Stochastic Programming and Financial Analysis IE447

Midterm Review

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Forming a Mathematical Programming Model

The general form of a mathematical programming model is:

$$\begin{aligned} \min\ & f(x) \\ \text{s.t.}\ & g(x) \leq 0 \\ & h(x) = 0 \\ & x \in X \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$, $h : \mathbb{R}^n \to \mathbb{R}^l$, and X may be a discrete set, such as $\mathbb{Z}^n$.

Notes:

- There is an important assumption here that all input data are known and fixed.
- Such a mathematical program is called *deterministic*.
- Is this realistic?

Categorizing Mathematical Programs

- Deterministic mathematical programs can be categorized along several fundamental lines.
  - Constrained vs. Unconstrained
  - Convex vs. Nonconvex
  - Linear vs. Nonlinear
  - Discrete vs. Continuous
- What is the importance of these categorizations?
  - Knowing what category an instance is in can tell us something about how difficult it will be to solve.
  - Different solvers are designed for different categories.

Unconstrained Optimization

- When $M = \emptyset$ and $X = \mathbb{R}^n$, we have an *unconstrained optimization problem*.
- Unconstrained optimization problems will not generally arise directly from applications.
- They do, however, arise as *subproblems* when solving mathematical programs.
- In unconstrained optimization, it is important to distinguish between the *convex* and *nonconvex* cases.
- Recall that in the convex case, optimizing globally is "easy."

Linear Programs

- A linear program is one that can be written in a form in which the functions f and g_i, $i \in M$ are all linear and $X = \mathbb{R}^n$.
- In general, a linear program is one that can be written as

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{s.t.} & a_i^\top x \geq b_i \; \forall i \in M_1 \\ & a_i^\top x \leq b_i \; \forall i \in M_2 \\ & a_i^\top x = b_i \; \forall i \in M_3 \\ & x_j \geq 0 \;\; \forall j \in N_1 \\ & x_j \leq 0 \;\; \forall j \in N_2 \end{array}$$

- Equivalently, a linear program can be written as

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{s.t.} & Ax \geq b \end{array}$$

- Generally speaking, linear programs are also "easy" to solve.

Nonlinear Programs

- A *nonlinear program* is any mathematical program that cannot be expressed as a linear program.
- Usually, this terminology also assumes $X = \mathbb{R}^n$.
- Note that by this definition, it is not always obvious whether a given instance is really nonlinear.
- In general, nonlinear programs are difficult to solve to global optimality.

Special Case: Convex Programs

- A *convex program* is a nonlinear program in which the objective function f is convex and the feasible region is a convex set.
- In practice, convex programs are usually "easy" to solve.

Special Case: Quadratic Programs

- If all of the functions f and g_i for $i \in M$ are quadratic functions, then we have a *quadratic program*.
- Often, the term quadratic program refers specifically to a program of the form

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^\top Qx + c^\top x \\ \text{s.t.} & Ax \geq b \end{array}$$

- Because $x^\top Qx = \frac{1}{2}x^\top(Q + Q^\top)x$, we can assume without loss of generality that Q is *symmetric*.
- The objective function of the above program is then convex if and only if Q is *positive semidefinite*, i.e., $y^\top Qy \geq 0$ for all $y \in \mathbb{R}^n$.
- There are specialized methods for solving convex quadratic programs efficiently.

Special Case: Integer Programs

- When $X = \mathbb{Z}^n$, we have an *integer program*.
- When $X = \mathbb{Z}^r \times \mathbb{R}^{n-r}$, we have a *mixed integer program*.
- By convention, all functions are assumed to be linear in these cases unless otherwise specified.
- If some of the functions are nonlinear, then we have a *mixed integer nonlinear program*.
- All mathematical program with integer variables are difficult to solve in general.

Stochastic Optimization

- In the real world, little is known ahead of time with certainty.
- Most of the applications we look at in this class will involve some degree of uncertainty.
- A *risky investment* is one whose return is not known ahead of time.
- a *risk-free* investment is one whose return is fixed.
- To make decisions involving risky investments, we need to incorporate some degree of *stochasticity* into our models.
- This can be done in a variety of ways.

Probability

- Stochastic optimization involves various "random" phenomena.
- To describe these phenomena, we need a little probability theory.
- The symbol ω will denote the *outcome* of a random experiment.
- The set of all possible outcomes, called the *sample space*, will generally be denoted Ω.
- Subsets of Ω are called *events*.

Probability spaces

- Let $\mathcal{A}$ be a set of events.
- A probability measure (or distribution) P is a function that indicates the probability that each event $A \in \mathcal{A}$ will occur.
- Probability measures must satisfy certain axioms and have the following basic properties.
  - $0 \leq P(A) \leq 1$
  - $P(\Omega) = 1$, $P(\emptyset) = 0$
  - $P(A_1 \cup A_2) = P(A_1) + P(A_2)$ if $A_1 \cap A_2 = \emptyset$.
- The triple $(\Omega, \mathcal{A}, P)$ is called a *probability space*.

Random Variables

- A random variable ξ on a probability space $(\Omega, \mathcal{A}, P)$ is a function $\xi : \Omega \to \mathbb{R}$ such that $\{\omega | \xi(\omega) \leq x\} \in \mathcal{A}$ for all finite x.
- ξ has a *cumulative distribution* given by $F_\xi(x) = P(\xi \leq x)$.
- *Discrete random variables* are those that take on a finite number of values $\xi^k, k \in K$
- Random variables have an associated *probability density function*.
- For a discrete random variable the density function $f(\xi^k) \equiv P(\xi = \xi^k)$
- A continuous random variables has density f with the property

$$
\begin{aligned}
P(a \leq \xi \leq b) &= \int_a^b f(\xi) d\xi \\
&= \int_a^b dF(\xi) \\
&= F(b) - F(a)
\end{aligned}
$$

Expectation and Variance

- The *Expected value* of ξ is
  - $\mathbb{E}(\xi) = \sum_{k \in K} \xi^k f(\xi^k)$ (Discrete)
  - $\mathbb{E}(\xi) = \int_{-\infty}^{\infty} f(\xi) d\xi = \int_{-\infty}^{\infty} dF(\xi)$. (Continuous)
- *Variance* of ξ is $\text{Var}(\xi) = \mathbb{E}(\xi - \mathbb{E}(\xi)^2)$.

Describing Uncertainty

- One of the challenges of dealing with uncertainty is how to describe it.
- In general, there are an infinite number of ways the future could turn out, but we must describe future possibilities succinctly if we hope to discern anything from them.
- The scenario approach assumes that there are a finite number of possible future outcomes of uncertainty.
- Each of these possible outcomes is called a *scenario*.
  - Demand for a product is "low, medium, or high."
  - Weather is "dry or wet."
  - The market will go "up or down."
- Even if this is not reality, such a discrete approximation is often "good enough."
- Using discrete approximations also results in discrete probability spaces, which are sometimes easier to deal with.

Linear Programming Models: Short Term Financing

- Short term financing models are to make provisions for a series of known cash flows over a period of T months.
- For example, suppose the following sources of funds are available:
  - Bank credit
  - Issue of zero-coupon bonds
  - Cash reserves in an interest-bearing account.
- How should a series of cash flows be provided for at minimum cost if no payment obligations are to remain at the end of the period?
- Such questions can be answered using a linear programming model.

Linear Programming Models: Portfolio Dedication

Definition 1. Dedication *or* cash flow matching *refers to the funding of known future liabilities through the purchase of a portfolio of risk-free non-callable bonds.*

- Suppose a pension fund faces liabilities totalling ℓ_j for years $j = 1, ..., T$.
- The fund wishes to dedicate these liabilities via a portfolio comprised of n different types of bonds.
- Bond type i costs c_i, matures in year j_i, and yields a yearly coupon payment of d_i up to maturity.
- The principal paid out at maturity for bond i is p_i.
- How should the fund invest in these bonds at minimum cost while covering all liabilities?
- This question can again be answered with a linear programming model.

Linear Programming Duality Theory

- Consider a linear program in standard form

$$\begin{aligned} \min c^\top x& \\ s.t. \quad Ax &= b \\ x &\geq 0 \end{aligned}$$

- The dual is then

$$\begin{aligned} \max \ p^\top b& \\ s.t. \quad p^\top A &\leq c^\top \end{aligned}$$

From the Primal to the Dual

We can dualize general LPs as follows

PRIMAL	minimize	maximize	DUAL
constraints	$\geq b_i$ $\leq b_i$ $= b_i$	≥ 0 ≤ 0 free	variables
variables	≥ 0 ≤ 0 free	$\leq c_j$ $\geq c_j$ $= c_j$	constraints

Relationship of the Primal and the Dual

The following are the possible relationships between the primal and the dual:

	Finite Optimum	**Unbounded**	**Infeasible**
Finite Optimum	Possible	Impossible	Impossible
Unbounded	Impossible	Impossible	Possible
Infeasible	Impossible	Possible	Possible

Optimality Conditions

Let's consider an LP in standard form. We have now shown that the optimality conditions for (nondegenerate) x are

1. $Ax = b$ (primal feasibility)
2. $x \geq 0$ (primal feasibility)
3. $x_i = 0$ if $p^\top a_i \neq c_i$ (complementary slackness)
4. $p^\top A \leq c$ (dual feasibility)

- In standard form, the complementary slackness condition is simply $x^\top \bar{c} = 0$.
- This condition is always satisfied during the simplex algorithm, since the reduced costs of the basic variables are zero.

Dual Variables and Marginal Costs

- Consider an LP in standard form with a nondegenerate, optimal basic feasible solution x^* and optimal basis B.
- Suppose we wish to perturb the right hand side slightly by replacing b with $b + d$.
- As long as d is "small enough," we have $B^{-1}(b + d) > 0$ and B is still an optimal basis.
- The optimal cost of the perturbed problem is

$$c_B^\top B^{-1}(b + d) = p^\top (b + d)$$

- This means that the optimal cost changes by $p^\top d$.
- Hence, we can interpret the optimal dual prices as the marginal cost of changing the right hand side of the i^{th} equation.

Economic Interpretation

- The dual prices, or *shadow prices* allow us to put a value on “resources” (broadly construed).
- Alternatively, they allow us to consider the sensitivity of the optimal solution value to changes in the input.
- Consider the bond portfolio problem from Lecture 3.
- By examining the dual variable for the each constraint, we can determine the value of an extra unit of the corresponding “resource”.
- We can then determine the maximum amount we would be willing to pay to have a unit of that resource.
- Note that the reduced costs can be thought of as the shadow prices associated with the nonnegativity constraints.

Local Sensitivity Analysis

- For changes in the right-hand side,
  - Recompute the values of the basic variables, $B^{-1}b$.
  - Re-solve using dual simplex if necessary.
- For a changes in the cost vector,
  - Recompute the reduced costs.
  - Re-solve using primal simplex.
- For changes in a nonbasic column A_j
  - Recompute the reduced cost, $c_j - c_B B^{-1} A_j$.
  - Recompute the column in the tableau, $B^{-1} A_j$.
- For all of these changes, we can compute ranges within which the current basis remains optimal.

Optimality Conditions for Nonlinear Programs

- The KKT conditions provide a set of necessary conditions for optimality in the case of a nonlinear program.
- These conditions apply only when the gradients of the binding constraints are linearly independent.
- In this case, we get that x^* locally optimal $\Rightarrow$ there exists $u \in \mathbb{R}^m$ such that

$$\begin{aligned} \nabla f(x^*) + \sum u_i \nabla g_i(x^*) &= 0 \\ u_i g_i(x^*) &= 0 \; \forall i \in [1, m] \\ u &\geq 0 \end{aligned}$$

Remarks on the KKT conditions

- As in the LP case, we have PF, DF and CS conditions.
- x^* is a *KKT point* if the KKT conditions are satisfied at x^*.
- For a linear program, the KKT conditions are simply the standard optimality conditions for LP.
- Furthermore, x^* is a KKT point if and only if x^* is the solution to the first-order LP approximation to the NLP

  $$\min\{f(x^*) + \nabla f(x^*)^\top (x - x^*) \mid g_i(x^*) + \nabla g_i(x^*)^\top (x - x^*) \leq 0, i \in [1, m]\}$$

- The KKT conditions are sufficient for convex programs.
- The KKT conditions are necessary and sufficient for convex programs with all linear constraints.

Optimality Conditions for Quadratic Programs

- Consider the quadratic program

$$\begin{aligned} \min\ & \frac{1}{2}x^\top Q x + c^\top x \\ \text{s.t.}\ & Ax = b \\ & x \geq 0 \end{aligned}$$

- KKT conditions are that there exists a solution to the system

$$F(x,y,s) = \begin{bmatrix} Ax - b \\ A^\top y - Qx + s - c \\ s^\top x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, (x,s) \geq 0. \qquad (1)$$

- If Q is positive semi-definite, then we have a convex program and the above conditions are necessary and sufficient.

Market Model

- A *market* consists of a set of n risky assets, denoted generally by $S^1, \ldots, S^n$, and a risk-free asset S^0 and a probability space describing possible future states of the market.
- The price of investment i at time 0 is known and denoted by S_0^i.
- The price of investment i at time t is a random variable denoted S_t^i.
- We will frequently wish to describe the state of prices at a future time $t = 1$ in terms of scenarios.
- We then let $\Omega = \cup_{j=1}^m \Omega_j$ be a partition of Ω into m events such that the prices at time 1 are constants $S_1^i(\Omega_j)$ for all $\omega \in \Omega_j$, $i = 0, \ldots, n$.
- Since S^0 is risk-free, we have

$$S_1^0(\Omega_1) = \cdots = S_1^0(\Omega_m) = RS_0^0 \tag{2}$$

Arbitrage

- *Arbitrage* is getting something for nothing. It is the fabled "free lunch."
- More formally, there are two types of arbitrage

Definition 2. Type A arbitrage *is a trading strategy that has positive initial cash flow and nonnegative payoff under all future scenarios.*

Definition 3. Type B arbitrage *is a trading strategy that costs nothing initially, has nonnegative payoff under all future scenarios and has a strictly positive expected payoff.*

- Obviously finding and exploiting arbitrage opportunities can be very lucrative.
- Because market forces are quick to adjust, arbitrage opportunities do not usually exist for long.

Risk-Neutral Probability Measures

The existence of arbitrage is intrinsically linked to the existence of a *risk-neutral probability measure*.

Definition 4. *A risk-neutral probability measure (RNPM) for the market* $(S^0, \ldots, S^n)$ *is a vector* $p = (p_1, \ldots, p_m) > 0$ *such that*

$$S_0^i = R^{-1} \sum_{j=1}^{m} p_j S_1^i(\Omega_j), \quad (i = 0, \ldots, n).$$

Note that because of (2), the constraint corresponding to $i = 0$ is equivalent to

$$\sum_{j=1}^{m} p_j = 1,$$

which, together with $p \geq 0$, means that p must be a probability measure on Ω.

Fundamental Theorem of Asset Pricing

Theorem 1. *(Fundamental Theorem of Asset Pricing) An RNPM for the market* $(S^0, \ldots, S^n)$ *exists if and only if the market is arbitrage-free.*

Idea of Proof: Consider the following LP,

$$
\begin{aligned}
\text{(P)} \quad & \min_x \sum_{i=0}^{n} x_i S_0^i \\
\text{s.t.} \quad & \sum_{i=0}^{n} x_i S_1^i(\Omega_j) \geq 0, \quad (j = 1, \ldots, m).
\end{aligned}
$$

Note that since $x = 0$ is a feasible solution, the optimal objective value must be nonpositive.

Asset Pricing Using the Risk Neutral Probabilities

- The Fundamental Theorem of Asset Pricing introduces a very general notion of risk-neutral probabilities.
- As in the simple case of two scenarios and two underlying assets, we can use the risk-neutral probabilities to price assets whose prices are linear functions of the prices of known assets.
- The prices are again simply the discounted expected value of the asset with respect to the risk neutral probabilities.
- This is the same as adding an extra (linearly dependent) row to the arbitrage detection LP.
- If an asset to be added is linearly dependent on existing assets, but its price is not not equal to the same combination of the prices of the other assets, this makes the dual infeasible, i.e., introduces arbitrage.

Assets with Piecewise Linear Payoffs

- Consider a portfolio

$$S^x := \sum_{i=1}^{n} x_i S^i$$

of assets $S^i, i = 1, \ldots, n$ whose payoffs S_1^i are piecewise linear functions of a single underlying asset S_1^0 (not necessarily risk-free!).

- When the payoff functions of each security is a piecewise linear function of the underlying security, so is the payoff function of S^x.

$$S_1^x(\omega) = \Psi^x(S_1^0(\omega)) \quad \forall\, \omega \in \Omega,$$

where

$$\Psi^x(s) = \sum_{i=1}^{n} x_i \Psi^i(s)$$

has breakpoints among the set $\{K_j^i : j = 1, \ldots, k_i; \; i = 1, \ldots, n\}$.

- Let $0 < K_1 < \cdots < K_m$ be these breakpoints listed in ascending order, and let $K_0 := 0$.

Detecting Arbitrage

- The following optimization problem is designed to identify arbitrage opportunities in this case, if they exist.

$$\text{(P)} \quad \min_{x} \sum_{i=1}^{n} x_i S_0^i$$
$$\text{s.t.} \;\; \Psi^x(s) \geq 0 \quad \forall\, s \in [0, \infty).$$

- The problem is to find a minimum cost portfolio with nonnegative payoff for all realizations of S_1^0.
- If there exists such a portfolio with negative cost, then an arbitrage opportunity of type A exists, as before.
- By utilizing the fact that a piecewise linear function defined on $[0, \infty)$ is nonnegative if and only if it has a nonnegative value at each breakpoint and its last piece has nonnegative slope, we can rewrite this as a linear program.

Another Theorem on Asset Pricing

In analogy to the first fundamental theorem of asset pricing, LP duality can be used to prove the following result.

Theorem 2. *There is no arbitrage of type A if and only if the optimal objective value of the following LP is zero.*

$$
\begin{aligned}
(P') \quad \min_x \; & \sum_{i=1}^n S_0^i x_i \\
\text{s.t.} \; & \sum_{i=1}^n \Psi^i(0) x_i \geq 0, \\
& \sum_{i=1}^n \Psi^i(K_\ell^x) x_i \geq 0, \quad (\ell = 1, \ldots, m), \\
& \sum_{i=1}^n \big(\Psi^i(K_m + 1) - \Psi^i(K_m)\big) x_i \geq 0.
\end{aligned}
$$

Furthermore, if there is no arbitrage of type A, then there is no arbitrage of type B if and only if the dual of (P') has a strictly feasible solution.

The Portfolio Optimization Problem

Decision variables: x_i, proportion of wealth invested in asset i.

Constraints:

- the entire wealth is assumed invested, $\sum_i x_i = 1$,
- if short-selling of asset i is not allowed, $x_i \geq 0$,
- bounds on exposure to groups of assets, $\sum_{i \in \mathcal{G}} x_i \leq b$, . . .

Objective function: The investor wants to maximize expected return while minimizing risk. What to do?

- Let $R = [R_1 \ \ldots \ R_n]^\top$ be the random vector of asset returns and $\mu = \mathbb{E}[R]$ the vector of their expectations.
- Then the random return of the portfolio y is

$$\frac{\sum_i y_i S_1^i - \sum_i y_i S_0^i}{\sum_i y_i S_0^i} = \sum_i \frac{y_i S_0^i}{\sum_i y_i S_0^i} \cdot \frac{S_1^i - S_0^i}{S_0^i} = R^\top x.$$

The Konno & Yamazaki Model

- The expected portfolio return is

$$\mathbb{E}[R^\top x] = \sum_i x_i \mathbb{E}[R_i] = \mu^\top x.$$

- How do we measure risk?
- Konno & Yamazaki proposed an LP model for portfolio optimization by measuring risk based on the ℓ_1 norm.

$$\ell(x) := \mathbb{E}\left[|(R-\mu)^\top x|\right],$$

- In other words, we consider the mean absolute deviation of the portfolio return from its mean.

The Konno & Yamazaki Model

- We assume the self-financing constraint $\sum_i x_i = 1$ is taken into account among the constraints $Bx = b$
- The target return constraint $r_{\min} \leq \mu^\top x$ can be modelled among the constraints $Ax \geq a$.
- Subject to these constraints, we now want to minimize the risk,

$$
\begin{aligned}
\text{(KY)} \quad & \min_x \ \ell(x) \\
\text{s.t.} \quad & Ax \geq a, \\
& Bx = b.
\end{aligned}
$$

A Linear Portfolio Optimization Model

- The main motivation for using $\ell(x)$ as a risk measure instead of the variance of the portfolio return

$$\sigma^2(x) := \sigma^2(R^\top x) = \mathbb{E}\left[\left((R-\mu)^\top x\right)^2\right]$$

is that the resulting model is linear rather than quadratic.

- This allows us to handle a much larger number of assets.
- If $R \sim \mathbb{N}(\mu, Q)$ is a multivariate normal random vector with covariance matrix Q, then $\sigma^2(x) = x^\top Q x$ and

$$\ell(x) = \frac{1}{\sqrt{2\pi\sigma^2(x)}} \int_{-\infty}^{+\infty} |\vartheta| \exp\left(-\frac{\vartheta^2}{2\sigma^2(x)}\right) d\vartheta = \sqrt{\frac{2\sigma^2(x)}{\pi}}.$$

- Thus, minimizing $\sigma^2(x)$ is the same as minimizing $\ell(x)$ in this case.

Analyzing Tradeoffs

- In the case of the above portfolio optimization problem, there is an obvious tradeoff to be analyzed between *risk* and *return*.
- The general framework of *biobjective programming* can be used to analyze such tradeoffs.
- A *biobjective* or *bicriterion mathematical program* is an optimization problem of the form

$$\begin{array}{ll} \text{vmin} & f(x) \\ \text{subject to} & x \in X, \end{array}$$

where

- $f : \mathbb{R}^n \to \mathbb{R}^2$ is the *(bicriterion) objective function*, and
- X is the *feasible region*, usually defined to be

$$\{x \in \mathbb{Z}^p \times \mathbb{R}^{n-p} \mid g_i(x) \geq 0, i = 1, \ldots, m\}$$

for functions $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$.

- The vmin operator indicates that we are interested in generating the *efficient solutions* (defined next).

The Parametric Simplex Method

The parametric simplex method is a variant of the simplex method that can be used to analyze bicriteria LPs. We assume the objective vector is of the form $c + \theta d$ and we want to know the set of optimal solutions for all values of θ.

- Determine an initial feasible basis.
- Determine the interval $[\theta_1, \theta_2]$ for which this basis is optimal.
- Determine a variable j whose reduced cost is nonpositive for $\theta \geq \theta_2$.
- If the corresponding column has no positive entries, then the problem is unbounded for $\theta > \theta_2$.
- Otherwise, rotate column j into the basis.
- Determine a new interval $[\theta_2, \theta_3]$ in which the current basis is optimal.
- Iterate to find all breakpoint $\geq \theta_1$.
- Repeat the process to find breakpoints $\leq \theta_1$.

General Markowitz Portfolio Model

- We next consider a portfolio optimization model that uses variance as the risk measure.
- A general Markovitz portfolio optimization problem would thus take the following form,

$$
\begin{aligned}
\text{(M)} \quad & \min_x \; x^\top Q x \\
\text{s.t.} \quad & \mu^\top x \geq r, \\
& Ax \geq a, \\
& Bx = b,
\end{aligned}
$$

- We've singled out the inequality constraint $\mu^\top x \geq r$ because it contains the extra parameter r.
- Note that since the covariance matrix Q is positive semidefinite, (M) is a convex QP.

- Since $Q \succ 0$ (positive definite), we have $\sigma_{\min} > 0$, where

$$\begin{aligned} \sigma_{\min}^2 := \min_x \; & x^\top Q x \\ \text{s.t.} \quad & Ax \ge a \\ & Bx = b. \end{aligned}$$

- Note that here the constraint $\mu^\top x \ge r$ has been dropped.
- Let

$$\begin{aligned} \text{(R)} \quad r(\sigma) = \max_x \; & \mu^\top x \\ \text{s.t.} \quad & Ax \ge a \\ & Bx = b \\ & x^\top Q x \le \sigma^2, \end{aligned}$$

and note that for $\sigma \ge \sigma_{\min}$ the function $r(\sigma)$ is well-defined, as (R) has feasible solutions.

Efficient Portfolios

Note that $\mu^\top x \leq r(\sqrt{x^\top Q x})$ for all feasible x, and that it can never make sense to hold a portfolio x for which

$$\mu^\top x < r\left(\sqrt{x^\top Q x}\right),$$

since the portfolio x^* obtained from solving problem (R) with $\sigma^2 = x^\top Q x$ would yield the more desirable expected return

$$\mu^\top x^* = r\left(\sqrt{x^\top Q x}\right).$$

Definition 5. *Portfolios that satisfy the relation*

$$\mu^\top x = r\left(\sqrt{x^\top Q x}\right)$$

are called efficient. The curve $\sigma \mapsto r(\sigma)$, defined for $\sigma \geq \sigma_{\min}$, is called the efficient frontier.

The Market Price of Risk

- We now consider the situation where the universe of investable assets contains
  - One risk-free asset S^0 with return r_f and
  - n risky assets $S^1, \ldots, S^n$ with random return vector R and $\mathbb{E}[R] = \mu$.
- We write

$$\tilde{x} = \begin{bmatrix} x_0 \\ x \end{bmatrix}, \qquad \tilde{R} = \begin{bmatrix} r_f \\ R \end{bmatrix}, \qquad \tilde{\mu} = \begin{bmatrix} r_f \\ \mu \end{bmatrix}.$$

- The covariance matrix of $\tilde{R}$ then has the block structure

$$\tilde{Q} = \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix},$$

where $Q \succ 0$ is the covariance matrix of R.

Markowitz Model with Risk-free asset

We consider the Markowitz problem

$$
\begin{aligned}
\text{(P)} \quad & \min_{\tilde{x}} \tilde{x}^\top \tilde{Q} \tilde{x} \\
\text{s.t.} \quad & \tilde{A}\tilde{x} \geq \tilde{a} \\
& \tilde{B}\tilde{x} = b, \\
& \tilde{\mu}^\top \tilde{x} \geq \tilde{r},
\end{aligned}
$$

where the constraint matrices are of the form

$$
\begin{aligned}
\tilde{A} &= \begin{bmatrix} 0 & e \\ a & A \end{bmatrix}, \\
\tilde{B} &= [b \; B], \\
\tilde{a} &= \begin{bmatrix} 0 \\ a \end{bmatrix}
\end{aligned}
$$

where $e = [1 \; \ldots \; 1]$, and $\tilde{B}$ has first row $[1 \; e]$.

The Efficient Frontier

Theorem 3. *Under the above assumptions there exists a portfolio* $x^m \in \mathbb{R}^n$ *on the efficient frontier of (M) such that the efficient frontier of (P) is given by the ray* $\{\tilde{x}(\theta) : \theta \leq 1\}$, *where*

$$\tilde{x}(\theta) = \begin{pmatrix} \theta e_0 \\ (1-\theta)x^m \end{pmatrix}.$$

- In other words, for any $\tilde{\sigma}^2 \geq 0$ there exists a $\theta \leq 1$ such that the portfolio $\tilde{x}(\theta)$ achieves the maximum return $\tilde{\mu}^\top \tilde{x}(\theta) = \tilde{r}(\tilde{\sigma})$ at the risk level $\tilde{\sigma}^2$.
- Note that this means that the relative proportions of wealth allocation among the risky assets alone is the same for all investors, no matter how risk-averse they are.

The Maximum Sharpe Ratio Problem

- In the proof of theorem 3, the portfolio x^m is chosen as an optimal solution of the *maximum Sharpe ratio problem*

$$\text{(SR)} \quad \max_x \frac{\mu^\top x - r_f}{\sqrt{x^\top Q x}} \quad \text{s.t.} \quad Ax \geq a, \quad Bx = b$$

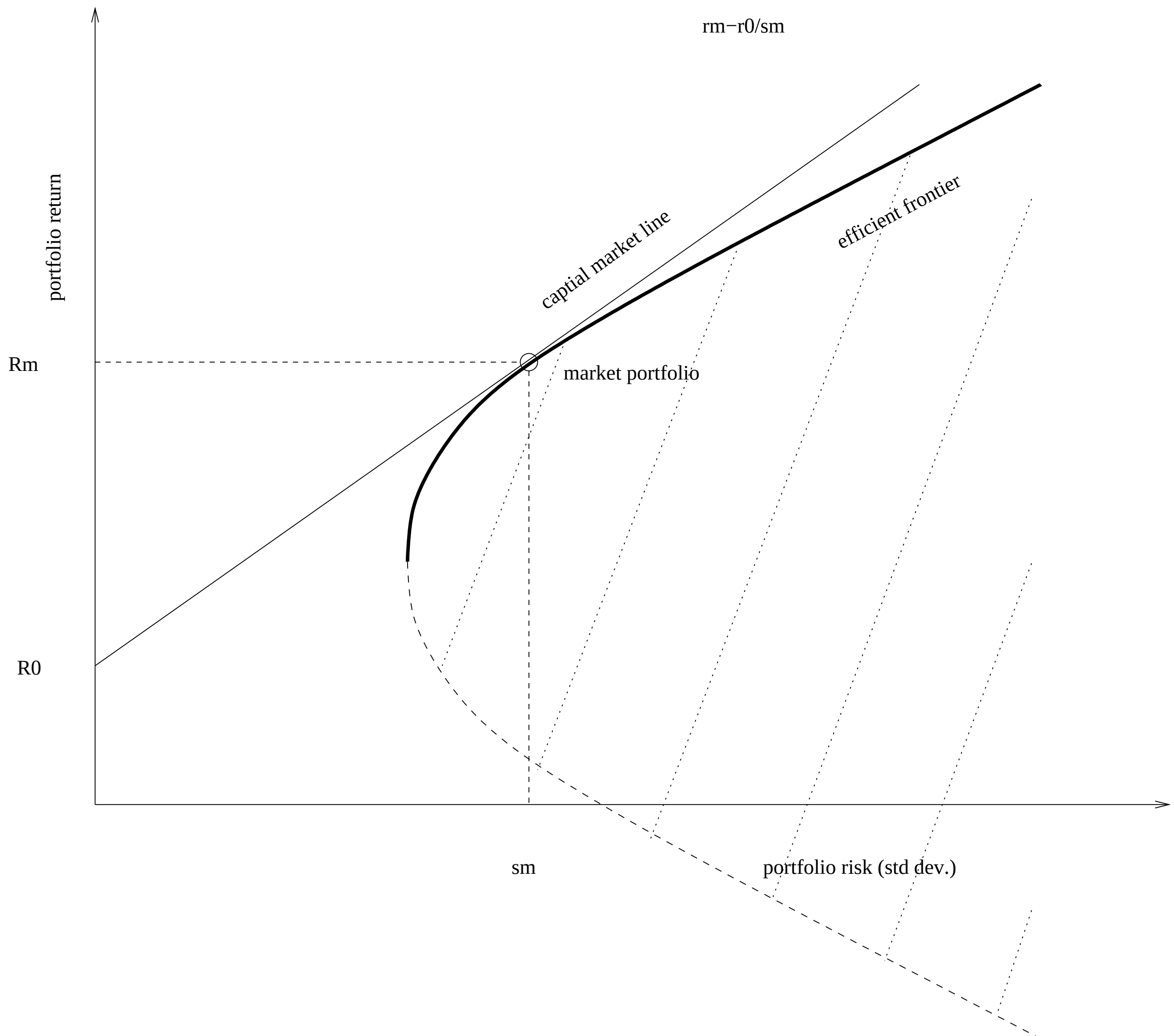
- Thus, another implicit assumption on the problem data A, a, B, b is that the feasible set $\{x \in \mathbb{R}^n : Ax \geq a, \ Bx = b\}$ is nonempty and that a (finite) optimal solution of (SR) exists.

Reformulating

Therefore, (SR) can be solved by solve the convex QP,

$$\begin{aligned} \text{(HSR)} \quad & \min_{(y,\tau)} y^\top Q y \\ \text{s.t.} \quad & (\mu - r_f e)^\top y = 1 \\ & Ay \geq \tau a \\ & By = \tau b \\ & \tau \geq 0, \end{aligned}$$

and converting an optimal solution (y^*, τ^*) of (HSR) into an optimal solution of (SR) by setting $x^* = \tau^{-1} y^*$. Note that τ cannot be zero in any feasible solution.


rm−r0/sm
portfolio return
captial market line
efficient frontier
Rm
market portfolio
R0
sm
portfolio risk (std dev.)