Financial Optimization
ISE 347/447

Lecture 6

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Reading for This Lecture

- C&T Chapter 4

Derivative Securities

- A *derivative security* is one whose price depends on the value of an underlying security.
- Options are a common example of a derivative security.
- A *European call option* gives the holder the right to purchase a given security at a future time (the *exercise date*) for a fixed price (the *strike price*).
- An *American call option* is similar except that it allows the holder to purchase the security anytime before expiration.
- *Put options* allow the holder to sell an underlying security.
- Options can be used for either speculation or hedging.
- Primarily, options are used to hedge against movements in the price of the underlying security.

Arbitrage

- *Arbitrage* is getting something for nothing. It is the fabled "free lunch."
- More formally, there are two types of arbitrage

Definition 1. Type A arbitrage *is a trading strategy that has positive initial cash flow and nonnegative payoff under all future scenarios.*

Definition 2. Type B arbitrage *is a trading strategy that costs nothing initially, has nonnegative payoff under all future scenarios and has a strictly positive expected payoff.*

- Obviously finding and exploiting arbitrage opportunities can be very lucrative.
- Because market forces are quick to adjust, arbitrage opportunities do not usually exist for long.

Pricing Derivatives

- Suppose we want to buy an option. What is a fair price?
- Let's take a very simple example involving stock XYZ.
- What we know
  - The price today is 40.
  - With probability p, the price in one year will be 80.
  - With probability $1-p$, the price in one year will be 20.
- How much should we pay for an option to buy XYZ one year from today at a price of 50?
- To answer, we construct a portfolio of cash and the underlying security that will have the same payoffs as the option.
- This is called *replication*.

Example: Pricing Derivatives

- Suppose the risk-free rate of return is 0 and that our portfolio consists of Δ shares of XYZ and B dollars in cash.
- To perform the replication, we solve a system that equates the payoffs in each of the two scenarios:

$$
\begin{aligned}
80\Delta + B &= 30 \\
20\Delta + B &= 0
\end{aligned}
$$

- The solution is $\Delta = 1/2$ and $B = -10$.
- Hence, the portfolio is worth $40\Delta - 10 = 10$ today and $\$10$ is the fair price of the option for one share.

Notation

It is easy to generalize the above example. First, we need some notation.

- Let (Ω, P) be a probability space.
- We will consider a partition of Ω into two disjoint events Ω_1 and Ω_2.
- We can think of Ω_1 as being "the market goes up" and Ω_2 being a "the market goes down."
- We will set $p = P(\Omega_1) = 1 - P(\Omega_2)$.
- The risk-free rate of return will r and we will set $R = 1 + r$.
- S_j^i will denote a random variable on Ω corresponding to the price of security i at time j.

Generalizing

- S^0 is a risk-free security with distribution

$$\begin{aligned} S_0^0(\omega) &= \alpha_0, && \forall \omega \in \Omega, \\ S_1^0(\omega) &= R S_0^0, && \forall \omega \in \Omega, \end{aligned}$$

- S^1 is a risky security with distribution

$$S_0^1(\omega) = \alpha_1, \quad \forall \omega \in \Omega,$$
$$S_1^1(\omega) = \begin{cases} u S_0^1 & \text{if } \omega \in \Omega_1, \\ d S_0^1 & \text{if } \omega \in \Omega_2. \end{cases}$$

- S^2 is a European call option on S^1 with distribution

$$\begin{aligned} S_0^2(\omega) &= \alpha_2, && \forall \omega \in \Omega, \\ S_1^2(\omega) &= (S_1^1(\omega) - K^2)_+, && \forall \omega \in \Omega, \end{aligned}$$

where K^2 is the strike price.

Pricing the Derivative

If we assume everything but S_0^2 is known, then we get the system of equations

$$\begin{aligned} \Delta u S_0^1 + BRS_0^0 &= C_1^u := (uS_0^1 - K^2)_+ \\ \Delta d S_0^1 + BRS_0^0 &= C_1^d := (dS_0^1 - K^2)_+ \end{aligned}$$

The solution is

$$\begin{aligned} \Delta &= \tfrac{C_1^u - C_1^d}{S_0^1(u-d)} \\ B &= \tfrac{uC_1^d - dC_1^u}{RS_0^0(u-d)} \end{aligned}$$

Hence, we get

$$S_0^2 = \frac{C_1^u - C_1^d}{u-d} + \frac{uC_1^d - dC_1^u}{R(u-d)} = \frac{1}{R}\left[\frac{R-d}{u-d}C_1^u + \frac{u-R}{u-d}C_1^d\right]$$

Risk Neutral Probabilities

- Notice that in this simplified case, we must have $d < R < u$ or there is an arbitrage opportunity (we will see why later).
- If there is no arbitrage, then if we define

$$p_u = \frac{R-d}{u-d} \text{ and } p_d = \frac{u-R}{u-d},$$

we must have $p_u > 0$, $p_d > 0$, $p_u + p_d = 1$.

- Hence, we can interpret p_u and p_d as probabilities.
- We call these the *risk-neutral probabilities*.
- The price of any derivative security can now be interpreted as the present value of the expected payoff with respect to the risk-neutral probabilities.
- Note that the risk-neutral probabilities are independent of the actual probabilities.

Another Interpretation

Let us again consider a risk-free asset S^0 and a risky asset S^1.

$$S_0^0(\omega) = 100, \quad \forall \omega \in \Omega,$$
$$S_1^0(\omega) = 101 = RS_0^0, \quad \forall \omega \in \Omega,$$

where R is the risk-free rate over the period from time 0 to time 1.

$$S_0^1(\omega) = 100,$$
$$S_1^1(\omega) = \begin{cases} 150R & \text{if } \omega \in \Omega_1, \\ 50R & \text{if } \omega \in \Omega_2. \end{cases}$$

Which asset should we invest in?

Risk Measures

- If $p = 0.5$, $\mathbb{E}[S_1^0] = 101 = \mathbb{E}[S_1^1]$, i.e., the expected payoff is the same for both assets.
- However, the risk-free asset has a sure payoff of 101, while the payoff of asset S^1 is stochastic, i.e., there is risk.
- How do we assess risk?
- Risk can be modeled as a function of the probabilities of future outcomes (in this case p).
- A common risk measure is the variance, in this case

$$\rho(p) = \mathbb{E}\left[(S_1^1 - \mathbb{E}[S_1^1])^2\right] = p(1-p)(100R)^2.$$

- Note that the expected payoff for asset S^1 is also a function of p:

$$\eta(p) = \mathbb{E}[S_1^1] = R(p150 + (1-p)50)$$

Risk Aversion

- For investment S^1 to be equally attractive as S^0, it has to have a higher return than S^0 to compensate for the risk.
- To model an investor's behavior, it is common assume that an investor would be willing to pay

$$S_0^1 = \frac{\eta(p) - \lambda\rho(p)}{R}, \tag{1}$$

where $\lambda > 0$ is a *risk aversion parameter*.

- Division by R converts back to present value.

Risk Measures

- In general, we do not know how investors choose their λ.
- However, they must all agree on the market price S_0^1.
- This introduces a *functional dependence* of λ on p obtained by solving equation (1) for λ.

$$\lambda(p) = \frac{\eta(p) - S_0^1 R}{\rho(p)}. \tag{2}$$

- In other words, the price set by the market implicitly determines the value of λ.
- All investors thus implicitly agree on a lambda.
- In our example, we find

$$\lambda(p) = \frac{100p + 50 - S_0^1}{10^4 Rp(1-p)}.$$

Risk Neutral Probabilities Revisited

- Using equation (2) from the previous slide, we can determine a p^* that would yield $\lambda = 0$, i.e., would cause investors to be risk-neutral.
- This probability p^* is determined by the equation

$$\eta(p) - S_0^1 R = 0,$$

which is *independent* of the specific risk measure $\rho(p)$ used!

- This equation equates the expected payoffs of the risky asset with that of the risk-free asset, as we did before.
- After a little algebra, we can get exactly the same risk-neutral probabilities.
- Note that we did not make reference to a particular derivative security this time.

Using the Risk Neutral Probabilities

- Using the risk-neutral probabilities, we can price any derivative whose payout can be expressed as a linear function of the payouts of two underlying securities.
- The price of any such asset can be determined as the discounted expected payoff with respect to the risk-neutral probabilities.
- It bears repeating that these "probabilities" have nothing to do with the actual probabilities.
- They are a mathematical construction that helps to price assets.

Back to the Example

- Let S^2 be a European call option on S^1 with strike price $110R$ at time 1.
- Such an asset yields the payoff

$$S_1^2 = (S_1^1 - 110R)_+ = \begin{cases} 40R & \text{if } S_1^1 = 150R, \\ 0 & \text{if } S_1^1 = 50R. \end{cases}$$

- That is,

$$\begin{bmatrix} S_1^2(\omega_1) \\ S_1^2(\omega_2) \end{bmatrix} = -0.2 \begin{bmatrix} S_1^0(\omega_1) \\ S_1^0(\omega_2) \end{bmatrix} + 0.4 \begin{bmatrix} S_1^1(\omega_1) \\ S_1^1(\omega_2) \end{bmatrix},$$

  for all $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$.

- Therefore,

$$S_0^2 = R^{-1}\mathbb{E}_{p^*}[S_1^2] = \frac{(0.5)40R + (0.5)0}{R} = 20.$$

Arbitrage Opportunities

Next, consider a market in which the following three investment opportunities are given:

$$S_0^0(\omega) = 100, \qquad \forall \omega \in \Omega$$
$$S_1^0(\omega) = 101 = RS_0^0, \qquad \forall \omega \in \Omega,$$

$$S_0^1(\omega) = 100, \quad \forall \omega \in \Omega,$$
$$S_1^1(\omega) = \begin{cases} 120 & \text{if } \omega \in \Omega_1, \\ 80 & \text{if } \omega \in \Omega_2, \end{cases}$$

$$S_0^2(\omega) = 100, \quad \forall \omega \in \Omega,$$
$$S_1^2(\omega) = \begin{cases} 110 & \text{if } \omega \in \Omega_1, \\ 85 & \text{if } \omega \in \Omega_2. \end{cases}$$

Arbitrage Opportunities

- Proceeding as before, for assets S^1 and S^2 to be acceptable investments, we find that the equations

$$S_0^i = \frac{\mathbb{E}[S_1^i] - \lambda_i \sigma^2(S_1^i)}{R}$$

have to be satisfied for $i = 1, 2$, i.e.,

$$\lambda_1 = \lambda_1(p), \qquad \lambda_2 = \lambda_2(p).$$

- If we fix λ_1, this fixes the risk-neutral probabilities, which in turn fixes λ_2.
- In other words, the risk-aversion parameters are functionally linked and may not be chosen independently!
- If these parameters are chosen inconsistently, this creates an *arbitrage opportunity*.

Arbitrage Opportunities

- When does there there exist $p^* \in (0,\ 1)$ such that $\lambda_1(p^*) = 0 = \lambda_2(p^*)$ jointly?

- The existence of such a p^* implies

$$100 = S_0^1 = R^{-1}\mathbb{E}_{p^*}[S_1^1] = R^{-1}\left(p^* 120 + (1-p^*)80\right),$$
$$100 = S_0^2 = R^{-1}\mathbb{E}_{p^*}[S_1^2] = R^{-1}\left(p^* 110 + (1-p^*)85\right).$$

- Solving the first equation yields $p^* = \frac{21}{40}$, while the second equation yields $p^* = \frac{16}{25}$. Thus, λ_1 and λ_2 cannot be jointly zero!

Arbitrage Opportunities

- Observe that an arbitrage opportunity exists in this market.
- Consider the portfolio $S^3 = 7.57S^0 + 16.9S^1 - 24.47S^2$.
- This portfolio costs nothing at time 0, since

$$S_0^3 = 7.57S_0^0 + 16.9S_0^1 - 24.47S_0^2 = 0,$$

but the value at time 1 is positive for all outcomes,

$$S_1^3 = \begin{cases} 7.57 \cdot 101 + 16.9 \cdot 120 - 24.47 \cdot 110 = 100.8 & \text{for } \omega \in \Omega_1, \\ 7.57 \cdot 101 + 16.9 \cdot 80 - 24.47 \cdot 85 = 36.62 & \text{for } \omega \in \Omega_2. \end{cases}$$

Arbitrage Opportunities

- The existence of a risk-neutral probability measure is intrinsically linked to the existence of arbitrage opportunities.
- Let us now consider the more general framework of a market consisting of n risky assets $S^1, \ldots, S^n$ and a risk-free asset S^0.
- The prices at time 0 are $S_0^0, \ldots, S_0^n$ for all $\omega \in \Omega$.
- Let $\Omega = \cup_{j=1}^m \Omega_j$ be a partition of Ω into m events such that the prices at time 1 are fixed at $S_1^i(\Omega_j)$ for all $\omega \in \Omega_j$, $i = 0, \ldots, n$.
- Since S^0 is risk-free, we have

$$S_1^0(\Omega_1) = \cdots = S_1^0(\Omega_m) = RS_0^0 \tag{3}$$

Risk-Neutral Probability Measures

Definition 3. *A* risk-neutral probability measure *(RNPM) for the market* $(S^0, \ldots, S^n)$ *is a vector* $p = (p_1, \ldots, p_m) > 0$ *such that*

$$S_0^i = R^{-1} \sum_{j=1}^{m} p_j S_1^i(\Omega_j), \quad (i = 0, \ldots, n).$$

Note that because of (3), the constraint corresponding to $i = 0$ is equivalent to

$$\sum_{j=1}^{m} p_j = 1,$$

which, together with $p \geq 0$, means that p must be a probability measure on Ω.

Fundamental Theorem of Asset Pricing

Theorem 1. *(Fundamental Theorem of Asset Pricing) An RNPM for the market* $(S^0, \ldots, S^n)$ *exists if and only if the market is arbitrage-free.*

Idea of Proof: Consider the following LP,

$$\text{(P)} \quad \min_x \sum_{i=0}^{n} x_i S_0^i$$
$$\text{s.t.} \quad \sum_{i=0}^{n} x_i S_1^i(\Omega_j) \geq 0, \quad (j = 1, \ldots, m).$$

Note that since $x = 0$ is a feasible solution, the optimal objective value must be nonpositive.

Proof Sketch (cont.)

- If there exists a feasible x such that

$$\sum_{i=0}^{n} x_i S_0^i < 0,$$

then the portfolio $\sum_{i=0}^{n} x_i S^i$ is a type A arbitrage opportunity and (P) is unbounded.

- If there exists a feasible x such that $\sum_{i=0}^{n} x_i S_0^i = 0$ and

$$\sum_{i=0}^{n} x_i S_1^i(\Omega_j) > 0$$

for at least one index j, then the portfolio $\sum_{i=0}^{n} x_i S^i$ is a type B arbitrage opportunity.

- Thus, (P) is designed to detect arbitrage opportunities if they exist.

Proof Sketch (cont.)

- The dual of (P) is the following LP,

$$\text{(D)} \quad \max_q \sum_{j=1}^{m} q_j \cdot 0$$

$$\text{s.t.} \quad \sum_{j=1}^{m} q_j S_1^i(\Omega_j) = S_0^i, \quad (i = 0, \ldots, n),$$

$$q_j \geq 0, \quad (j = 1, \ldots, m).$$

- The constraints of (D) are essentially equivalent to requiring $p = (Rq_1, \ldots, Rq_m)$ to be a RNPM for the market $(S^0, \ldots, S^n)$, except the requirement $p > 0$ was replaced by $p \geq 0$.
- So (D) is a feasibility problem that is designed to find an RNPM if one exists.

Proof Sketch (cont.)

Proof of the "If" part: The market is assumed to be type A arbitrage-free. Hence, $\sum_{i=0}^{n} x_i S_0^i \geq 0$ for all primal-feasible x, and since $x = 0$ is feasible, it is the case that

$$\sum_{i=0}^{n} x_i^* S_0^i = 0$$

for all primal-optimal x^*.

The market is also assumed to be type B arbitrage-free, so that for all primal-optimal x^*,

$$\sum_{i=0}^{n} x_i^* S_1^i(\Omega_j) = 0, \quad (j = 1, \ldots, m).$$

Proof Sketch (cont.)

LP duality and strict complementarity now imply the existence of a dual-optimal vector q^* such that $q_j^* > 0$ for all j. By the earlier remarks, $p^* = Rq^*$ is a RNPM.

Proof of the Only if part: If p^* is a RNPM, then $q^* := R^{-1}p^*$ is feasible for (D) and $q > 0$. By LP duality, (P) is not unbounded and the market is therefore free from type A arbitrage.

Since $q^* > 0$, strict complementarity implies that for x^* primal optimal, $\sum_{i=0}^{n} x_i^* S_1^i(\Omega_j) = 0$ for all j. Hence, the market is type B arbitrage free.

An LP model to Detect Type A Arbitrage

Here is an LP designed to detect Type A arbitrage

```
set secs;                        # securities in the market

param S > 1;                     # number of scenarios;
param current {secs};            # price at time zero
param future {secs, 1..S};       # price in each scenario

var buy {secs};                  # amount of each in portfolio

minimize present_value : sum {i in secs} current[i] * buy[i];

subject to future_value {s in 1..S}:
           sum {i in secs} future[i, s] * buy[i] >= 0;
```

This LP has an optimal value of zero if and only there is no Type A arbitrage. It is unbounded otherwise.

An LP model to Detect Type B Arbitrage

- There is no Type B arbitrage if and only if there is a strictly feasible solution to the dual of the LP above.
- In practice, detecting type B arbitrage would be done by solving the following LP:

```
set secs;                          # securities in the market

param S > 1;                       # number of scenarios;
param current {secs};              # price at time zero
param future {secs, 1..S};         # price in each scenario

var buy {secs};                    # amount of each in portfolio
var future_values {1..S} >= 0; # future value in each scenario

maximize expected_value : sum {s in 1..S} future_values[s];

subject to current_value : sum {i in secs} current[i]*buy[i]=0;
subject to future_value {s in 1..S} :
sum {i in secs} future[i, s] * buy[i] - future_values[s] = 0;
```

What Is the General Principle?

- We wrote the problem of maximizing profit subject to a guarantee of no loss in any scenario as a mathematical program.
- Detecting type A arbitrage is then the same as determining whether this mathematical program is unbounded.
- In the case of linear programs, duality gives us a succinct condition for unboundedness.
- If a linear program is feasible, then it is unbounded if and only if the dual is infeasible.
- This principal can be extended to other settings, as we will see in the next lecture.
- The risk-neutral probabilities can be interpreted most simply as a solution to the dual that proves the boundedness of the primal.
- This principal can be generalized in various ways.

Asset Pricing Using the Risk Neutral Probabilities

- The Fundamental Theorem of Asset Pricing introduces a very general notion of risk-neutral probabilities.
- As in the simple case of two scenarios and two underlying assets, we can use the risk-neutral probabilities to price assets whose prices are linear functions of the prices of known assets.
- The prices are again simply the discounted expected value of the asset with respect to the risk neutral probabilities.
- This is the same as adding an extra (linearly dependent) row to the arbitrage detection LP.
- If an asset to be added is linearly dependent on existing assets, but its price is not not equal to the same combination of the prices of the other assets, this makes the dual infeasible, i.e., introduces arbitrage.

More On Arbitrage Detection

- In Lecture 6, we saw a method for detecting arbitrage opportunities in a finite state-space setting.
- We now discuss a similar technique that applies in a continuous setting.
- Consider a portfolio

$$S^x := \sum_{i=1}^{n} x_i S^i$$

of assets $S^i, i = 1, \ldots, n$ whose payoffs S_1^i are piecewise linear functions of a single underlying asset S_1^0 (not necessarily risk-free!).

Piecewise Linear Payoffs

That is, for each i there exist breakpoints

$$0 < K_1^i < K_2^i < \cdots < K_{k_i}^i < \infty$$

and continuous functions $\Psi^i : [0, \infty) \rightarrow \infty$ such that

- Ψ^i is a linear function on each of the intervals

$$\left[0, K_1^i\right], \ \left[K_1^i, K_2^i\right], \ \ldots \left[K_{k_i}^i, \infty\right), \text{ and}$$

- $S_1^i(\omega) = \Psi^i(S_1^0(\omega))$ for all $\omega \in \Omega$.

Example: European Call Option

- Let S^i be a European call option on the underlying security S^0 with strike price K^i.
- Then $S_1^i = (S_1^0 - K^i)_+$ is piecewise linear with a single breakpoint at K^i.
- Likewise, if S^i is a European put option.
- Thus, we are talking about detection of arbitrage opportunities in portfolios of different options on the same underlying stock.

The Portfolio Payoff Function

- When the payoff functions of each security is a piecewise linear function of the underlying security, so is the payoff function of S^x.

$$S_1^x(\omega) = \Psi^x(S_1^0(\omega)) \quad \forall\, \omega \in \Omega,$$

where

$$\Psi^x(s) = \sum_{i=1}^{n} x_i \Psi^i(s)$$

has breakpoints among the set $\{K_j^i :\ j = 1, \ldots, k_i;\ i = 1, \ldots, n\}$.

- Let $0 < K_1 < \cdots < K_m$ be these breakpoints listed in ascending order, and let $K_0 := 0$.

Detecting Arbitrage

- The following optimization problem is designed to identify arbitrage opportunities in this case, if they exist.

$$
\begin{aligned}
\text{(P)} \quad & \min_x \sum_{i=1}^{n} x_i S_0^i \\
& \text{s.t. } \Psi^x(s) \geq 0 \quad \forall\, s \in [0, \infty).
\end{aligned}
$$

- The problem is to find a minimum cost portfolio with nonegative payoff for all realizations of S_1^0.
- If there exists such a portfolio with negative cost, then an arbitrage opportunity of type A exists, as before.
- What's different is that (P) is not a linear programming problem because it has *infinitely many* constraints!

Reformulating

- Since Ψ^x is piecewise linear, the infinite set of constraints can be replaced by a finite set.

- The constraints $\Psi^x(s) \geq 0 \ (s \geq 0)$ are satisfied if and only if the following constraints are satisfied:

$$\begin{aligned} \Psi^x(K_\ell) &\geq 0, \quad (\ell = 0, \ldots, m), \\ D_+\Psi^x(K_m) &\geq 0, \end{aligned}$$

  where D_+ denotes the right-handed derivative.

- Since Ψ^x is linear on $[K_m, \infty)$, we have

$$D_+\Psi^x(K_m) = \Psi^x(K_m + 1) - \Psi^x(K_m)$$

A Linear Programming Formulation

The problem (P) can thus be reformulated as the following LP,

$$
\begin{aligned}
(\mathsf{P}') \quad & \min_x \sum_{i=1}^n S_0^i x_i \\
\text{s.t.} \quad & \sum_{i=1}^n \Psi^i(0) x_i \ge 0, \\
& \sum_{i=1}^n \Psi^i(K_\ell^x) x_i \ge 0, \quad (\ell = 1, \ldots, m), \\
& \sum_{i=1}^n \big(\Psi^i(K_m + 1) - \Psi^i(K_m)\big) x_i \ge 0.
\end{aligned}
$$

Another Theorem on Asset Pricing

In analogy to the first fundamental theorem of asset pricing, LP duality can be used to prove the following result.

Theorem 2. *There is no arbitrage of type A if and only if the optimal objective value of* (P') *is zero.*

Furthermore, if there is no arbitrage of type A, then there is no arbitrage of type B if and only if the dual of (P') *has a strictly feasible solution.*