Financial Optimization
ISE 347/447

Lecture 20

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Reading for This Lecture

- C&T Chapter 16

Stochastic Programming

- A *stochastic program* is a type of mathematical program in which we explicitly model uncertainty in the input.
- The uncertainty is described by specifying an explicit probability distribution for the input parameters.
- All variables and constraints are expressed in the usual way, except that the variables are divided (roughly) into two classes
  - *Anticipative variables*: Decisions that must be made before the uncertainties are resolved.
  - *Adaptive variables*: Decisions that are made after uncertainty in the parameters has been resolved.
- All stochastic programs have some anticipative variables, as they would otherwise become deterministic.
- Stochastic programs that include both types of variables are generally called *recourse models*.

Stochastic Programs with Recourse

- Recourse models have an explicit concept of time.
- In a recourse model, uncertainty is resolved in discrete stages over time.
- Between stages, certain decisions must be made, i.e., the values of certain variables must be fixed.
- Solving such a stochastic program really only involves determining the optimal values for the anticipative variables.
- After some uncertainty has been resolved, we have another stochastic program in which variables that were previously considered adaptive become anticipative.

Example: My Retirement

- When I retire in 10 years or so :-), I would like to have a comfortable income.
- I'll need enough savings to generate the income I'll need to support my lavish lifestyle.
- One approach would be to simply formulate a mean-variance portfolio optimization problem, solve it, and then "buy and hold."
- This doesn't explicitly take into account the fact that I can periodically rebalance my portfolio.
- Even solving a
- I may make a different investment decision today if I explicitly take into account that I will have *recourse* at a later point in time.
- This is the central idea of stochastic programming.

Modeling Assumptions

- In Y years, I would like to reach a savings goal of G.
- I will rebalance my portfolio every v periods, so that I need to have an investment plan for each of $T = Y/v$ periods (stages).
- We are given a universe $\mathcal{N} = \{1, \ldots, n\}$ of assets to invest in.
- Let $\mu_{it}, i \in \mathcal{N}, t \in \mathcal{T} = \{1, \ldots, T\}$ be the (mean) return of investment i in period t.
- For each dollar by which I exceed my goal of G, I get a reward of q.
- For each dollar I am short of G, I get a penalty of p.
- I have $\$B$ to invest initially.

Variables

- $x_{it}, i \in \mathcal{N}, t \in \mathcal{T}$: Amount of money to invest in asset i at beginning of period t t.
- z : Excess money at the end of horizon.
- w : Shortage in money at the end of the horizon.

A Naive Formulation

minimize

$$qz + pw$$

subject to

$$\begin{aligned} \sum_{i \in \mathcal{N}} x_{i1} &= B \\ \sum_{i \in \mathcal{N}} x_{it} &= \sum_{i \in \mathcal{N}} (1 + \mu_{it}) x_{i,t-1} \qquad \forall t \in \mathcal{T} \\ \sum_{i \in \mathcal{N}} (1 + \mu_{iT}) x_{iT} - z + w &= G \\ x_{it} &\geq 0 \qquad \forall i \in \mathcal{N}, t \in \mathcal{T} \\ z, w &\geq 0 \end{aligned}$$

A Better Model

- What are some weaknesses of the model on the previous slide?
- Well, there are many...
- For one, it doesn't take into account the variability in returns (i.e., risk).
- Another is that it doesn't take into account my ability to rebalance my portfolio *after* observing returns from previous periods.
- I can and would change my portfolio after observing the market outcome.
- Let's use our standard notation for a market consisting of n assets with the price of asset i at the end of period t being denoted by the random variable S_t^i.
- Let $R_{it} = S_t^i / S_{t-1}^i$ be the return of asset i in period t.
- As we have done previously, let's take a scenario approach to specifying the distribution of R_{it}.

Scenarios

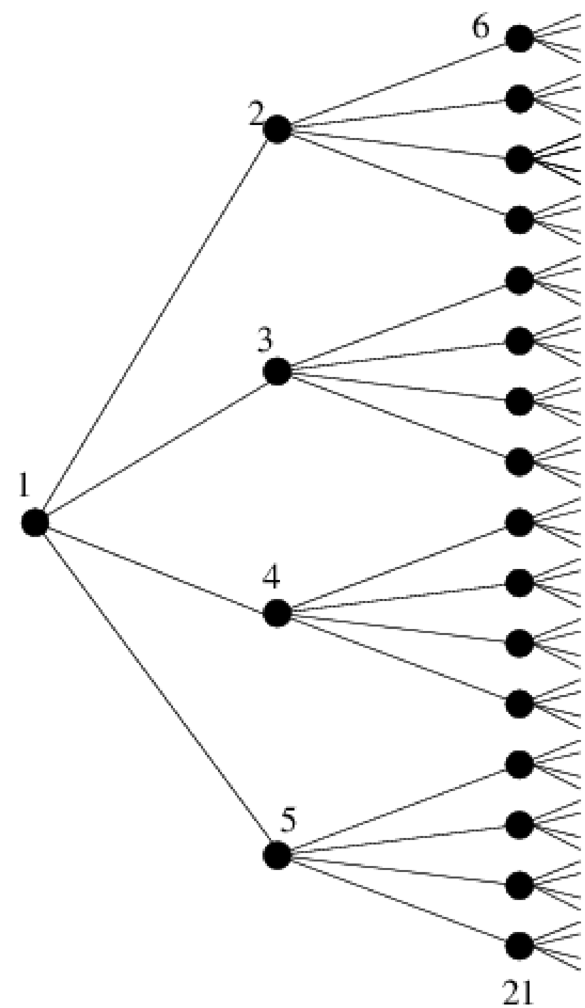
- We let the scenarios consist of all possible sequences of outcomes.
- Generally, we assume that for a particular realization of returns in period t, there will be M possible realizations for returns in period $t+1$.
- We then have M^T possible scenarios indexed by a set S.
- As before, we can then assume that we have a probability space (P^t, Ω^t) for each period t and that Ω^t is partitioned into $|S|$ subsets $\Omega^t_s, s \in S$.
- We then let $p^t_s = P(\Omega^t_s)\ \forall s \in S, t \in \mathcal{T}$.
- For instance, if $M = 4$ and $T = 3$, then we might have...

$t=1$	$t=2$	$t=3$
1	1	1
1	1	2
1	1	3
1	1	4
1	2	1
	⋮	
4	4	4

- $|S| = 64$
- We can specify any probability on this outcome space that we would like.
- The time period outcomes don't need to be equally likely and returns in different time periods need not be mutually independent.

A Scenario Tree

- Essentially, we are approximating the continuous probability distribution of returns using a discrete set of outcomes.
- Conceptually, the sequence of random events (returns) can be arranged into a tree



Making it Stochastic

- Once we have a distribution on the returns, we could add uncertainty into our previous model simply by considering each scenario separately.
- The variables now become
  - $x_{its}, i \in \mathcal{N}, t \in \mathcal{T}$: Amount of money to reinvest in asset i at beginning of period t in scenario s.
  - $z_s, s \in S$: Excess money at the end of horizon in scenario s.
  - $w_s, s \in S$: Shortage in money at the end of the horizon in scenario s.
- Note that the return μ_{its} is now indexed by the scenario s.

A Stochastic Version: First Attempt

minimize

$$??????????????$$

subject to

$$
\begin{aligned}
\sum_{i \in \mathcal{N}} x_{i1} &= B \\
\sum_{i \in \mathcal{N}} x_{its} &= \sum_{i \in \mathcal{N}} (1 + \mu_{its}) x_{i,t-1,s} \qquad \forall t \in \mathcal{T}, \forall s \in S \\
\sum_{i \in \mathcal{N}} \mu_{iTs} x_{iTs} - z_s + w_s &= G \qquad \forall s \in S \\
x_{its} &\geq 0 \qquad \forall i \in \mathcal{N}, t \in \mathcal{T}, \forall s \in S \\
z_s, w_s &\geq 0 \qquad \forall s \in S
\end{aligned}
$$

Easy, Huh?

- We have just converted a multi-stage stochastic program into a deterministic model.
- However, there are some problems with our first attempt.
- What are they?

One Way to Fix It

- What we did to create our *deterministic equivalent* was to create copies of the variables for every scenario at every time period.
- One missing element is that we still do not have a notion of a probability distribution on the scenarios.
- But there's an even bigger problem...
- We need to enforce *nonanticipativity*...
- Let's define E_s^t as the set of scenarios with the same outcomes as scenario s up to time t.
- At time t, the copies of all the anticipative decision variables corresponding to scenarios in E_s^t must have the same value.
- Otherwise, we will essentially be making decision at time t using information only available in periods after t.

A Stochastic Version: Explicit Nonanticipativity

minimize

$$\sum_{s\in S} p_s\left(qz_s - pw_s\right)$$

subject to

$$\begin{aligned}
\sum_{i\in\mathcal{N}} x_{i1} &= B \\
\sum_{i\in\mathcal{N}} x_{its} &= \sum_{i\in\mathcal{N}}(1+\mu_{its})x_{i,t-1,s} \qquad \forall t\in\mathcal{T},\forall s\in S \\
\sum_{i\in\mathcal{N}} \mu_{iTs}x_{iTs} - z_s + w_s &= G \qquad \forall s\in S \\
x_{its} &= x_{its'}\ \forall i\in\mathcal{N},\forall t\in T,\forall s\in S,\forall s'\in E_s^t \\
x_{its} &\geq 0 \qquad \forall i\in\mathcal{N}, t\in\mathcal{T},\forall s\in S \\
z_s, w_s &\geq 0 \qquad \forall s\in S
\end{aligned}$$

Another Way

- We can also enforce nonanticipativity by using the "right" set of variables.
- We have a vector of variables for each node in the scenario tree.
- This vector corresponds to what our decision would be, given the realizations of the random variables we have seen so far.
- Index the nodes $\mathcal{L} = \{1, 2, \ldots \mathcal{L}\}$.
- We will need to know the "parent" of any node.
- Let $A(l)$ be the ancestor of node $l \in \mathcal{L}$ in the scenario tree.
- Let $N(t)$ be the set of all nodes associated with decisions to be made at the beginning of period t.

Another Multistage Formulation

maximize

$$\sum_{l \in N(T)} p_l \left(q z_l + p w_l \right)$$

subject to

$$\begin{aligned}
\sum_{i \in \mathcal{N}} x_{i1} &= B \\
\sum_{i \in \mathcal{N}} x_{il} &= \sum_{i \in \mathcal{N}} (1 + \mu_{il}) x_{i,A(l)} \qquad \forall l \in \mathcal{L} \\
\sum_{i \in \mathcal{N}} \mu_{il} x_{il} - z_l + w_l &= G \qquad \forall l \in N(T) \\
x_{il} &\geq 0 \qquad \forall i \in \mathcal{N}, l \in \mathcal{L} \\
z_l, w_l &\geq 0 \qquad \forall l \in N(T)
\end{aligned}$$

AMPL

```
set Investments;
param NumNodes := 21;
param NumScen := 64;
param NumOutcome := 4;
param Return{0 .. NumOutcome-1, Investments};

var x{1 .. NumNodes,Investments} >= 0;
var z{1 .. NumScen} >= 0;
var w{1 .. NumScen} >= 0;

param A{k in 2 .. NumNodes} = ceil((k-1)/NumOutcome);
param A2{s in 1 .. NumScen} = 5 + ceil(s/NumOutcome);
param O{k in 2 .. NumNodes} = (k-2) mod NumOutcome;
param O2{s in 1 .. NumScen} = (s-1) mod NumOutcome;

maximize ExpectedWealth:
 sum{s in 1 .. NumScen} 1/NumScen * (q * z[s] - r * w[s]);

subject to InvestAllMoney:
sum{i in Investments} x[1,i] = b;

subject to WealthBalance1{k in 2 .. NumNodes}:
sum{i in Investments} x[k,i] = sum{i in Investments} Return[O[k],i] * x[A[k],i];

subject to Shortage{s in 1 .. NumScen}:
sum{i in Investments} Return[O2[s],i] * x[A2[s],i] - z[s] + w[s] = G;
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