Financial Optimization
ISE 347/447

Lecture 19

Dr. Ted Ralphs

Reading for This Lecture

- C&T Chapter 16

Mathematical Programming and Uncertainty

- Many of the models we have seen up until now have involved some degree of uncertainty.
  - Markowitz mean-variance model for portfolio optimization
  - Binomial lattice model for option pricing
  - Similarity model for constructing an index fund
- How did we take account of the inherent randomness in each of these cases?

Modeling Uncertainty

- In all of the previous cases, the stochasticity was handled essentially by creating a deterministic mathematical program that implicitly captures some of the randomness.
- A more direct approach is to create a modeling framework that allows stochasticity to be modeled explicitly.
- There are a number of such frameworks, but we will consider two of them.
  - Stochastic Programming
  - Robust Optimization
- The type of stochasticity addressed in both of these frameworks is uncertainty in the input parameters.
- The ability to model this type of uncertainty is enough to capture the richness of many real-world problems.

A Random Linear Optimization Problem

minimize

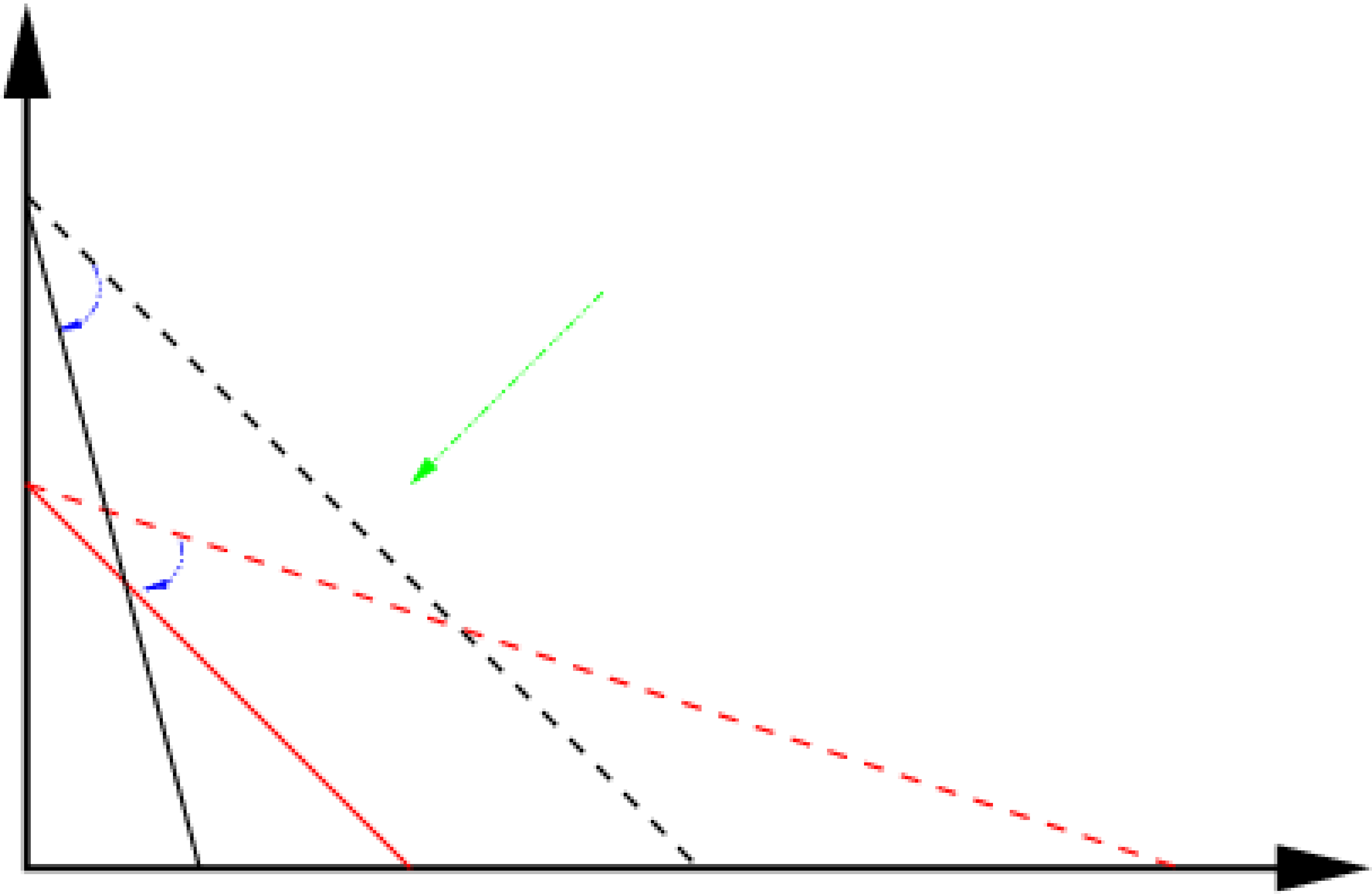
$$x_1 + x_2$$

subject to

$$\begin{aligned} \omega_1 x_1 + x_2 &\geq 7 \\ \omega_2 x_1 + x_2 &\geq 4 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

- $\omega_1 \sim \mathcal{U}[1, 4]$
- $\omega_2 \sim \mathcal{U}[1/3, 1]$

Worth 1000 Words?



What To Do?

- How do we solve this problem?
- What do we *mean* by solving this problem?
- Suppose it is possible to make a decision *after* the observation of the random vector ω?
- This could be called a "wait-and-see" approach.
- Can we solve the problem then?
- I sure hope so—it's just a simple deterministic linear optimization problem!

Here and Now

- Generally, "wait-and-see" is not an appropriate model of how things work.
- We generally need to decide on a course of action *before* knowing the outcome of randomness.
- In order for the problem to make sense in this case, we need to decide what to do about not knowing ω_1 and ω_2.
- Three suggestions
  - Guess at uncertainty
  - Probabilistic Constraints
  - Penalize Shortfall

First Approach: Guess Away!

- Often, this is what's done in practice for lack of anything better.
- We simply guess reasonable values for ω_1, ω_2
- What should we guess?
- Three (obvious) ideas
  - Unbiased: Choose mean values
  - Pessimistic: Choose worst case values
  - Optimistic: Choose best case values
- The choice depends on our level of risk aversion.

Guess: Unbiased

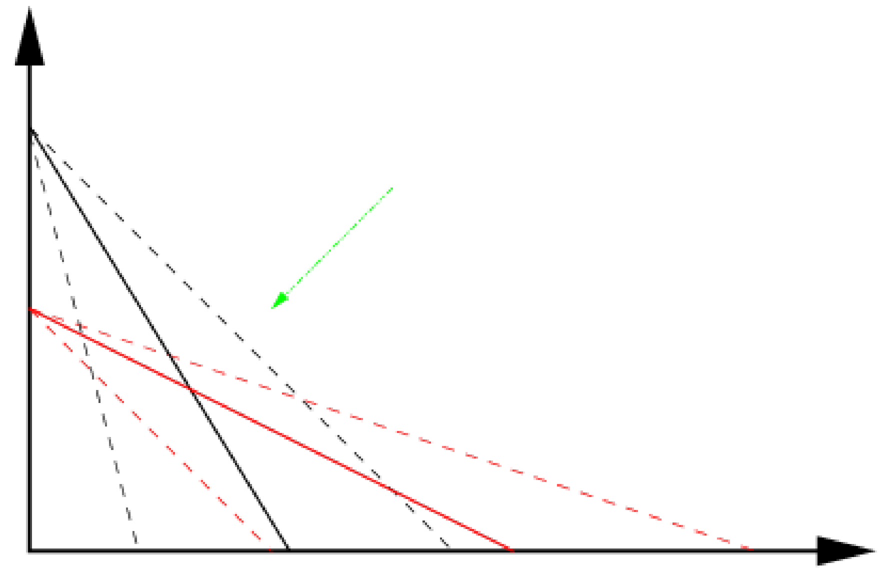
$\hat{\omega} \equiv \mathbb{E}(\omega) = (5/2, 3/2)$

minimize

$$x_1 + x_2$$

subject to

$$\begin{aligned} \frac{5}{2}x_1 + x_2 &\geq 7 \\ \frac{3}{2}x_1 + x_2 &\geq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$



- OPT $= 50/11$
- $(\hat{x}_1, \hat{x}_2) = (18/11, 32/11)$

Guess: Pessimistic

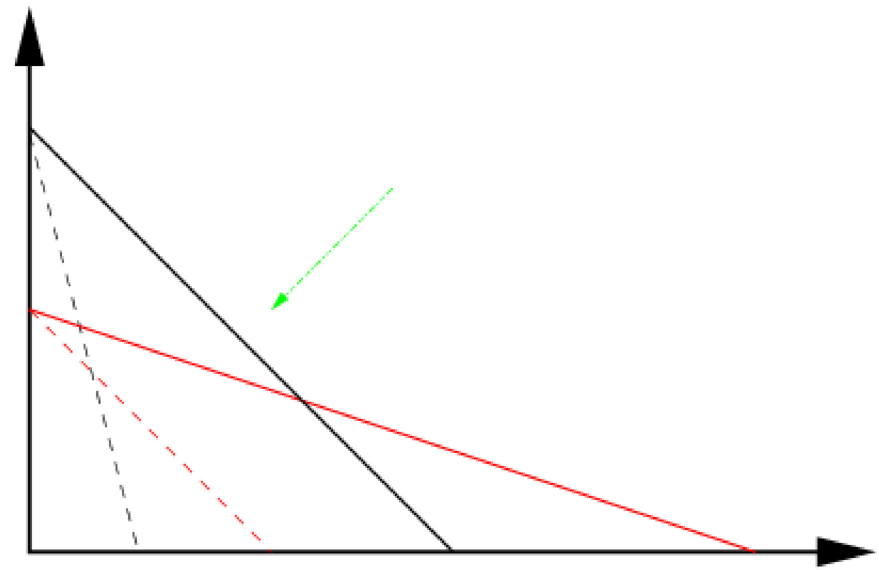
$\hat{\omega} = (1, 1/3)$

minimize

$$x_1 + x_2$$

subject to

$$\begin{aligned} 1x_1 + x_2 &\geq 7 \\ 1/3x_1 + x_2 &\geq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$



Picture...

- OPT $= 7$
- $(\hat{x}_1, \hat{x}_2) = (0, 7)$

Guess: Optimistic

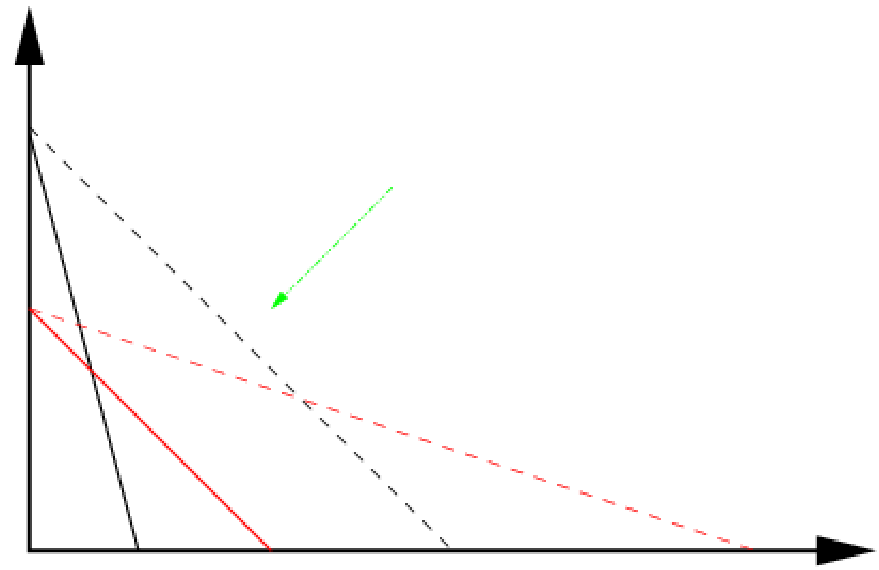
$\hat{\omega} = (4, 1)$

minimize

$$x_1 + x_2$$

subject to

$$\begin{aligned} 4x_1 + x_2 &\geq 7 \\ 1x_1 + x_2 &\geq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$



- OPT $= 4$
- $(\hat{x}_1, \hat{x}_2) = (4, 0)$

Pros and Cons

- Pros
  - All of these approaches are easy
  - We just solve a deterministic problem of the same size as the original random problem.
  - Only "rough" information about the randomness ω is needed.
- Cons
  - Only takes into account one possible outcome of future uncertainty.
  - There might even be an outcome for which the chosen "solution" is infeasible.

Second Approach: Chance Constrained

Another (perhaps more reasonable) approach is to enforce that the *probability* of a constraint being satisfied is sufficiently large.

Let's add the constraints

$$\begin{aligned} \mathsf{P}\{\omega_1 x_1 + x_2 \geq 7\} &\geq \alpha_1 \\ \mathsf{P}\{\omega_2 x_1 + x_2 \geq 4\} &\geq \alpha_2 \end{aligned}$$

Or maybe the constraint

$$\mathsf{P}\{\omega_1 x_1 + x_2 \geq 7, \omega_2 x_1 + x_2 \geq 4\} \geq \alpha$$

Chance Constraints

- Note that for $\alpha_1, \alpha_2, \alpha = 1$, this is equivalent to a normal (deterministic) problem.
- Question: How do we solve probabilistically constrained problems?
- Answer: It's extremely difficult
- We will put this method aside for now.

Third Approach: Penalize Shortfall

- We accept infeasibility, but penalize the expected violation.
- Notation:
  - $x^+ \equiv \max(0, z)$: The positive part of z.
  - $x^- \equiv \max(0, -z)$: The negative part of z.
- Then, for the constraint $\omega_1 x_1 + x_2 \geq 7$, the shortfall is $(\omega_1 x_1 + x_2 - 7)^-$.
- For each constraint, assign (unit) shortfall costs q_1, q_2.
- Optimization problem becomes...

$$\min_{x \in \mathbb{R}^2_+} \{x_1 + x_2 + q_1 \mathbb{E}_{\omega_1} \left[(\omega_1 x_1 + x_2 - 7)^-\right] + q_2 \mathbb{E}_{\omega_2} \left[(\omega_2 x_1 + x_2 - 4)^-\right]\}$$

Yikes!

- The function we are trying to optimize looks ugly.
- However, it is convex.
- In fact, it is not too hard to see that the problem is equivalent to the following:

$$\min_{x \in \mathbb{R}^2_+} \left\{ x_1 + x_2 + \mathbb{E}_\omega \left[\min_{y \in \mathbb{R}^2_+} \left\{ q_1 y_1 + q_2 y_2 : \begin{array}{l} \omega_1 x_1 + x_2 + y_1 \geq 7 \\ \omega_2 x_1 + x_2 + y_2 \geq 4 \end{array} \right\} \right] \right\}$$

Recourse Function

Let's write the problem in terms of only the original variables:

$$\min_{x \in \mathbb{R}^2_+} \{x_1 + x_2 + \mathcal{Q}(x_1, x_2)\}$$

where

$$\mathcal{Q}(x_1, x_2) = \mathbb{E}_\omega \left[\min_{y \in \mathbb{R}^2_+} \left\{ q_1 y_1 + q_2 y_2 : \begin{array}{l} y_1 \geq 7 - \omega_1 x_1 - x_2 \\ y_2 \geq 4 - \omega_2 x_1 - x_2 \end{array} \right\} \right]$$

- $\mathcal{Q}(x_1, x_2)$ is called the *recourse function*.
- For a given decision x_1, x_2, what do we do (recourse)?
- In this case, the answer is simply to penalize the shortfall.
- y_1, y_2 will be exactly the shortfall in constraints 1 and 2.

Decisions, Stages, and Recourse

Instead of penalizing shortfall, we might be able to take "corrective action," i.e., recourse!

Consider a planning problem with two periods. The following sequence of events occurs.

1. We make a decision now (first-period decision)
2. A random event occurs ("stuff" happens)
3. We make a second period decision that attempts to repair the havoc wrought by the random event . (recourse)

Example: Farmer Fred's Magic Beans

- Farmer Fred can grow Wheat, Corn, or Beans on his 500 acres.
- Farmer Fred requires 200 tons of wheat and 240 tons of corn to feed his cattle
  - These can be grown on his land or bought from a wholesaler.
  - Any production in excess of these amounts can be sold for $\$170/ton$ (wheat) and $\$150/ton$ (corn)
  - Any shortfall must be bought from the wholesaler at a cost of $\$238/ton$ (wheat) and $\$210/ton$ (corn).
- Farmer Fred can also grow beans
  - Beans sell at $\$36/ton$ for the first 6000 tons
  - Due to economic quotas on bean production, beans in excess of 6000 tons can only be sold at $\$10/ton$

The Data

There are 500 acres available for planting

	Wheat	Corn	Beans
Yield (Tons/acre)	2.5	3	20
Planting Cost ($/acre)	150	230	260
Selling Price	170	150	36 ($\leq$ 6000T) 10 ($>$6000T)
Purchase Price	238	210	N/A
Minimum Requirement	200	240	N/A

LP Formulation: Decision Variables

- $x_{W,C,B}$: Acres of Wheat, Corn, Beans Planted
- $w_{W,C,B}$: Tons of Wheat, Corn, Beans sold (at favorable price).
- e_B: Tons of beans sold at lower price
- $y_{W,C}$: Tons of Wheat, Corn purchased.
- Notes:
  - Farmer Fred has *recourse*.
  - After he observes the weather, he can decide how much of each crop to sell or purchase!

LP Formulation: Objective and Constraints

maximize

$$-150x_W - 230x_C - 260x_B - 238y_W + 170w_W - 210y_C + 150y_C + 36w_B + 10e_B$$

subject to

$$\begin{aligned}
x_W + x_C + x_B &\leq 500 \\
2.5x_W + y_W - w_W &= 200 \\
3x_C + y_C - w_C &= 240 \\
20x_B - w_B - e_B &= 0 \\
w_B &\leq 6000 \\
x_W, x_C, x_B, y_W, y_C, e_B, w_W, w_C, w_B &\geq 0
\end{aligned}$$

Solution with (expected) yields

	Wheat	Corn	Beans
Plant (acres)	120	80	300
Production	300	240	6000
Sales	100	0	6000
Purchase	0	0	0

Profit: $118,600

Planting Intuition

The LP solution corresponds to Farmer Fred's intuition.

- Plant the land necessary to grow up to his quota limit of beans.
- Plant land necessary to meet his requirements for wheat and corn
- Plant remaining land with wheat – sell excess.

It's the Weather, Stupid!

- Farmer Fred knows enough to know that his yields aren't always precisely $Y = (2.5, 3, 20)$.
- He decides to run two more scenarios:
  - Good weather: $1.2Y$
  - Bad weather: $0.8Y$

Formulation: Good weather

maximize

$$-150x_W - 230x_C - 260x_B - 238y_W + 170w_W - 210y_C + 150y_C + 36w_B + 10e_B$$

subject to

$$\begin{aligned}
x_W + x_C + x_B &\leq 500 \\
3x_W + y_W - w_W &= 200 \\
3.6x_C + y_C - w_C &= 240 \\
24x_B - w_B - e_B &= 0 \\
w_B &\leq 6000 \\
x_W, x_C, x_B, y_W, y_C, e_B, w_W, w_C, w_B &\geq 0
\end{aligned}$$

Solution: Good Weather

	Wheat	Corn	Beans
Plant (acres)	183.33	66.67	250
Production	550	240	6000
Sales	350	0	6000
Purchase	0	0	0

Profit: $167,667

Formulation: Bad Weather

maximize

$$-150x_W - 230x_C - 260x_B - 238y_W + 170w_W - 210y_C + 150y_C + 36w_B + 10e_B$$

subject to

$$\begin{aligned}
x_W + x_C + x_B &\leq 500 \\
2x_W + y_W - w_W &= 200 \\
2.4x_C + y_C - w_C &= 240 \\
16x_B - w_B - e_B &= 0 \\
w_B &\leq 6000 \\
x_W, x_C, x_B, y_W, y_C, e_B, w_W, w_C, w_B &\geq 0
\end{aligned}$$

Solution: Bad Weather

	Wheat	Corn	Beans
Plant (acres)	100	25	375
Production	200	60	6000
Sales	0	0	6000
Purchase	0	180	0

Profit: $59,950

What To Do?

- Obviously the answer is quite dependent on the weather and the respective yields.
- Without knowing the weather, he can't determine the proper acreage of beans to plant.
- It's impossible to make a perfect decision, since planting decisions must be made now, but purchase and sales decisions can be made later.

Answer: Maximize Expected Profit

- Assume that the three scenarios occur with equal probability.
- Attach a scenario subscript $s = 1, 2, 3$ to each of the purchase and sale variables (1=Good, 2=Average, 3=Bad).
  - w_{C2} : Tons of corn sold at favorable price in scenario 2
  - e_{B3} : Tons of beans sold at unfavorable price in scenario 3.

Expected Profit

An expression for Farmer Fred's Expected Profit is the following:

$$
\begin{aligned}
& 150x_W - 230x_C - 260x_B \\
& +1/3(-238y_{W1} + 170w_{W1} - 210y_{C1} + 150y_{C1} + 36w_{B1} + 10e_{B1}) \\
& +1/3(-238y_{W2} + 170w_{W2} - 210y_{C2} + 150y_{C2} + 36w_{B2} + 10e_{B2}) \\
& +1/3(-238y_{W3} + 170w_{W3} - 210y_{C3} + 150y_{C3} + 36w_{B3} + 10e_{B3})
\end{aligned}
$$

Expected Value Problem: Constraints

$$
\begin{aligned}
x_W + x_C + x_B &\leq 500 \\
3x_W + y_{W1} - w_{W1} &= 200 \\
2.5x_W + y_{W2} - w_{W2} &= 200 \\
2x_W + y_{W3} - w_{W3} &= 200 \\
3.6x_C + y_{C1} - w_{C1} &= 240 \\
3x_C + y_{C2} - w_{C2} &= 240 \\
2.4x_C + y_{C3} - w_{C3} &= 240 \\
24x_B - w_{B1} - e_{B1} &= 0 \\
20x_B - w_{B2} - e_{B2} &= 0 \\
16x_B - w_{B3} - e_{B3} &= 0 \\
w_{B1}, w_{B2}, w_{B3} &\leq 6000 \\
\text{All vars} &\geq 0
\end{aligned}
$$

Expected Value Problem: Solution

	Wheat	Corn	Beans	
s	Plant (acres)	170	80	250
1	Production	510	288	6000
1	Sales	310	48	6000
1	Purchase	0	0	0
2	Production	425	240	5000
2	Sales	225	0	5000
2	Purchase	0	0	0
3	Production	340	192	4000
3	Sales	140	0	4000
3	Purchase	0	48	0

Profit: $108,390

Examining the Solution

- Best solution allocates land for beans to always avoid having to sell them at the unfavorable price.
- Corn is planted so that the requirement is met in the average scenario.
- The remaining land is allocated to wheat.
- Again, it is impossible to find a solution that is ideal under all circumstances.
- Decisions in stochastic models are balanced, or *hedged* against the various scenarios.

Fortune Telling

- Suppose Farmer Fred could *with certainty* tell whether or not the upcoming growing season was going to have good weather, average weather, or bad weather.
  - His bursitis was acting up
  - Consulting the Farmer's Almanac
  - Hire a fortune teller
- One question here is how much Farmer Fred would be willing to pay for this "perfect" information.
- In real-life problems, how much is it "worth" to invest in better (or perfect) forecasting technology?

What's it worth?

- With perfect information, Farmer Fred's would plant (wheat, corn, beans).
  - Good yield: (183.33, 66.67, 250), Profit: $167,667
  - Average yield: (120, 80, 300), Profit: $118,600
  - Bad yield: (100, 25, 375), Profit: $59,950
- Assuming each of these scenarios occurs with probability 1/3, his long run average profit would be

$$(1/3)(167667) + (1/3)(118600) + (1/3)(59950) = 115406$$

- With his (optimal) "here-and-now" decision of (170, 80, 250), he would make a long run profit of 108390
- This difference (115406-108390) is the *expected value of perfect information* (EVPI)