Financial Optimization
ISE 347/447

Lecture 9

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Reading for This Lecture

- C&T Chapter 8

General Markowitz Portfolio Model

- Recall from last time that a general Markovitz portfolio optimization problem takes the following form.

$$
\begin{aligned}
\text{(M)} \quad & \min_x \; x^\top Q x \\
\text{s.t.} \quad & \mu^\top x \geq r, \\
& Ax \geq a, \\
& Bx = b,
\end{aligned}
$$

- Recall also that since the covariance matrix Q is positive semidefinite, (M) is still a convex QP.
- We are now interested in examining the tradeoff between risk and return in more detail.

The Efficient Frontier

- In a given market, if S^j can be replicated by a linear combination $S^j = \sum_{i \neq j} \xi_i S^i$, then any portfolio $\sum_i y_i S^i$ can be replaced by the equivalent portfolio
$$\sum_{i \neq j} (y_i + y_j \xi_i) S^i.$$
- Thus, we may assume w.l.o.g. that none of the S^j can be replicated.
- Furthermore, if none of the assets S^i is risk-free, we may assume that Q is positive definite.
- For now, we will make this assumption and add a risk-free asset later.

- Since $Q \succ 0$ (positive definite), we have $\sigma_{\min} > 0$, where

$$\sigma_{\min}^2 := \min_x x^\top Q x \quad \text{s.t.} \quad Ax \geq a, \quad Bx = b.$$

- Note that here the constraint $\mu^\top x \geq r$ has been dropped.
- Let

$$\text{(R)} \quad r(\sigma) = \max_x \mu^\top x \quad \text{s.t.} \quad Ax \geq a, \quad Bx = b, \quad x^\top Q x \leq \sigma^2,$$

and note that for $\sigma \geq \sigma_{\min}$ the function $r(\sigma)$ is well-defined, as (R) has feasible solutions.

- An alternative view of the same relation is to consider any $r \geq r_{\min} := r(\sigma_{\min})$ and to compute

$$\begin{aligned} \sigma^2(r) = \min_x \; & x^\top Q x \\ \text{s.t.} \quad & Ax \geq a \\ & Bx = b \\ & \mu^\top x \geq r. \end{aligned}$$

- Using optimality conditions, one can show that the relations $\sigma(r)$ and $r(\sigma)$ are inverses of each other.

Efficient Portfolios

Note that $\mu^\top x \leq r(\sqrt{x^\top Q x})$ for all feasible x, and that it can never make sense to hold a portfolio x for which

$$\mu^\top x < r\left(\sqrt{x^\top Q x}\right),$$

since the portfolio x^* obtained from solving problem (R) with $\sigma^2 = x^\top Q x$ would yield the more desirable expected return

$$\mu^\top x^* = r\left(\sqrt{x^\top Q x}\right).$$

Definition 1. *Portfolios that satisfy the relation*

$$\mu^\top x = r\left(\sqrt{x^\top Q x}\right)$$

are called efficient. The curve $\sigma \mapsto r(\sigma)$, defined for $\sigma \geq \sigma_{\min}$, is called the efficient frontier.

Proposition 1. *The efficient frontier is a concave function.*

Proof: This is equivalent to claiming that $r \mapsto \sigma(r)$ is a convex function for $r \geq r_{\min}$. Let x^1, x^2 be efficient portfolios corresponding to the risk levels $\sigma(r^1), \sigma(r^2)$, where $r^1, r^2 \geq r_{\min}$, and consider the portfolio

$$x = \theta x^1 + (1-\theta)x^2,$$

where $\theta \in (0,1)$. Then $Ax \geq a$, $Bx = b$ and

$$\mu^\top x = \theta r^1 + (1-\theta)r^2 =: r.$$

Thus, x is feasible (but possibly suboptimal) for the problem

$$\begin{aligned} \sigma^2(r) = \min_y \ & y^\top Q y \\ \text{s.t.} \quad & Ay \geq a, \\ & By = b, \\ & \mu^\top y \geq r. \end{aligned}$$

This shows that

$$\begin{aligned}
\sigma^2(r) &= \sigma^2\big(\theta r^1 + (1-\theta) r^2\big) \\
&\le x^\top Q x \\
&= \theta^2 x^{1\top} Q x^1 + 2\theta(1-\theta) x^{1\top} Q x^2 + (1-\theta)^2 x^{2\top} Q x^2 \\
&\le \theta^2 x^{1\top} Q x^1 + 2\theta(1-\theta)\sqrt{x^{1\top} Q x^1}\sqrt{x^{2\top} Q x^2} + (1-\theta)^2 x^{2\top} Q x^2 \\
&= \left(\theta\sqrt{x^{1\top} Q x^1} + (1-\theta)\sqrt{x^{2\top} Q x^2}\right)^2 \\
&= \big(\theta\sigma(r^1) + (1-\theta)\sigma(r^2)\big)^2,
\end{aligned}$$

where the fourth line follows from the Cauchy-Schwartz inequality. Taking square roots on both sides shows that σ is a convex function of r, as claimed.

The Market Price of Risk

- We now consider the situation where the universe of investable assets contains
  - One risk-free asset S^0 with return r_f and
  - n risky assets $S^1, \ldots, S^n$ with random return vector R and $\mathbb{E}[R] = \mu$.
- We write

$$\tilde{x} = \begin{bmatrix} x_0 \\ x \end{bmatrix}, \qquad \tilde{R} = \begin{bmatrix} r_f \\ R \end{bmatrix}, \qquad \tilde{\mu} = \begin{bmatrix} r_f \\ \mu \end{bmatrix}.$$

- The covariance matrix of $\tilde{R}$ then has the block structure

$$\tilde{Q} = \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix},$$

where $Q \succ 0$ is the covariance matrix of R.

Markowitz Model with Risk-free asset

We consider the Markowitz problem

$$
\begin{aligned}
\text{(P)} \quad & \min_{\tilde{x}} \tilde{x}^\top \tilde{Q} \tilde{x} \\
\text{s.t.} \quad & \tilde{A}\tilde{x} \geq \tilde{a} \\
& \tilde{B}\tilde{x} = b, \\
& \tilde{\mu}^\top \tilde{x} \geq \tilde{r},
\end{aligned}
$$

where the constraint matrices are of the form

$$
\begin{aligned}
\tilde{A} &= \begin{bmatrix} 0 & e \\ a & A \end{bmatrix}, \\
\tilde{B} &= [b \; B], \\
\tilde{a} &= \begin{bmatrix} 0 \\ a \end{bmatrix}
\end{aligned}
$$

where $e = [1 \; \ldots \; 1]$, and $\tilde{B}$ has first row $[1 \; e]$.

Balancing Risk

- Let $e_0 = [1\ 0]^\top$ be the portfolio that corresponds to investing the entire wealth in the risk-free asset, and note that e_0 is feasible.
- Note also that the constraints $\tilde{B}\tilde{x} = b$ contain the self-financing condition $\sum_i \tilde{x}_i = 1$.
- The constraint structure assumed above implies that for all $x \in \mathbb{R}^n$ satisfying $Ax \geq a$ and $Bx = b$ and for all $\theta \leq 1$, the portfolio

$$\begin{pmatrix} \theta e_0 \\ (1-\theta)x \end{pmatrix}$$

is feasible for (P).

Markowitz Model without Risk-free Asset

- Now consider the associated Markowitz problem

$$
\begin{aligned}
\text{(M)} \quad & \min_x \; x^\top Q x \\
\text{s.t.} \quad & Ax \geq a \\
& Bx = b, \\
& \mu^\top x \geq r
\end{aligned}
$$

  obtained from (P) by restricting investments to risky assets only.

- Let $r(\sigma)$, $\sigma(r)$, $r_{\min}$ and $\sigma_{\min}$ be as introduced earlier, and let $\tilde{r}(\tilde{\sigma})$, $\tilde{\sigma}(\tilde{r})$, $\tilde{r}_{\min}$ and $\tilde{\sigma}_{\min}$ be the corresponding objects for problem (P).

- Note that $\tilde{\sigma}_{\min} = 0$ and $\tilde{r}_{\min} = r_f$.

- We assume that (M) is feasible for some $r > r_f$, since otherwise investing a positive amount in risky assets is pointless.

The Efficient Frontier

Theorem 1. *Under the above assumptions there exists a portfolio* $x^m \in \mathbb{R}^n$ *on the efficient frontier of (M) such that the efficient frontier of (P) is given by the ray* $\{\tilde{x}(\theta) : \theta \leq 1\}$, *where*

$$\tilde{x}(\theta) = \begin{pmatrix} \theta e_0 \\ (1-\theta)x^m \end{pmatrix}.$$

- In other words, for any $\tilde{\sigma}^2 \geq 0$ there exists a $\theta \leq 1$ such that the portfolio $\tilde{x}(\theta)$ achieves the maximum return $\tilde{\mu}^\top \tilde{x}(\theta) = \tilde{r}(\tilde{\sigma})$ at the risk level $\tilde{\sigma}^2$.
- Note that this means that the relative proportions of wealth allocation among the risky assets alone is the same for all investors, no matter how risk-averse they are.

The Maximum Sharpe Ratio Problem

- In the proof of the theorem, the portfolio x^m is chosen as an optimal solution of the *maximum Sharpe ratio problem*

$$
\begin{aligned}
\text{(SR)} \quad & \max_x \frac{\mu^\top x - r_f}{\sqrt{x^\top Q x}} \\
\text{s.t.} \quad & Ax \geq a \\
& Bx = b
\end{aligned}
$$

- Thus, another implicit assumption on the problem data A, a, B, b is that the feasible set $\{x \in \mathbb{R}^n : Ax \geq a, Bx = b\}$ is nonempty and that a (finite) optimal solution of (SR) exists.

Solving the The Maximum Sharpe Ratio Problem

- It is easy to see that (SR) can be solved via homogenization.
- Since the first constraint in $Bx = b$ reads $e^\top x = 1$, we can rewrite $\mu^\top x - r_f$ as $(\mu - r_f e)^\top x$.
- Now note that if $y = \tau x$ for some $\tau > 0$, then

$$\frac{(\mu - r_f e)^\top x}{\sqrt{x^\top Q x}} = \frac{(\mu - r_f e)^\top y}{\sqrt{y^\top Q y}}.$$

- In particular, if $\mu^\top x > r_f$ we may choose $\tau > 0$ such that

$$(\mu - r_f e)^\top y = 1.$$

Reformulating

Therefore, (SR) can be solved by solve the convex QP,

$$\text{(HSR)} \quad \min_{(y,\tau)} y^\top Q y$$
$$\text{s.t.} \quad (\mu - r_f e)^\top y = 1$$
$$Ay \geq \tau a$$
$$By = \tau b$$
$$\tau \geq 0,$$

and converting an optimal solution (y^*, τ^*) of (HSR) into an optimal solution of (SR) by setting $x^* = \tau^{-1} y^*$. Note that τ cannot be zero in any feasible solution.

The Market Portfolio

- The portfolio corresponding to x^m is called the *market portfolio*, the ray $\{\tilde{x}(\theta) : \theta \leq 1\}$ the *efficient frontier*, and the gradient

$$\frac{\mu^\top x^m - r_f}{\sqrt{x^{m\top} Q x^m}}$$

  the *Maximum Sharpe Ratio*.

- The MSR has an interpretation of *market price of risk*.

- An investor is willing to take on an additional unit of risk (measured in terms of the standard deviation of the portfolio return) only if the expected portfolio return increases by the Maximum Sharpe Ratio.

Proof of Theorem 1: Let $\tilde{x} = [\theta \; x^\top]^\top$ be an optimal portfolio for (P) with return $\tilde{r}$. Since $\sum_i \tilde{x}_i = 1$ and $\sum_{i=1}^n \tilde{x}_i \geq 0$, we have $\theta \leq 1$. Without loss of generality, we may assume that $\theta < 1$.

Further, the structure of $\tilde{A}$ implies that $Ax \geq a$ and hence, $Ax^* \geq a$, where $x^* = (1-\theta)^{-1}x$, and the structure of B implies that

$$Bx^* = (1-\theta)^{-1}Bx = (1-\theta)^{-1}(B\tilde{x} - \theta b) = b.$$

Thus, x^* is feasible for (M) at the target return level $r^* = \mu^\top x^*$.

Furthermore, we have

$$\tilde{r} = \tilde{\mu}^\top \tilde{x} = \theta r_f + (1-\theta)r^*,$$

and

$$\sigma^2(\tilde{r}) = \tilde{x}^\top \tilde{Q}\tilde{x} = x^\top Q x = (1-\theta)^2 \sigma_*^2,$$

where $\sigma_*^2 = x^{*\top} Q x^*$.

Now let x^m an optimal solution of the Maximum Sharpe Ratio Problem, let $r_m = \mu^\top x^m$, $\sigma_m^2 = x^{m\top} Q x^m$ and $\vartheta = (r_m - \tilde{r})/(r_m - r_f)$.

Then $\tilde{r}, r_m > r_f$ implies $\vartheta < 1$ and the portfolio

$$\tilde{x}(\vartheta) = \begin{bmatrix} \vartheta & (1-\vartheta)x^{m\top} \end{bmatrix}^\top$$

also has expected return $\tilde{r} = \vartheta r_f + (1-\vartheta) r_m$ by construction of ϑ, and its associated risk is $(1-\vartheta)^2 \sigma_m^2$.

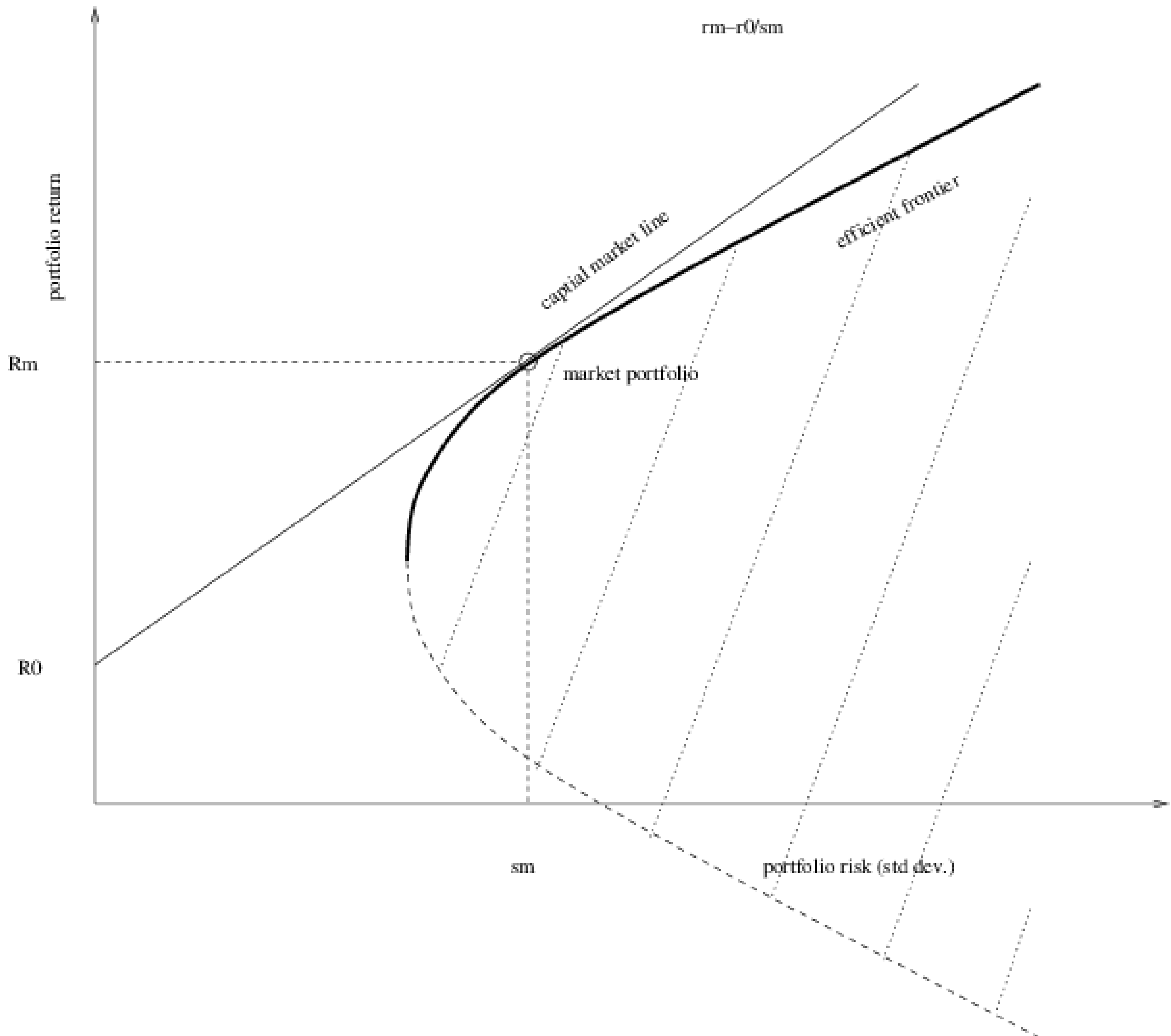
Furthermore, since

$$\frac{(r^* - r_f)^2}{\sigma_*^2} \leq \frac{(r_m - r_f)^2}{\sigma_m^2},$$

we have

$$\begin{aligned}(1-\vartheta)^2\sigma_m^2 &\leq (1-\vartheta)^2\left(\frac{r_m - r_f}{r_* - r_f}\right)^2\sigma_*^2 \\ &= \left(\frac{r_f - r_m - \tilde{r} + r_m}{r_* - r_f}\cdot\frac{r_m - r_f}{r_* - r_f}\right)^2\sigma_*^2 \\ &= \left(\frac{\tilde{r} - r_f}{r_* - r_f}\right)^2\sigma_*^2 \\ &= \left(\frac{\theta r_f + (1-\theta)r_* - r_f}{r_* - r_f}\right)^2\sigma_*^2 \\ &= (1-\theta)^2\sigma_*^2.\end{aligned}$$

Therefore, unless x^* also solves the Maximum Sharpe Ratio problem, $\tilde{x}(\vartheta)$ contradicts the optimality of $\tilde{x}$.


rm−r0/sm
capital market line
efficient frontier
portfolio return
Rm
market portfolio
R0
sm
portfolio risk (std dev.)