w \in \mathbb{R}^\ell \mid w^\top V \tilde{D}^{\frac{1}{2}} \tilde{J} \tilde{D}^{\frac{1}{2}} V^\top w + 2(p^\top V \tilde{D}^{-\frac{1}{2}})(\tilde{D}^{\frac{1}{2}} V^\top w) + \rho \leq 0 \right\}. \tag{32}$$

For this normalization we define the affine transformation $\mathsf{L} : \mathbb{R}^\ell \mapsto \mathbb{R}^\ell$ as follows:

$$\mathsf{L}(w) = \tilde{D}^{\frac{1}{2}} V^\top w. \tag{33}$$

Recall that V is an orthonormal matrix, and that by construction $\tilde{D}$ is non-singular. Hence, the matrix $\tilde{D}^{\frac{1}{2}} V^\top$ is non-singular.

To complete the description of the second normalization we need to examine the value of ρ. Consider the case $\rho \neq 0$, and define

$$u = \frac{1}{\sqrt{|\rho|}} \mathsf{L}(w), \quad \bar{p} = \frac{1}{\sqrt{|\rho|}} \tilde{D}^{-\frac{1}{2}} V^\top p, \quad \omega = \frac{\rho}{|\rho|}. \tag{34}$$

Then, since $\tilde{D}^{\frac{1}{2}} V^\top$ is non-singular, using (34) we obtain a one-to-one mapping between $\mathcal{Q}$ and the set

$$\tilde{\mathcal{Q}} = \left\{ u \in \mathbb{R}^\ell \mid u^\top \tilde{J} u + 2\tilde{p}^\top u + \omega \leq 0 \right\}. \tag{35}$$

Now, for the case $\rho = 0$ define

$$u = \mathsf{L}(w), \quad \tilde{p} = \bar{D}^{-\frac{1}{2}} V^\top p, \quad \omega = 0. \tag{36}$$

In this case, using (36) we obtain a one-to-one mapping between $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$. Hence, the set $\tilde{\mathcal{Q}}$ in (35) defines our normalization when $\mathcal{Q}$ is a paraboloid.

Note that in the two cases considered P and $\tilde{J}$ have always the same inertia. Hence, the classification of the quadrics $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$ is the same. Additionally, if we apply the affine transformations L as given in (27) and (33) to two parallel hyperplanes, then the resulting hyperplanes are still parallel. Finally, by construction the transformation L has an inverse in both normalizations. These three features show that the results obtained in Section 3 using the normalized quadrics are both valid and applicable in the original space.

C Definitions and known results

Definition 5 (Base of a Convex Cylinder)**.** Let $\mathcal{C} \subset \mathbb{R}^n$ be a convex cylinder with the direction $d^0 \in \mathbb{R}^n$. A set $\mathcal{D} \subset \mathcal{C}$ is called a *base* of $\mathcal{C}$ if for every vector $x \in \mathcal{C}$, there is a unique $d \in \mathcal{D}$ and $\sigma \in \mathbb{R}$ such that $x = d + \sigma d_0$.

C.1 Eigenvalues of a rank one update

Recall that the eigenvalues of $P + \tau a a^\top$ can be computed by finding the roots of the equation

$$\det(P + \tau a a^\top - \lambda I) = 0.$$

This equation is shown [Golub, 1973] to be equivalent to the characteristic equation

$$\prod_{i=1}^{n} (P_{i,i} - \lambda) + \tau \sum_{i=1}^{n} a_i^2 \prod_{\substack{j=1 \\ j \neq i}}^{n} (P_{i,i} - \lambda) = 0. \tag{37}$$

We use this result in the proofs of this paper.

D Results for proofs with the unbounded intersections in Section 3.4

D.1 Proof of Theorem 3

We first show that

$$\bar{\tau} \geq -\frac{1}{(1 - 2a_1^2)}.$$

From (19) we have that the most negative value $\bar{\tau}$ can take is achieved when $\alpha\beta < 0$. We have

$$\frac{-4\,|\alpha\beta|}{(1 - 2a_1^2)(\alpha - \beta)^2} = \left(\frac{-1}{(1 - 2a_1^2)}\right)\left(\frac{4\,|\alpha\beta|}{(\alpha - \beta)^2}\right) \geq \frac{-1}{(1 - 2a_1^2)}. \tag{38}$$

The last inequality follows because if $\alpha\beta < 0$, then $\alpha^2 - 2\alpha\beta + \beta^2 \geq 4\,|\alpha\beta|$, since $\alpha^2 + \beta^2 \geq 2\,|\alpha\beta|$. From Lemma 5 we know that $P(\bar{\tau})$ has one negative eigenvalue and $n - 1$ positive eigenvalues if the inequality (38) is strict. If (38) is satisfied with equality, then $P(\bar{\tau})$ has one negative eigenvalue, one zero eigenvalue, and $n-2$ positive eigenvalues.

If $\bar{\tau} > -\frac{1}{(1-2a_1^2)}$, then we obtain that $\mathcal{Q}(\bar{\tau})$ is a cone. Now, we analyze the case when $\bar{\tau} = -\frac{1}{(1-2a_1^2)}$, which by (38) can happen only when $\beta = -\alpha$. In this case $P(\bar{\tau})$ is singular, $p(\bar{\tau}) = 0$, and $\rho(\bar{\tau}) > 0$. Recall that since $P(\bar{\tau})$ is symmetric, then there exist $D(\bar{\tau}) \in \mathbb{R}^{\ell\times\ell}$ and $V(\bar{\tau}) \in \mathbb{R}^{\ell\times\ell}$ such that $P(\bar{\tau}) = V(\bar{\tau})^\top D(\bar{\tau})V(\bar{\tau})$.

Let us now characterize the shape of the quadric $\mathcal{Q}(\bar{\tau})$. First, recall that when $\bar{\tau} = -\frac{1}{(1-2a_1^2)}$ then $P(\bar{\tau})$ has one negative eigenvalue, one zero eigenvalue, and $\ell-2$ positive eigenvalues. We may assume w.l.o.g. that $D_{1,1}(\bar{\tau}) < 0$, $D_{2,2}(\bar{\tau}) = 0$, and $D_{i,i}(\bar{\tau}) > 0$, $i \in \{3, \dots n\}$. Then

$$P(\bar{\tau}) = V(\bar{\tau})\hat{D}(\bar{\tau})^{\frac{1}{2}}\hat{J}\hat{D}(\bar{\tau})^{\frac{1}{2}}V(\bar{\tau})^\top,$$

where $\hat{D}(\bar{\tau})$ is a diagonal matrix with $\hat{D}_{i,i}(\bar{\tau}) = |D_{i,i}(\bar{\tau})|$, $i \in \{1, \dots n\} \setminus \{2\}$, and $\hat{D}_{2,2}(\bar{\tau}) = 1$. Additionally, $\hat{J}$ is a diagonal matrix defined as $\hat{J}_{1,1} = -1$, $\hat{J}_{2,2} = 0$, and $\hat{J}_{i,i} = 1$, $i \in \{3, \dots n\}$. Thus, using the transformation

$$u = \frac{\hat{D}(\tau)^{\frac{1}{2}}V(\tau)^\top w}{\sqrt{\rho(\bar{\tau})}}, \ \forall w \in \mathcal{Q}(\bar{\tau}),$$

we obtain that $\mathcal{Q}(\bar{\tau})$ is an affine transformation of the set

$$\{u \in \mathbb{R}^\ell \mid u^\top \hat{J} u \leq -1\}, \tag{39}$$

which is a hyperbolic cylinder of two sheets. The right hand side of the quadratic equation in (39) is -1 because

$$\rho(\bar{\tau}) = -\frac{\beta\alpha}{(1-2a_1^2)} = \frac{\alpha^2}{(1-2a_1^2)} > 0.$$

Finally, given that $\hat{J}$ and $P(\bar{\tau})$ have the same inertia, we have shown that $\mathcal{Q}(\bar{\tau})$ is a hyperbolic cylinder of two sheets. This proves the result. □

D.2 Proof of Theorem 4

First, to facilitate the discussion let $f : \mathbb{R} \mapsto \mathbb{R}$ be such that $f(\tau) = \tau^2(1-2a_1^2)\frac{(\alpha-\beta)^2}{4} - \tau((1-2a_1^2)+\alpha\beta) - 1$, which is the numerator of (20). We need to compare the roots of f with the critical value $\hat{\tau} = -\frac{1}{(1-2a_1^2)}$. Based on this comparison, we then classify the shapes of the quadrics $\mathcal{Q}(\tau)$ at these two roots.

Recalling that $\alpha \neq \beta$, the roots $\bar{\tau}_1$ and $\bar{\tau}_2$ of f are

$$\frac{2(1-2a_1^2+\alpha\beta \pm \sqrt{(1-2a_1^2+\alpha\beta)^2 + (1-2a_1^2)(\alpha-\beta)^2})}{(1-2a_1^2)(\alpha-\beta)^2}. \tag{40}$$

Hence, since $(1-2a_1^2)(\alpha-\beta)^2 > 0$, we have that one root is positive and the other is negative. We may assume w.l.o.g. that $\bar{\tau}_1 \leq \bar{\tau}_2$. Also, observe that the roots are always different, since the discriminant of (40) is never zero for $a_1^2 < 1/2$.

Let us compare these two roots with the critical value $\hat{\tau} = -\frac{1}{(1-2a_1^2)}$. First of all, note that $f(\hat{\tau}) > 0$, and that the coefficient of τ^2 in $f(\tau)$ is positive since $a_1^2 < \frac{1}{2}$. Hence, $\hat{\tau} \in (\bar{\tau}_1, \bar{\tau}_2)$.

Additionally, if $\alpha \neq -\beta$, then $f(\tau) > 0$. To complete the comparison we need to check the value of the derivative $f'(\hat{\tau})$ to verify in which branch of f the value $\hat{\tau}$ lies. We have that

$$f'(\hat{\tau}) = -\frac{(\alpha-\beta)^2}{2} - (1 - 2a_1^2 + \alpha\beta) = -\frac{(\alpha^2+\beta^2)}{2} - (1-2a_1^2) \leq 0.$$

Hence, the inequality $\hat{\tau} \leq \bar{\tau}_1$ is always satisfied, and it is strict if $\alpha \neq -\beta$.

From Lemma 5, we know that if $\hat{\tau} < \bar{\tau}_1$, then $P(\bar{\tau}_1)$ and $P(\bar{\tau}_2)$ have $\ell - 1$ positive eigenvalues and one negative eigenvalue. As a result, $\mathcal{Q}(\tau_1)$ and $\mathcal{Q}(\tau_2)$ are two different scaled second order cones. On the other hand, if $\alpha = -\beta$, then the roots of f are given by

$$\frac{1 - 2a_1^2 - \alpha^2 \pm \sqrt{(1-2a_1^2+\alpha^2)^2}}{2(1-2a_1^2)\alpha^2}.$$

Thus, $\hat{\tau} = \bar{\tau}_1$ when the hyperplanes are symmetric with respect to the origin. From Lemma 5 we know that $P(\bar{\tau}_1)$ has one negative eigenvalue, one zero eigenvalue, and $\ell - 2$ positive eigenvalues. Additionally, note that

$$\rho(\bar{\tau}_1) = 1 + \frac{\alpha^2}{(1-2a_1^2)} > 0.$$

Thus, similarly to the proof of Theorem 3, one can use the eigenvalue decomposition of $P(\bar{\tau}_1)$ to show that $\mathcal{Q}(\bar{\tau}_1)$ is an affine transformation of the set (39). Thus, $\mathcal{Q}(\bar{\tau}_1)$ is a cylindrical hyperboloid of two sheets. Finally, since $\bar{\tau}_1 < \bar{\tau}_2$, we have that $P(\bar{\tau}_2)$ has one negative eigenvalue and $\ell - 1$ positive eigenvalues, and we obtain that $\mathcal{Q}(\bar{\tau}_2)$ is a cone. This proves the result. $\square$

D.3 Additional lemmas for Section 3.4.2

Lemma 8. *Let $\bar{\tau}$ be the the smaller root of the numerator of (17). In the first cases of Theorems 3 and 4, one has that in Lemma 7 the vertex $x(\bar{\tau})$ of the quadric $\mathcal{Q}(\bar{\tau})$ is either in $\mathcal{A}$ or $\mathcal{B}$.*

Proof. From Theorem 1 we have that

$$a^\top x(\bar{\tau}) = -a^\top P(\bar{\tau})^{-1} p(\bar{\tau}) = \bar{\tau}\frac{(\alpha+\beta)(1-2a_1^2)}{2(1+\bar{\tau}(1-2a_1^2))}.$$

First of all, for Lemma 7 we obtain that $a^\top x(\bar{\tau}) = 0$. Note that if α and β have opposite signs, then each hyperplane is intersecting a different branch of $\mathcal{Q}$. This is not possible for MISOCO problems, because the feasible set of its SOCO relaxation would be non-convex. Now note that $\alpha \neq 0$ and $\beta \neq 0$, since otherwise one of the intersections $\mathcal{Q} \cap \mathcal{A}^= = \emptyset$ or $\mathcal{Q} \cap \mathcal{B}^= = \emptyset$. Hence, we have that $x(\bar{\tau}) \in \mathcal{A}$ or $x(\bar{\tau}) \in \mathcal{B}$.

Now, recall also from Sections D.1 and D.2 that $-\frac{1}{(1-2a_1^2)} \leq \bar{\tau}$. Hence,

$$\lim_{\bar{\tau}\to\infty} a^\top x(\bar{\tau}) = \frac{(\alpha+\beta)}{2}.$$

On the other hand, we have

$$\lim_{\bar{\tau} \searrow -\frac{1}{(1-2a_1^2)}} a^\top x(\bar{\tau}) = \begin{cases} -\infty & \text{if } \alpha+\beta > 0, \\ +\infty & \text{if } \alpha+\beta < 0. \end{cases}$$

Thus, if $\alpha+\beta>0$ then $a^\top x(\bar\tau)<\alpha$. Now, if $a^\top x(\bar\tau)\le\beta$ is true, then we obtain that

$$\bar\tau\frac{(\alpha+\beta)(1-2a_1^2)}{2(1+\bar\tau(1-2a_1^2))}\le\beta\quad\text{which implies}\quad\bar\tau\le\frac{2\beta}{(\alpha-\beta)(1-2a_1^2)}.$$

On the other hand, if $\alpha+\beta<0$, then $a^\top x(\bar\tau)>\beta$. Now, if $a^\top x(\bar\tau)\ge\alpha$ is true, then we obtain that

$$\bar\tau\frac{(\alpha+\beta)(1-2a_1^2)}{2(1+\bar\tau(1-2a_1^2))}\ge\alpha \text{ that implies } \bar\tau\le\frac{-2\alpha}{(\alpha-\beta)(1-2a_1^2)}.$$

Recall that $\beta<\alpha$. Then, $\alpha+\beta>0$ implies that $\alpha>0$ and $\alpha>|\beta|$. Additionally, $\alpha+\beta<0$ implies that $\beta<0$ and $\beta<-|\alpha|$.

For the first case of Theorem 3 we need to consider two cases. On one hand if $\alpha\beta\ge 0$, then $\bar\tau=0$. In this case, if $\alpha+\beta>0$ then $\frac{2\beta}{(\alpha-\beta)(1-2a_1^2)}\ge 0$, and $x(\bar\tau)\in\mathcal{B}$. Additionally, if $\alpha+\beta<0$ then $\frac{-2\alpha}{(\alpha-\beta)(1-2a_1^2)}\ge 0$ and $x(\bar\tau)\in\mathcal{A}$. On the other hand, if $\alpha\beta\le 0$ then $\bar\tau=\frac{4\alpha\beta}{(1-2a_1^2)(\alpha-\beta)^2}\le 0$. Hence, if $\alpha+\beta>0$ then

$$\frac{4\alpha\beta}{(1-2a_1^2)(\alpha-\beta)^2}=\left(\frac{2\beta}{(1-2a_1^2)(\alpha-\beta)}\right)\left(\frac{2\alpha}{(\alpha-\beta)}\right)\le\frac{2\beta}{(\alpha-\beta)(1-2a_1^2)},$$

and the vertex $x(\bar\tau)\in\mathcal{B}$. Additionally, if $\alpha+\beta<0$ then

$$\frac{4\alpha\beta}{(1-2a_1^2)(\alpha-\beta)^2}=\left(\frac{2\alpha}{(1-2a_1^2)(\alpha-\beta)}\right)\left(\frac{2\beta}{(\alpha-\beta)}\right)\le\frac{-2\alpha}{(\alpha-\beta)(1-2a_1^2)},$$

and the vertex $x(\bar\tau)\in\mathcal{A}$.

For the first case of Theorem 4 recall that

$$\begin{aligned}\bar\tau&=\frac{2\left(1-2a_1^2+\alpha\beta-\sqrt{(1-2a_1^2+\alpha\beta)^2+(1-2a_1^2)(\alpha-\beta)^2}\right)}{(1-2a_1^2)(\alpha-\beta)^2}\\&=\frac{2\left(1-2a_1^2+\alpha\beta-\sqrt{(1-2a_1^2+\alpha^2)(1-2a_1^2+\beta^2)}\right)}{(1-2a_1^2)(\alpha-\beta)^2}.\end{aligned}$$

Hence, if $\alpha+\beta>0$, then

$$\frac{2\left(1-2a_1^2+\alpha\beta-\sqrt{(1-2a_1^2+\alpha^2)(1-2a_1^2+\beta^2)}\right)}{(1-2a_1^2)(\alpha-\beta)^2}\le\frac{2\beta}{(\alpha-\beta)(1-2a_1^2)},$$

and the vertex $x(\bar\tau)\in\mathcal{B}$. Additionally, if $\alpha+\beta<0$ then

$$\frac{2\left(1-2a_1^2+\alpha\beta-\sqrt{(1-2a_1^2+\alpha^2)(1-2a_1^2+\beta^2)}\right)}{(1-2a_1^2)(\alpha-\beta)^2}\le\frac{-2\alpha}{(\alpha-\beta)(1-2a_1^2)},$$

and the vertex $x(\bar\tau)\in\mathcal{A}$. This shows that $x(\bar\tau)$ is contained in one of the sets $\mathcal{A}$ or $\mathcal{B}$. ☐

Lemma 9. *Let $\bar\tau$ be the smaller root of the numerator of* (17)*. In the first cases of Theorems 3 and 4, and in Lemma 7, we have that $\mathcal{Q}\cap(\mathcal{A}\cup\mathcal{B})\subseteq\mathcal{Q}(\bar\tau)$.*

Proof. Recall that $\mathcal{Q}(\bar\tau) = \{x \in \mathbb{R}^\ell \mid x^\top P(\bar\tau)x + 2p(\bar\tau)^\top x + \rho(\bar\tau) \le 0\}$, then from Theorem 1 we have for the first case of Theorem 3 that

$$x^\top P(\bar\tau)x + 2p(\bar\tau)^\top x + \rho(\bar\tau) = x^\top Jx + \bar\tau_1\left((a^\top x)^2 - \alpha a^\top x - \beta a^\top x + \alpha\beta\right),$$

and for the first case of Theorem 4 and Lemma 7 we have that

$$x^\top P(\bar\tau)x + 2p(\bar\tau)^\top x + \rho(\bar\tau) = x^\top Jx + 1 + \bar\tau\left((a^\top x)^2 - \alpha a^\top x - \beta a^\top x + \alpha\beta\right).$$

Recall that in the case of Lemma 7, we have that $\alpha \neq 0$ and $\beta \neq 0$ have the same sign. From (19), (40), and (21) we know that $\bar\tau_1 \le 0$ and for $\tilde x \in \mathcal{Q}$ we have $\tilde x^\top J\tilde x \le 0$ or $\tilde x^\top J\tilde x + 1 \le 0$. Now, observe that $(a^\top x)^2 - \alpha a^\top x - \beta a^\top x + \alpha\beta = (a^\top x - \alpha)(a^\top x - \beta)$. On one hand, if $\tilde x \in \mathcal{B} \cap \mathcal{Q}$, then $(a^\top\tilde x - \alpha) \le 0$ and $(a^\top\tilde x - \beta) \le 0$. On the other hand, if $\tilde x \in \mathcal{A} \cap \mathcal{Q}$, then $(a^\top\tilde x - \alpha) \ge 0$ and $(a^\top\tilde x - \beta) \ge 0$. Thus, if $\tilde x \in \mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})$, we have that

$$(a^\top\tilde x)^2 - \alpha(a^\top\tilde x) - \beta(a^\top\tilde x) + \alpha\beta \ge 0,$$

and we obtain that $\tilde x^\top P(\bar\tau)\tilde x + 2p(\bar\tau)^\top\tilde x + \rho(\bar\tau) \le 0$ for $\tilde x \in \mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})$. Thus, $\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B}) \subseteq \mathcal{Q}(\bar\tau)$. □

Lemma 10. *Let $\bar\tau$ be the smaller root of the numerator of (17). In the first case of Theorems 3 and 4, and Lemma 7, each of the subsets $\mathcal{Q}^+ \cap \mathcal{A}$, $\mathcal{Q}^+ \cap \mathcal{B}$, $\mathcal{Q}^- \cap \mathcal{A}$, and $\mathcal{Q}^- \cap \mathcal{B}$, is a subset of one of the branches $\mathcal{Q}^+(\bar\tau)$ or $\mathcal{Q}^-(\bar\tau)$.*

Proof. First, we show that either $\mathcal{Q}^+ \cap \mathcal{A} \subseteq \mathcal{Q}^+(\bar\tau)$ or $\mathcal{Q}^+ \cap \mathcal{A} \subseteq \mathcal{Q}^-(\bar\tau)$. We know from the definition of the sets in Section 3.4.2 that $\mathcal{Q}^+ \cap \mathcal{A}$, $\mathcal{Q}^+(\bar\tau)$, $\mathcal{Q}^-(\bar\tau)$ are convex sets and from Lemma 9 we have that $\mathcal{Q}^+ \cap \mathcal{A} \subseteq \mathcal{Q}(\bar\tau_1)$. Recall also that $\mathcal{Q}(\bar\tau)$ is a cone, which vertex is denoted by $x(\bar\tau)$, and that $\mathcal{Q}^+(\bar\tau) \cap \mathcal{Q}^-(\bar\tau) = x(\bar\tau)$. Then, observe that if $\mathcal{Q}^+ \cap \mathcal{A} \cap \mathcal{Q}^+(\bar\tau) \neq \emptyset$ and $\mathcal{Q}^+ \cap \mathcal{A} \cap \mathcal{Q}^-(\bar\tau) \neq \emptyset$, then $x(\bar\tau) \in \mathcal{Q}^+ \cap \mathcal{A}$, otherwise $\mathcal{Q}^+ \cap \mathcal{A} \nsubseteq \mathcal{Q}(\bar\tau)$. We have

$$\begin{aligned} x(\bar\tau) = -P(\bar\tau)^{-1}p(\bar\tau) &= -\left(J - \bar\tau\frac{Jaa^\top J}{1 + \bar\tau(1 - 2a_1^2)}\right)\left(-\bar\tau_1\frac{\alpha+\beta}{2}a\right) \\ &= \bar\tau\frac{\alpha+\beta}{2}\left(1 - \bar\tau\frac{(1-2a_1^2)}{1 + \bar\tau(1-2a_1^2)}\right)Ja \\ &= \bar\tau\frac{\alpha+\beta}{2(1 + \bar\tau(1-2a_1^2))}Ja. \end{aligned}$$

Then, we obtain that

$$x(\bar\tau)^\top Jx(\bar\tau) = \bar\tau^2\frac{(\alpha+\beta)^2(1-2a_1^2)}{4(1+\bar\tau(1-2a_1^2))^2} \ge 0.$$

Now, if $\bar\tau = 0$, then $\mathcal{Q}(\bar\tau) = \mathcal{Q}$, and it is clear that $\mathcal{Q}^+$ is a subset of $\mathcal{Q}^+(\bar\tau)$. On the other hand, if $\bar\tau \neq 0$, then $x(\bar\tau) \notin \mathcal{Q}$. For that reason $x(\bar\tau) \notin \mathcal{Q}^+ \cap \mathcal{A}$, and either $\mathcal{Q}^+ \cap \mathcal{A} \cap \mathcal{Q}^+(\bar\tau) = \emptyset$ or $\mathcal{Q}^+ \cap \mathcal{A} \cap \mathcal{Q}^-(\bar\tau) = \emptyset$. Hence, $\mathcal{Q}^+ \cap \mathcal{A}$ must be a subset of either $\mathcal{Q}^+(\bar\tau)$ or $\mathcal{Q}^-(\bar\tau)$. A similar argument can be built to show that each subsets $\mathcal{Q}^+ \cap \mathcal{B}$, $\mathcal{Q}^- \cap \mathcal{A}$, and $\mathcal{Q}^- \cap \mathcal{B}$, must be a subset of either $\mathcal{Q}^+(\bar\tau)$ or $\mathcal{Q}^-(\bar\tau)$. □

Lemma 11. *In the first case of Theorems 3 and 4 if $\mathcal{Q}^+ \cap \mathcal{A} \neq \emptyset$ and $\mathcal{Q}^+ \cap \mathcal{B} \neq \emptyset$, then we have either $\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}) \subseteq \mathcal{Q}^+(\bar\tau_1)$ or $\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}) \subseteq \mathcal{Q}^-(\bar\tau_1)$.*

Proof. From Lemma 10 we know that $\mathcal{Q}^+ \cap \mathcal{A}$ and $\mathcal{Q}^+ \cap \mathcal{B}$ are subsets of one of the branches $\mathcal{Q}^+(\tau_1)$ or $\mathcal{Q}^-(\tau_1)$. Recall that $\mathcal{Q}^+$, $\mathcal{Q}^-$, $\mathcal{Q}^+(\tau_1)$, and $\mathcal{Q}^-(\tau_1)$ are convex sets.

Now, assume to the contrary that $\mathcal{Q}^+ \cap \mathcal{A} \subseteq \mathcal{Q}^+(\bar{\tau}_1)$ and $\mathcal{Q}^+ \cap \mathcal{B} \subseteq \mathcal{Q}^-(\bar{\tau}_1)$. We need to consider two cases. First, if $\mathcal{Q}$ is a cone and $0 \in \mathcal{A} \cup \mathcal{B}$, then from (19) we obtain that $\bar{\tau} = 0$, i.e., $\mathcal{Q} = \mathcal{Q}(\bar{\tau})$. Hence it is clear that $\mathcal{Q}^+ \cap (\mathcal{A} \cup \mathcal{B}) \subseteq \mathcal{Q}^+(\bar{\tau}_1)$, which contradicts the assumption.

Second, if $\mathcal{Q}$ is a hyperboloid of two sheets, or $\mathcal{Q}$ is a cone and $0 \notin \mathcal{A} \cup \mathcal{B}$, then from the proof of Lemma 10 we know that $x(\bar{\tau}) \notin \mathcal{Q}$. Recall that $\mathcal{Q}^+(\bar{\tau}) \cap \mathcal{Q}^-(\bar{\tau}) = x(\bar{\tau})$. Hence, using the separation theorem we know that there exists a hyperplane $\mathcal{H} = \{x \in \mathbb{R}^\ell \mid h^\top x = \eta\}$ separating $\mathcal{Q}^+(\bar{\tau}_1)$ and $Q^-(\bar{\tau}_1)$, such that $x(\bar{\tau}_1) \in \mathcal{H}$. Given the assumption $\mathcal{Q}^+ \cap \mathcal{A} \subset \mathcal{Q}^+(\bar{\tau}_1)$ and $\mathcal{Q}^+ \cap \mathcal{B} \subset \mathcal{Q}^-(\bar{\tau}_1)$, we have that $\mathcal{H}$ must separate $\mathcal{Q}^+ \cap \mathcal{A}$ and $\mathcal{Q}^+ \cap \mathcal{B}$ as well. Hence, $\mathcal{H}$ must be parallel to $\mathcal{A}$ and $\mathcal{B}$, and $\beta \leq \eta \leq \alpha$. Now, if $\beta < \eta < \alpha$, then we obtain that $x(\bar{\tau}) \notin \mathcal{A} \cup \mathcal{B}$, which contradicts Lemma 8. On the other hand, if $\eta = \alpha$ or $\eta = \beta$, we obtain that $x(\bar{\tau}_1) \in \mathcal{Q}$, which is also a contradiction. This proves the lemma. $\square$