Capacitated Node Routing Problems
(Preliminary Progress Report)

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1 Introduction

This abstract describes a branch, cut, and price (BCP) algorithm for the solution of a class of network design problems with routing and packing constraints that can all be seen as special cases of a general model we will call the *Capacitated Node Routing Problem* (CNRP). We present several alternative integer programming formulation of the CNRP that can be shown to generalize a number of other important combinatorial models, including the Vehicle Routing Problem (VRP), the Capacitated Spanning Tree Problem (CSTP), and the Cable Trench Problem (CTP). These problems all have a common underlying flow structure which has not been exploited in previous studies from the literature.

Designing efficient networks for routing a commodity under capacity constraints is an important and fundamental problem with wide application. CNRPs such as the VRP arise commonly in logistics and the design of inventory distribution networks for manufacturing. The CSTP [26] addresses the layout of computer and telecommunications networks under capacity constraints. The CTP [104] models the problem of digging trenches and laying cable to connect a network of computers to a central communications node. Other variations of the model we will discuss have important applications in scheduling and routing.

In the CNRP model we will propose, the fixed and variable costs on the network, in combination with some connectivity constraints, impose a routing structure that determines which routes in the network can be used most efficiently. At the same time, the capacities impose a packing structure by determining how many units of a commodity can be routed along each network path. By varying the cost and capacity structure of the network, we can model each of the aforementioned problems.

Although there has been significant effort devoted to solving many of these design problems (particularly the VRP) in the last decade, little real progress has been made. Even with tremendous advances in computing power and the advent of affordable parallel computing platforms, the smallest unsolved instance of the VRP remains at just 50 customer nodes [95]. Attempts at solving the CSTP, a relaxation of the VRP, have not fared much better [58]. This is due in part to our myopic view of these problems. Because of the intense effort devoted to solving the well-studied Traveling Salesman Problem (TSP), most approaches in the literature have focused almost exclusively on the routing structure of these problems. However, our experience indicates that it is

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the additional packing structure that makes CNRPs difficult. In fact, most instances of the VRP become extremely easy when the capacity restriction is relaxed.

This preliminary study is motivated by previous work on the VRP [95] and explores the complex interaction between the routing and packing structures of these problems in the context of a general network design model. Our goal is to shed light on the relationship between the very difficult VRP and some more tractable relaxations. We will describe a number of alternative solution methods based on the general technique of BCP and provide preliminary computational results for these algorithms.

2 A Capacitated Node Routing Model

The CNRP is a variant of the well-known fixed-charge network flow problem (FCNFP), in which we have only one supply node (called the depot) and in which we may require the nodes in the network to have a specified in-degree or out-degree. By making this very simple modification to the FCNFP, we arrive at a model which subsumes each of the aforementioned problems. Since we will be focusing primarily on the Vehicle Routing Problem and some of its relaxations, we first describe that problem.

2.1 The Vehicle Routing Problem

The standard VRP model was introduced by Dantzig and Ramser [37] in 1959 and is easily seen to be $\mathcal{NP}$-complete. In this graph-based problem, a central depot $\{0\}$ uses k independent delivery vehicles, each of identical capacity C, to service integral demands d_i for a single commodity from customers $i \in N = \{1, \cdots, n\}$. Delivery is to be accomplished at minimum total cost, with c_{ij} denoting the transit cost from i to j, for $0 \le i, j \le n$. The cost structure is assumed *symmetric*, i.e., $c_{ij} = c_{ji}$ and $c_{ii} = 0$.

A solution for this problem consists of a partition $\{R_1, \ldots, R_k\}$ of N into k *routes*, each satisfying $\sum_{j \in R_i} d_j \le C$, and a corresponding permutation, or *tour*, σ_i, of each route specifying the service ordering. This problem is naturally associated with the complete undirected graph consisting of nodes $N \cup \{0\}$, edges E, and edge-traversal costs $c_{ij}, \{i, j\} \in E$. A solution is a cloverleaf pattern whose k petals correspond to the routes serviced by the k vehicles. The most common integer programming formulation for this problem is as follows:

$$\min \sum_{e \in E} c_e x_e$$

$$\sum_{e=\{0,j\} \in E} x_e = 2k \tag{1}$$

$$\sum_{e=\{i,j\} \in E} x_e = 2 \quad \forall i \in N \tag{2}$$

$$\sum_{\substack{e=\{i,j\} \in E \\ i \in S, j \notin S}} x_e \ge 2b(S) \quad \forall S \subset N,\ |S| > 1 \tag{3}$$

$$0 \le x_e \le 1 \quad \forall e = \{i, j\} \in E,\ i, j \ne 0 \tag{4}$$

$$0 \le x_e \le 2 \quad \forall e = \{0, j\} \in E \tag{5}$$

$$x_e \quad \mathit{integral} \quad \forall e \in E. \tag{6}$$

For ease of computation, we define $b(S) = \lceil (\sum_{i \in S} d_i)/C \rceil$, an obvious lower bound on the number of trucks needed to service the customers in set S. Thus, the constraints (3) implicitly enforce that no route can service total demand greater than the capacity C.

The VRP can be modeled as a *restriction* of the Traveling Salesman Problem by simply adjoining to the graph $k-1$ additional copies of node 0 and its incident edges (there are no edges among the k depot nodes). A tour in the resulting graph specifies a VRP solution provided the total demand along each of the k segments joining successive copies of the depot does not exceed C. Hence, we see that by relaxing the capacity constraints, i.e., setting $C = \infty$, we arrive at an instance of the TSP. Alternatively, the question of whether a given set of customer demands can be serviced with k vehicles of identical capacity is an instance of the *Feasibility Bin Packing Problem* (FBPP). By setting the edge costs to zero, we arrive at an instance of that problem.

2.2 A Flow Formulation

A flow formulation of the VRP, which is motivated by previous work [13, 83, 16], exhibits connections to other combinatorial models, including the FCNFP and the well-studied *Capacitated Network Design Problem*. To understand the flow formulation, consider first a directed version of the VRP. If x constitutes a feasible solution to a directed version of the formulation presented above, then it must be possible to route d_i units of flow from the depot to each node i in a network of capacity Cx. Similarly, in the undirected case, we form a network by replacing each undirected edge $\{i, j\}$ with two directed edges, each having capacity $\frac{1}{2}Cx_{ij}$. Then it must again be possible to route d_i units of flow to each node i from the depot.

Using these same ideas, we now present a new flow formulation of the VRP. In this formulation, the variables y_{ij} represent the flow of a commodity from node i to node j and x_{ij} is a fixed-charge variable that is 0 or 1 depending on whether or not edge $\{i, j\}$ is used in the solution. Note that this formulation is directed, although the underlying graph is not. Later, we will describe an alternative formulation in which the fixed-charge variables remain undirected. With the flow variables in the formulation, we can now remove the capacity constraints (3) and replace them with flow constraints. Letting the set A be the set of directed edges obtained by taking two oppositely oriented copies of each undirected edge in the original edge set E, we then get the following new formulation of the problem.

$$min \quad \sum_{e=\{i,j\} \in E} c_e(x_{ij} + x_{ji})$$

$$\sum_{(0,j) \in A} x_{0j} = k \tag{7}$$

$$\sum_{(j,0) \in A} x_{j0} = k \tag{8}$$

$$\sum_{(i,j) \in A} x_{ij} = 1 \quad \forall i \in N \tag{9}$$

$$\sum_{(j,i) \in A} x_{ji} = 1 \quad \forall i \in N \tag{10}$$

$$\sum_{(j,i) \in A} y_{ji} - \sum_{(i,j) \in A} y_{ij} = d_i \quad \forall i \in N \tag{11}$$

$$y_{ij} \leq Cx_{ij} \quad \forall (i,j) \in A \tag{12}$$

$$y_{ij} \geq 0 \quad \forall (i,j) \in A \tag{13}$$

$$0 \leq x_{ij} \leq 1 \quad \forall (i,j) \in A,\ i,j \neq 0 \tag{14}$$

$$x_{ij} \quad integral \quad \forall (i,j) \in A. \tag{15}$$

Here, the constraints (7)-(10) state that the in-degree and the out-degree of the depot should be k and that the in-degree and out-degree of all other nodes should be one. These constraints impose the routing structure. Constraints (11) are flow conservation constraints and (12) play the double role of enforcing the capacity restrictions and forcing the fixed-charge variables towards value one for edges that have flow on them. The role of these special constraints is discussed further below and in Section 3.1.2. The remaining constraints are variable bounds.

This formulation offers several advantages over that given by (1)-(6). To begin with, the number of constraints is polynomial in the size of the problem, whereas the previous formulation is not only exponential in size, but $\mathcal{NP}$-complete to solve. Furthermore, if we replace constraints (12) with the tighter form

$$y_{ij} \leq (C - d_i)x_{ij} \ \forall (i,j) \in A, \tag{16}$$

it is easy to show based on a result from [23] that any x satisfying this tightened formulation automatically satisfies all of the multistar inequalities, a class of valid inequalities originally proposed in [10]. Hence, this relaxation is necessarily stronger than that given by (1)-(2) and (4)-(6) without the constraints (3), which can be added dynamically to either model. We will briefly discuss methods for this in Section 3.1.

More importantly, this formulation provides the opportunity for generalization that we introduced earlier. This generalization was motivated by work we are doing on the CTP, so we briefly introduce that problem now. The CTP is the problem of minimizing the cost of digging trenches and laying cable for a communications network with a central hub. We are given a per-unit cost for digging the trench (τ) based on the length of the trench and also a per-unit cost of laying the cable (γ). There must be one continuous length of cable reaching from the hub (node 0) to each node in the network and traveling through existing trenches. This can be seen as an instance of the FCNFP in which there is a designated node which supplies $n-1$ units of a commodity to the $n-1$ other nodes in the network, each of which demands 1 unit (see Section 2.4.2 for more detail). Observe that there is both a fixed cost and a variable cost associated with each edge used in the solution. This form of the objective function is in essence what makes the problem $\mathcal{NP}$-complete.

The standard VRP model considers only a fixed charge for each edge traversed in the final solution. By introducing a variable cost for the flow, we get a more general model that contains the CTP and others as special cases. To achieve this, we simply add some terms to the objective function to obtain

$$min \ \ \tau \left(\sum_{e=\{i,j\} \in E} c_e(x_{ij} + x_{ji}) \right) + \gamma \left(\sum_{e=\{i,j\} \in E} c_e(y_{ij} + y_{ji}) \right) \tag{17}$$

and we get the desired CNRP model. In Section 2.4, we will discuss the relationship between this model and other important combinatorial models. First, we discuss a few alternative formulations.

2.3 Alternative Formulations

There are two alternative formulations, which we will not review in detail because of space constraints, but which are important alternatives to that presented above. First, we note that it is possible to develop a model for the VRP in which there is only one fixed-charge variable per undirected edge, instead of two. This is done essentially by sending half of the flow around each side of the “route” in question. Although this formulation does correctly model the standard VRP, it does not allow us to utilize the variable cost structure of our model since the flows no longer represent the actual demand flow through the network. As we will discuss in Section 4, however, this model has the advantage that it eliminates some of the symmetry that is naturally present in the directed formulation. This symmetry is a product of the fact that demand can equivalently flow either direction around a defined route. Hence, every feasible solution to the original undirected problem can be represented in 2^k different ways in the new model. This behavior only holds true for the VRP, however. Problems whose feasible solutions have a tree structure do not have this problem with symmetry.

A second alternative, which we have just begun to investigate is the multi-commodity flow formulation, in which each customer's demand is considered to be a separate commodity. A similar formulation has been successfully used for the solution of the Steiner Tree Problem in [14]. Although this formulation is $O(|N|)$ times as large as the flow formulation we first proposed, it has the advantage of being significantly tighter. Because the flow can be decomposed by commodity, we can easily tighten the capacity constraints (12) (the flow of each commodity in the network must be less than or equal to the demand for that commodity). In order to model the truck capacity, we must then add a bundling constraint which ensures that the total flow on each edge in the network is less than or equal to the capacity. Tightening of the capacity constraints (12) seems to be critical to effective solution of these models since these constraints effectively force the fixed-charge variables towards value one when an edge has flow on it. This can be seen more clearly if one rewrites the constraints (12) in the form

$$x_{ij} \geq \frac{y_{ij}}{C}. \tag{18}$$

It then becomes obvious that the closer the denominator comes to equaling the numerator on the right-hand side, the more effective this constraint will be. With a multi-commodity flow formulation, the numerator and denominator should be equal. This topic will be further discussed further in Section 3.1.2.

2.4 Related Models

2.4.1 Fixed-charge Network Flow and Capacitated Network Design Problems

We have already mentioned the close connection of our model to the FCNFP. In fact, by deleting the degree constraints (7)-(10) and setting the capacity C equal to the sum of the demands, we obtain the FCNFP. By further deleting the upper bounds on the fixed-charge variables, we obtain the model commonly known as the *Capacitated Network Design Problem* (CNDP). Hence, this models a situation in which capacity must be installed on a network in discrete units. Hence, our model can be viewed as a side-constrained version of either the FCNFP or CNDP in which the network is constrained to have a particular structure. We will explore this relationship more thoroughly later in this abstract.

2.4.2 The Cable Trench Problem

Very little is known about the CTP to date. As we have already mentioned, the problem is that of minimizing the cost of digging trenches and laying cable for a communications network with a central hub. We are given a per-unit cost for digging the trench (τ) based on the length of the trench and also a per-unit cost of laying the cable (γ). There must be one continuous length of cable reaching from the hub (node 0) to each node in the network and traveling through existing trenches.

Conceptually, this problem is a combination of the Minimum Spanning Tree Problem (MST) and the Shortest Path Problem (SPP) [104]. As in the VRP, we model the problem on a complete, undirected graph $G = (N \cup \{0\}, E)$, where $\{0\}$ is the specified depot node, with a specified edge length c_{ij} for each edge (i, j). Given a spanning tree T of G, denote the total length of the spanning tree by $l(T)$ and the sum of the path lengths p_i from node 0 to each node i in T by $s(T)$. Then the objective is to find T such that $\tau l(T) + \gamma s(T)$ is minimized, where τ and γ are the given per-unit costs.

In our generalized model, the CTP can be obtained simply by deleting the depot degree constraints (7) and (8) and the degree constraints (9). This leaves the flow-balance constraints and constraints (10) that require the in-degree of each non-depot node to be one, which ensure that we get a spanning tree. In fact, it is easily shown that constraints (10) are unnecessary, but may be left in the formulation to strengthen the LP relaxation. The demand at each node is set to 1 and the capacity C is $|N| - 1$ (this is an uncapacitated model).

What makes this problem interesting is the role of the ratio τ/γ. If this ratio is "large enough," then the solution to this problem will be a minimum spanning tree. If this ratio is "small enough," then we will simply get a shortest paths tree. However, for general ratios, it can be show that this problem is $\mathcal{NP}$-complete. Hence, this problem exhibits the very interesting property that different forms of the objective function can lead to wide variations in the difficulty of a particular instance.

2.4.3 Capacitated Spanning Tree Problem

The CSTP was originally proposed in [26] as a model for designing efficient computer networks. In this problem, as in the VRP, we again have only fixed edge costs ($\gamma = 0$), a general demand associated with each node and a capacity C in constraints (12) that represents the maximum demand that can be serviced by any particular subnet. As in the CTP, we remove the degree constraints (7) and (8), as well as the degree constraints (9). This model seems to share a great deal of structure with the VRP. In fact, it retains all of the packing structure from the VRP while relaxing only the routing structure. As we have already discussed, it appears that the packing structure is what makes these problems difficult to solve. Therefore, it is entirely consistent that instances of this problem are nearly as hard as instances of the VRP.

3 Solution Approach

In a typical branch and bound algorithm for combinatorial optimization, the bounding operation is accomplished using the tools of linear programming, a technique first described in full generality by Hoffman and Padberg [68]. When both variables and cutting planes are generated dynamically during LP-based branch and bound, the technique becomes known as *branch, cut, and price* (BCP). As a core technique, we propose to solve the described CNRP model using the branch, cut, and price (BCP) algorithm. This well-known algorithm has been very successful in solving difficult discrete optimization problems (DOPs) (for example, see [4]). However, it has received little attention in the literature, primarily because of the effort required to develop an effective implementation. Most previous research in this area has focused on the VRP and the CSTP. A number of authors have proposed implementations of branch and bound and branch and cut for these difficult problems (for example, see [9, 13, 23, 95, 58, 81]).

We implemented our algorithm within a generic parallel framework for BCP created by the author called SYMPHONY [98, 99, 97]. SYMPHONY achieves a "black box" structure by separating these problem-specific methods from the rest of the implementation. The internal library interfaces with the user's subroutines through a well-defined API and independently performs all the normal functions of BCP—tree management, LP solution, and pool management, as well as inter-process communication (when parallelism is employed). Although there are default options for most of the operations, the user can also assert control over the behavior of the algorithm by overriding the default methods and through a myriad of parameters.

3.1 Valid Inequalities and Separation

The most important and challenging aspect of any BCP algorithm is designing subroutines that effectively separate a given fractional point from the convex hull of integer solutions. This has been, and still remains, a very challenging aspect of applying BCP techniques to this class of problems. In our case, we begin by looking at valid inequalities that are already known from the study of various related models such as the CNDP, VRP, TSP, and CSTP. Overall, the strongest inequalities are likely to be those derived within the context of the VRP (see [77, 78, 10, 24, 30, 80, 13, 12, 23, 92, 11, 83]). Since our generalized model has the same underlying polyhedral structure as the standard VRP, we can apply all these known classes directly. In addition, the network design literature contains classes of valid inequalities known for the CNDP polyhedron and relaxations (see [2, 3, 21, 22]).

Despite the long list of known valid inequalities that we have to choose from, the separation problem for this polyhedron remains extremely difficult. Classes that *do* admit effective separation procedures tend to be ineffective in the context of branch and cut. Furthermore, while there are some polyhedral results related to these classes of inequalities, it is not possible in general to say whether they define facets for the polyhedra in which we are interested. In the remainder of the section, we will outline some ideas for addressing this problem.

3.1.1 Generalized Subtour Elimination Constraints

The most obvious class of cuts to begin working with are the constraints (3) from our original formulation, which we were able to drop by introducing the flow variables. In the formulation (7)-(15), these can be rewritten as

$$\sum_{i \notin S, j \in S} x_{ij} \geq \delta b(S) \;\; \forall S \subset N, \; 0 \notin S, \tag{19}$$

where δ is 1 for problems such as the CSTP in which feasible solutions take the form of a tree and 2 for problems such as the VRP and TSP in which feasible solutions take the form of a tour. These constraints help enforce the connectivity required by the network capacity constraints and are valid for all the problems we have discussed. They have been called by various names in the literature, but we will refer to them as the *generalized subtour elimination* (GSE) constraints since they can be seen as a generalization of the subtour elimination constraints for the TSP. For the VRP, the GSE constraints provide the link between the packing constraints of the FBPP and the routing constraints of the TSP. They are hence understandably important in describing the polyhedral structure of the polytope. However, the separation problem for these constraints was shown to be $\mathcal{NP}$-complete by Harche and Rinaldi (see [13]) even if $b(S)$ is taken to be $\lceil (\sum_{i \in S} d_i)/C \rceil$.

Because of the apparent intractability of solving the separation problem, a good deal of effort has been devoted in the literature to developing effective separation heuristics for these constraints. However, until the paper by Augerat, et al. [13], most known algorithms were not very effective at locating violated GSE constraints. In [12], it is shown that the fractional version of these constraints (i.e., with $b(S)$ replaced by $\left(\sum_{i \in S} d_i\right)/C$) is polynomially separable. This method can be used as a heuristic for finding violated constraints of the form (3). Other heuristics that we have implemented for separating these constraints are described in [95].

3.1.2 Flow Linking and Other Constraint Classes

Because this model is related to several well-studied problems, the existing literature is a rich source of valid inequalities. Many known classes of valid inequalities for the VRP are generalizations of classes known for the TSP [6]. In our previous experience, these seem to have limited effectiveness because they focus almost entirely on the routing structure of the problem and do not account for the packing structure. Further generalizations of the GSE constraints, such as framed capacity constraints show promise of effectiveness, but cannot currently be separated effectively. We are also considering many known classes of inequalities that have been used to solve the FCNFP and the CNDP. However, since these are both relaxations of our model, these classes may or may not prove to help.

One obvious set of constraints useful for cutting off certain fractional solutions is what we call the *edge cuts*, given by

$$x_{ij} + x_{ji} \le 1 \;\forall \{i,j\} \in E. \tag{20}$$

We have already mentioned that one of the main roles of the flow variables in the model we have proposed is to "force" the fixed-charge variables towards value one when an edge is used. As before, this can be seen more clearly if one rewrites the constraints (16) in the form

$$x_{ij} \ge \frac{y_{ij}}{C - d_i}. \tag{21}$$

One way to tighten these constraints is to use a multi-commodity flow formulation. This idea was discussed in Section 2.3. Another way is to consider alternative forms of these flow linking constraints. Another form is

$$y_{ij} - \sum_{k \in \delta(\{j\})} y_{jk} \le x_{ij} d_j \;\forall j \in N. \tag{22}$$

Like the multi-commodity flow formulation, these constraints attempt to decompose the flow in order to force the fixed-charge variable to one.

The most effective classes, though, are likely to be those yet to be found that will explicitly account for the packing structure of the problem. One such class is known already—the path-bin inequalities, proposed in [92] and [11]. This class is a generalization of the comb inequalities that attempts to explicitly forbid routing structures that are not possible because of the packing structure of the problem. During this work, we hope to find more such classes of inequalities, as well as methods for separating them. In the next section, we discuss a general technique we developed in previous work on the VRP, which we hope will allow us to separate whatever new classes of inequalities we do find.

3.2 Branching

Although effective branching is critical to the success of any branch and bound scheme, comparatively little effort has been devoted in the literature to finding better methods of branching for these models. In previous work, we experimented with branching strategies involving branching on the fixed-charge variables and on GSE constraints within the strong branching framework provided by SYMPHONY. The strong branching paradigm has proven an extremely effective tool for reducing the size of the search tree. However, it is still dependent on choosing a "good" list of branching candidates. In general terms, "good" candidates are ones that impose enough additional structure on each new subproblem to increase the lower bound significantly or force the solution to become feasible (integral). In the VRP, almost all branching schemes proposed to date have focused on imposing *routing* structure on each of the subproblems. This is natural—the edge variables are obvious candidates for branching and fixing their value gives structure to the emerging routes.

We noticed the extreme bias of these branching schemes towards imposing routing structure during preliminary experiments in which we attempted to solve the FBPP using our VRP solver by setting all the edge costs to zero. This approach actually performed very poorly. In fact, it was generally faster to solve the original VRP instance to optimality. Without the edge costs to guide the branching decisions, the branching became random and meaningless. Our conclusion is that we are doing almost nothing to impose *packing* structure on these problems during branching. We conjecture that this is one reason why many extremely difficult instances of the VRP become easy when the capacity is relaxed—branching becomes much more effective when the packing structure is relaxed.

Approximate solution techniques for the *Number Partitioning Problem* (NPP) based on incomplete branch and bound yield clues we can use to develop better branching schemes for the VRP [103]. Recall that the NPP is that of dividing a set of positive integers into two subsets such that the two subset sums are as close as possible to each other. A simple greedy approximation scheme begins by sorting the numbers in decreasing order and placing the largest number in one of the two subsets. Then place each remaining number in the subset with the smaller total sum thus far. This generalizes naturally to the set-differencing method of Karmarkar and Karp [72]. At each step, the Karmarkar-Karp heuristic (KK) places the two largest numbers in opposite subsets by placing an arc between them in a corresponding graph. It then replaces these numbers with their difference and continues until there is only one node left. Finally, the resulting graph is two-colored in order to determine the assignment of nodes. As opposed to the greedy heuristic, KK defers the decision about which subset each element will go in until the end of the algorithm. This heuristic is very efficient and effective.

One can naturally associate complete branch and bound schemes with these two heuristics. Interestingly, they each branch in fundamentally different ways. The greedy approach branches simply by choosing an item and putting it into one of the two bins in each branch. The KK approach instead branches by choosing a pair of items and specifying that in one branch, they should be in the same bin and in the other, they should be in different bins. It seems obvious that the second branching scheme will be more effective and that turns out to be true in practice.

In future work, we would like to implement a similar branching scheme within our CNDP solver. However, doing so is not straightforward. In light of the formulation given in Section 2 for the VRP, we observe the following. When branching on a binary edge variables in this formulation, the branch in which the variable corresponding to that edge is fixed to "1" does in fact correspond to imposing that the pair of customers corresponding to the endpoints of that edge in the graph should be on the same route. But the *other* branch, in which the edge variable is fixed to zero, does *not* correspond to that particular set of customers *not* being on the same route. In fact, it imposes very little structure if any on the problem. This means we will need a radical departure from what we are currently doing to make this scheme work. We also envision experimenting with other schemes such as branching on sets of variables and constraints.

4 Preliminary Computational Results

We have implemented a preliminary version of this algorithm using the SYMPHONY library described earlier and used it to solve instances of the TSP, VRP, CSTP, and CTP. This preliminary solver includes support for both the directed and undirected formulations discussed in Sections 2.2 and 2.3 and is already roughly competitive with previously reported results for the VRP and CSTP. The solver currently includes dynamic generation of GSE constraints and edge cuts, as well as the option to dynamically generate the constraints 16 and (22). We also have some basic TSP cutting plane routines borrowed from the CONCORDE TSP Solver [4].

Empirical evidence so far indicates that the formulation yielded by (7)-(15), plus constraints (20) and (22) is no stronger than that obtained by adding dynamically generated cuts of the form (19) to the formulation obtained by deleting the flow variables (and the corresponding constraints) from our new formulation (this is equivalent to the classical formulation). This is not altogether surprising, since we conjecture that ultimately, the formulation (1)-(6), is stronger than (7)-(15) for models without variable costs on the flows in the network. In practice, however, this depends on the effectiveness of separation routines for constraints (19), which appears to be quite good.

The most interesting aspect of our preliminary results is our ability to compare directly the difficulty of the VRP, TSP, and CSTP on problem instances from the literature. Figure 1 contains initial computational results for these three problems. The data for these instances is available at `www.BranchAndCut.org/VRP`. As we predicted, the undirected formulation seems to perform much better for the TSP and VRP due to the reduction in symmetry. However, the directed formulation performs vastly better for the CSTP. Although we were not surprised by this result, it is interesting that simply relaxing the out-degree constraints creates such a difference in behavior of the solver. The preliminary results strongly support our hypothesis that it is the packing structure of these capacitated models that makes them so difficult. After relaxing this packing structure to obtain the TSP, these instances became vastly easier. Relaxing the routing structure to obtain the CSTP did not change the difficulty very much. Versions of these problems with positive variable cost on the edges are extremely difficult for our current methods. We therefore plan to concentrate future efforts on explaining this behavior.

5 Conclusions and Future Work

In this abstract, we defined a new general model for capacitated routing and packing problems called the Capacitated Node Routing Problem and developed a corresponding flow-based integer programming formulation. Our initial computational results have shown that this new flow-based formulation by itself does not dominate the classical formulation. We have shown empirically that it is the packing structure that makes these problems so difficult for our current methods. We conjecture that by using the additional information yielded by the presence of flow variables will allow us to generate stronger valid inequalities and perform more intelligent branching which better

	TSP		$CSTP$		VRP	
problem	Tree Size	CPU sec	Tree Size	CPU sec	Tree Size	CPU sec
eil13	1	0.00	13	0.09	1	0.00
eil22	1	0.11	2	0.10	1	0.02
eil33	1	0.02	69	3.97	2	0.44
bayg29	1	0.12	1	0.04	4	0.32
bays29	1	0.17	15	1.12	5	0.55
ulysses16.tsp	1	0.00	1	0.03	1	0.01
ulysses22.tsp	1	0.00	1	0.06	1	0.03
gr17	1	0.01	5	0.05	1	0.01
gr21	1	0.00	1	0.02	1	0.03
gr24	1	0.02	5	0.27	4	0.40
fri26	1	0.02	1	0.07	8	0.39
swiss42	1	0.02	35	3.66	10	2.45
att48	2	0.30	92	5.04	193	30.10
gr48	2	1.38	1	0.07	16	4.17
hk48	1	0.19	209	22.88	45	21.19
eil51	1	0.16	77	15.11	11	10.79
A − n32 − k5	1	0.02	1	0.07	2	0.20
A − n33 − k5	3	0.81	3	0.21	7	0.90
A − n34 − k5	6	2.06	4	0.40	9	2.63
A − n36 − k5	1	0.03	52	5.17	51	7.95
A − n37 − k5	1	0.03	5	0.22	11	0.97
A − n38 − k5	1	0.10	1	0.13	111	21.80
A − n39 − k5	1	0.30	11	0.99	480	310.92
A − n44 − k6	3	1.72	586	84.08	1185	1525.78
A − n45 − k6	2	0.27	47	6.19	133	145.59
A − n46 − k7	1	1.25	3	0.20	2	1.95
A − n48 − k7	2	2.01	775	507.41	1949	1620.57
A − n53 − k7	1	0.62	115	19.99	619	881.05
B − n31 − k5	1	0.01	3	0.63	1	0.08
B − n38 − k6	1	0.04	5	0.56	14	1.73
B − n39 − k5	1	0.03	188	9.67	1	0.05
B − n41 − k6	1	0.08	216	18.96	20	2.89
B − n43 − k6	1	0.09	1	0.36	138	34.92
B − n45 − k5	1	0.09	22	1.13	18	5.81
B − n51 − k7	1	0.36	1	0.13	129	32.48
B − n52 − k7	1	0.19	1	0.20	26	0.76
B − n56 − k7	1	0.10	38	2.20	1	0.29
Total	50	12.73	2606	711.48	5211	4670.22

Figure 1: Preliminary Computational Results

accounts for this structure. We also conjecture that the multi-commodity flow formulation will yield improved bounds. This is what we hope to accomplish in future work.

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