Financial Optimization
ISE 347/447

Lecture 2

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Reading for This Lecture

- C&T Chapter 1

Review from Last Time

- Recall that a mathematical model consists of:
  - Decision variables (with domains)
  - Constraints (functions of the variables with domains)
  - Objective Function (maximize or minimize)
  - Parameters and Data

The general form of a *mathematical optimization model* is:

$$\begin{array}{lrcl} \min \text{ or } \max & f(x_1, \ldots, x_n) & & \\ s.t. & g_i(x_1, \ldots, x_n) & \left\{ \begin{array}{c} \leq \\ = \\ \geq \end{array} \right\} & b_i \\ & (x_1, \ldots, x_n) & \in & X \end{array}$$

where X may be a discrete set. Without loss of generality, we may assume that all constraints are of the "$\geq$" type.

Categorizing Mathematical Optimization Problems

- Mathematical optimization problems can be categorized along several fundamental lines.
  - Constrained vs. Unconstrained
  - Convex vs. Nonconvex
  - Linear vs. Nonlinear
  - Discrete vs. Continuous
- What is the importance of these categorizations?
  - Knowing what category an instance is in can tell us something about how difficult it will be to solve.
  - Different solvers are designed for different categories.

Unconstrained Optimization

- When $M = \emptyset$ and $X = \mathbb{R}^n$, we have an *unconstrained optimization problem*.
- Unconstrained optimization problems will not generally arise directly from applications.
- They do, however, arise as *subproblems* when solving mathematical optimization problems.
- In unconstrained optimization, it is important to distinguish between the *convex* and *nonconvex* cases.

Linear Optimization Problems

- A linear optimization problem is one that can be written in a form in which the functions f and g_i, $i \in M$ are all linear and $X = \mathbb{R}^n$.
- In general, a linear optimization problem is one that can be written as

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{s.t.} \quad & a_i^\top x \geq b_i \ \forall i \in M_1 \\ & a_i^\top x \leq b_i \ \forall i \in M_2 \\ & a_i^\top x = b_i \ \forall i \in M_3 \\ & x_j \geq 0 \ \ \forall j \in N_1 \\ & x_j \leq 0 \ \ \forall j \in N_2 \end{aligned}$$

- Equivalently, a linear optimization problem can be written as

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{s.t.} \quad & Ax \geq b \end{aligned}$$

- Generally speaking, linear optimization problems are "easy" to solve.

Nonlinear Optimization Problems

- A *nonlinear optimization problem* is any mathematical optimization problem that cannot be expressed as a linear optimization problem.
- Usually, this terminology also assumes $X = \mathbb{R}^n$.
- Note that by this definition, it is not always obvious whether a given instance is really nonlinear.
- In general, nonlinear optimization problems are difficult to solve to global optimality.

Special Case: Convex Optimization Problems

- A *convex optimization problem* is a nonlinear optimization problem in which the objective function f is convex and the feasible region

$$\mathcal{F} = \{x \in \mathbb{R}^n \mid g_i(x) \geq b_i\}$$

is a convex set.

- In practice, convex optimization problems are usually "easy" to solve.

Special Case: Quadratic Optimization Problems

- A *quadratic optimization problem* is one in which all of the functions f and g_i for $i \in M$ are quadratic functions.
- Often, the term quadratic optimization problem refers specifically to a optimization problem of the form

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^\top Qx + c^\top x \\ \text{s.t.} & Ax \geq b \end{array}$$

- Because $x^\top Qx = \frac{1}{2}x^\top(Q + Q^\top)x$, we can assume without loss of generality that Q is *symmetric*.
- The objective function of the above optimization problem is then convex if and only if Q is *positive semidefinite*, i.e., $y^\top Qy \geq 0$ for all $y \in \mathbb{R}^n$.
- There are specialized methods for solving convex quadratic optimization problems efficiently.

Special Case: Integer Optimization Problems

- When $X = \mathbb{Z}^n$, we have an *integer optimization problem*.
- When $X = \mathbb{Z}^r \times \mathbb{R}^{n-r}$, we have a *mixed integer optimization problem*.
- By convention, all functions are assumed to be linear in these cases unless otherwise specified.
- If some of the functions are nonlinear, then we have a *mixed integer nonlinear optimization problem*.
- All mathematical optimization problem with integer variables are difficult to solve in general.

Probability Review

- Stochastic optimization is essentially mathematical optimization with "random" parameters.
- Therefore, we'll need to dig out just a little probability theory.
- The symbol ω will denote the *outcome* of a random experiment.
- The set of all possible outcomes, called the *sample space*, will generally be denoted Ω.
- Subsets of Ω are called *events*.

Probability spaces

- Let $\mathcal{A}$ be a set of events.
- A probability measure (or distribution) P is a function that indicates the probability that each event $A \in \mathcal{A}$ will occur.
- Probability measures must satisfy certain axioms and have the following basic properties.
  - $0 \leq P(A) \leq 1$
  - $P(\Omega) = 1$, $P(\emptyset) = 0$
  - $P(A_1 \cup A_2) = P(A_1) + P(A_2)$ if $A_1 \cap A_2 = \emptyset$.
- The triple $(\Omega, \mathcal{A}, P)$ is called a *probability space*.

Random Variables

- A random variable ξ on a probability space $(\Omega, \mathcal{A}, P)$ is a function $\xi : \Omega \to \mathbb{R}$ such that $\{\omega | \xi(\omega) \leq x\} \in \mathcal{A}$ for all finite x.
- ξ has a *cumulative distribution* given by $F_\xi(x) = P(\xi \leq x)$.
- *Discrete random variables* are those that take on a finite number of values $\xi^k, k \in K$
- Random variables have an associated *probability density function*.
- For a discrete random variable the density function $f(\xi^k) \equiv P(\xi = \xi^k)$
- A continuous random variables has density f with the property

$$\begin{aligned} P(a \leq \xi \leq b) &= \int_a^b f(\xi) d\xi \\ &= \int_a^b dF(\xi) \\ &= F(b) - F(a) \end{aligned}$$

Expectation and Variance

- The *Expected value* of ξ is
  - $\mathbb{E}(\xi) = \sum_{k \in K} \xi^k f(\xi^k)$ (Discrete)
  - $\mathbb{E}(\xi) = \int_{-\infty}^{\infty} f(\xi) d\xi = \int_{-\infty}^{\infty} dF(\xi)$. (Continuous)
- *Variance* of ξ is $\text{Var}(\xi) = \mathbb{E}(\xi - \mathbb{E}(\xi)^2)$.