Financial Optimization
ISE 347/447

Lecture 22

Dr. Ted Ralphs

Reading for This Lecture

- C&T Chapter 16

Solving the Deterministic Equivalent

Recall the deterministic equivalent (DE) version of the standard two-stage stochastic program.

$$
\begin{array}{lcccccccccc}
\text{minimize} \\
c^\top x & + & p_1 q^\top y_1 & + & p_2 q^\top y_2 & + & \cdots & + & p_s q^\top y_s \\
\text{subject to} \\
Ax & & & & & & & & & = & b \\
T_1 x & + & W y_1 & & & & & & & = & h_1 \\
T_2 x & & & + & W y_2 & & & & & = & h_2 \\
\vdots & & & + & & & \ddots & & & & \\
T_S x & & & & & & & + & W y_s & = & h_s \\
x \in X & & y_1 \in Y & & y_2 \in Y & & & & y_s \in Y
\end{array}
$$

Note the block angular structure. How do we take advantage of this?

Bender's Decomposition

- *Bender's Decomposition* is a technique for solving LPs with this kind of block angular structure.
- Note that if we fix the first-stage variables (x), then the LP decomposes neatly into $|S|$ smaller LPs, one for each scenario.
- Furthermore, these LPs are all identical except for the right-hand side.

Rewriting

- As before, let us rewrite the DE LP as

minimize

$$c^\top x + \sum_{s \in S} P_s(x)$$

subject to

$$\begin{aligned} Ax &= b \\ x &\in X \end{aligned}$$

where

$$P_s(x) = \min_{y \in Y} \{p_s(q^\top y) \mid Wy = h_s - T_s x\}$$

General Solution Approach

- We have already seen that $P_k(x) = v(h_k - T_k x)$ and thus is a convex function (in fact, it is piecewise linear).
- We will linearize the objective function by building up an approximation to it using linear inequalities.
- Essentially, we approximate the LP $P_s(x)$ associated with scenario $s \in S$ by
  minimize
  $$z_s$$
  subject to
  $$z_s \geq (u_s^j)^\top T_s(x^j - x) + P_s(x^j) \; \forall j \in J,$$
  where J indexes a collection of first-stage solutions and u_s^j is an optimal solution to the dual of the LP $P_s(x^j)$ when it exists.
- We will also have to make sure to eliminate any first-stage solution for which there is no feasible recourse in scenario s.

Optimality Cuts

- Let's consider the LP associated with scenario $s \in S$

$$P_s(x) = \min_{y \in Y} \{p_s(q^\top y) \mid Wy = h_s - T_s x\}$$

- By LP duality, we have

$$P_s(x) = \max\{u_s^\top (h_s - T_s x) \mid W^\top u_s \leq p_s q\}$$

- Let $\hat{x}$ be such that $P_s(\hat{x})$ is feasible and let $\hat{u}_s$ be an optimal dual solution.

Optimality Cuts (cont.)

- Then by LP duality, we have

$$P_s(x) \geq \hat{u}_s^\top (h_s - T_s x)$$

- Furthermore, since

$$P_s(\hat{x}) = \hat{u}_s^\top (h_s - T_s \hat{x}),$$

  we have

$$P_s(x) \geq \hat{u}_s^\top T_s(\hat{x} - x) + P_s(\hat{x})$$

Feasibility Cuts

- Suppose we uncover a first-stage solution $\hat{x}$ for which $P_s(\hat{x})$ is infeasible?
- In this case, we can obtain a direction $\hat{u}_s$ of unboundedness for the dual of the LP $P_s(\hat{x})$.
- For such a direction, we have

$$\hat{u}_s^\top (h_s - T_s \hat{x}) > 0$$

and

$$W^\top \hat{u}_s \leq p_s q$$

- So we have that $P_s(x)$ will be infeasible for any x such that $\hat{u}_s^\top (h_s - T_s x) > 0$.
- Since we are only interested in first-stage solutions with feasible recourse, the inequality

$$\hat{u}_s^\top T_s x \geq \hat{u}_s^\top h_s$$

must be satisfied by all first-stage solutions.

Reformulation with Benders Cuts

Conceptually, we can reformulate the DE LP as the following LP:

minimize

$$c^\top x + \sum_{s \in S} z_s$$

subject to

$$\begin{aligned} Ax &= b \\ x &\in X \\ z_s &\geq (u_s^j)^\top T_s(x^j - x) + P_s(x^j) \ \forall j \in J, s \in S, \end{aligned}$$

- This is an exact reformulation if J indexes the set of all all first-stage solutions.
- In practice, we maintain a set $\bar{J} = \{x^0, \ldots, x^k\}$ of the solutions generated in the first k iterations, as we see next.

Initializing the Algorithm

- We will start by solving the initial *master LP* with $\bar{J} = \emptyset$:
  minimize
  $$c^\top x$$
  subject to
  $$\begin{aligned} Ax &= b \\ x &\in X \end{aligned}$$
  to obtain x^0.
- We will then solve $P_s(x^0)$ for each $s \in S$.
- This will give us an upper bound
  $$c^\top x + \sum_{s \in S} P_s(x^0)$$
  on the optimal value of the stochastic program.

Iterating

- In iteration k, we solve $P_s(x^{k-1})$ to obtain either an optimality cut or a feasibility cut (with associated dual solution $\hat{u_s}^{k-1}$).
- We then solve

minimize

$$c^\top x + \sum_{s \in S} z_s$$

subject to

$$\begin{aligned} Ax &= b \\ x &\in X \\ z_s &\geq (u_s^j)^\top T_s(x^j - x) + P_s(x^j)\ \forall j \in \{0, \ldots, k-1\}, s \in S, \end{aligned}$$

- This is a relaxation of the original problem, so we get a lower bound and a new first-stage solution x^1 (which yields a new upper bound).
- This procedure is iterated until the upper and lower bounds are equal (or at least are “close enough” together).
- This is called the *L-shaped method*.