Financial Optimization
ISE 347/447

Lecture 18

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Reading for This Lecture

- C&T Chapter 15

The Mortgage Market

- Mortgages represent the largest single sector of the U.S. debt market, surpassing even the fedral government.
- Many financial instruments have therefore been created to provide credit to this market.
- The primary way this has been accomplished since the 1970s is the bundling together of individual mortgages into capital market instruments called *mortgage-backed securities* (MBSs).
- The principal and interest from the mortgages in the pool backing an MBS are passed through to investors in some fashion.
- By selling MBSs, banks can realize their fees up front and lay off their risk to the market.

Pass-through MBSs

- Initially, MBSs were simply packaged using a pass-through structure.
- Each investor received a pro rata share of principal and interest payments for mortgages in the pool.
- The problem with this approach is that the cash flows are very unpredictable due to *pre-payment risk*.
- Mortgage payers prepay for a variety of reasons, but for fixed-rate mortgages, this is usually associated with a drop in interest rates.
- This may force an unplanned reinvestment at a lower interest rate.

Collateralized Mortgage Obligations

- A *collateralized mortgage obligation* is a more sophisticated MBS that rearranges cash flows to make them more predictable.
- There are many ways of doing this, but here we focus on the creation of *consecutive tranches*.
- The basic idea is to package the cash flows into bonds with different maturities.
- Principal payments are funneled to investors in each tranche consecutively until the obligation is repaid.

Simple Two-Tranche Model

- Suppose we have an MBS consisting of $\$100$ million in mortgage loans.
- In a two-tranche model, we might divide the pool into two $\$50$ million tranches.
- Initially, investors in both tranches receive interest payments, but all principal payments are funneled to the investors in the first tranche (the *fast-pay tranche*) until it is repaid.
- After the fast-pay tranche is repaid, remaining principal payments go to the second tranche.
- By restructuring in this way, the fast-pay tranche reaches maturity much earlier than the *slow-pay tranche*.
- A byproduct of the restructuring is that the risk of default is much lower for the fast-pay tranche.
- This means that the interest rate paid on the fast-pay tranche can be reduced, resulting in additional profit.

A Model of Consecutive Tranches

- Early payments are more likely to be fully funded than later ones.
- Hence, *fast tranches* get a higher credit rating than slower ones and can be sold at lower interest rates.
- Overall, the interest that has to be paid to buyers of the tranches is lower than the interest paid by the mortgage holders.
- Hence, the bank issuing the restructured tranches earns money.
- A bond with payback p_t of principal at time t $(t = 1, \ldots, T)$ is priced with respect to its *weighted average life* (WAL)

$$WAL = \frac{\sum_{t=1}^{T} t p_t}{\sum_{t=1}^{T} p_t}.$$

- A bond with a WAL of n years will be priced like a treasury bond with a duration of n years plus a *spread* (extra interest) which depends on the *credit rating*.

Credit Ratings and Spreads

- The spot rates of future coupon payments and the spreads for different credit ratings can be looked up in a table.
- For example, the table could look as follows, where the spreads under the credit ratings (AAA etc.) are given in *basis points*, i.e., in $1/100$ of 1%.

duration	spot rate	AAA	AA	A	BBB	BB	B	CCC
1	4.74%	85	100	115	130	165	220	345
2	4.89%	90	105	125	140	190	275	425
3	5.05%	95	110	135	150	210	335	500
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Notation

- Q_0 is the amount of principal to be repaid.
- T is the horizon over which the principal is to be repaid.
- I_t is the interest to paid in year t.
- A_t is the scheduled amortization payment in year t.
- q_t is the pre-payment rate in year t.
- R_t is the pre-payment amount in year t.
- P_t is the total payment in year t (incuding pre-payment).
- Q_t is the amount of principal outstanding at the end of year t.
- $r \times 100\%$ is the compound yearly interest rate.

Example

Let's take $Q_0 = 100$, $r = 0.1$, $T = 10$ $q_1 = .01\%$. If the principal is to be repaid in equal installments, then the scheduled amortization payment in year 1 is

$$A_1 = \frac{Q_0 r}{(1+r)^T - 1} = 6.27.$$

So we have for year 1:

- Interest: $I_1 = rQ_0 = 10$.
- Scheduled amortization: $A_1 = Q_0 r/[(1+r)^T - 1] = 6.27$.
- Prepayment: $R_1 = q_1(Q_0 - 6.27) = 0.937$.
- Total principal pay down: $P_1 = R_1 + A_1 = 6.27 + 0.937 = 7.207$.
- Principal left after year 1: $Q_1 = Q_0 - P_1 = 92.793$.

Generalizing

In general, we have a given scenario $q_1, \ldots, q_T$ of prepayment rates in years $1, 2, \ldots, T$. In year t, we have

- Interest: $I_t = rQ_{k-1}$.
- Scheduled amortization: $A_t = Q_{k-1}r/[(1+r)^T - 1]$.
- Prepayment: $R_t = q_t(Q_{k-1} - A_t)$.
- Total principal pay down: $P_t = R_t + A_t$.
- Principal left after year 1: $Q_t = Q_{k-1} - P_t$.

One can thus recursively compute the corresponding (I_t, P_t, Q_t) $(t = 1, \ldots, T)$.

Packaging

- The model we have presented is a simplification of the real problem.
- In real CMOs, the *pay-back time* of the principal is also variable, rather than just the *amount* of pricipal paid back.
- Nevertheless, this model is a good approximation to the real one and leads to very similar results.
- Once the payouts P_t are known, the question is how to optimally package them into consecutive tranches

$$(P_1, \ldots, P_{T_1}), (P_{T_1+1}, \ldots, P_{T_2}), \ldots, (P_{\ldots}, \ldots, P_T).$$

Candidate Tranches

- Let us refer to the candidate tranche $(P_j, \ldots, P_t)$ as (j, t).
- Associated with the candidate tranche (j, t) is its *buffer*

$$B_{jt} = \frac{\sum_{k=t+1}^{T} P_k}{\sum_{k=1}^{T} P_k},$$

- The buffer is the proportion of principal left after the tranche expires.
- Each tranche also has its own WAL

$$WAL_{jt} = \frac{\sum_{k=j}^{t} kP_k}{\sum_{k=j}^{t} P_k}.$$

- Note that this is the WAL of a bond that has no repayment of principal for the first $j-1$ years, but interest (coupons) is still paid during this time.

Prepayment Scenarios

- In order to achieve a high quality ranking, a tranche must be able to sustain higher than expected default rates without compromizing payments to the tranche holders.
- The default rate is determined by the scenario of prepayment rates $q_1, \ldots, q_T$.
- Regulatory bodies require that several prescribed scenarios be tested.
- For example, the Public Securities Association (PSA) industry standard benchmark is $q_1 = .01$, $q_2 = .03$, $q_3 = .05$, and $q_t = .06$ for $t \geq 4$

Tranche Credit Ratings

- For a tranche to be given a certain credit rating, it must satisfy

$$B_{jt} \geq WAL_{jt} \cdot d \cdot L,$$

where L is the *loss multiple*, specified as follows,

Rating	AAA	AA	A	BBB	BB	B	CCC
L	6	5	4	3	2	1.5	0

- Hence, the earlier tranches naturally receive higher credit ratings.

Present Value of a Tranche

- From (B_{jt}, W_{jt}) and the above table one can thus compute the credit rating for each candidate tranche (j, t).
- This rating implies a coupon rate c_{jt} that can be read off the earlier table of spot rates and spreads.
- Using the coupon rates c_{jt}, the net present value Z_{jt} of tranche (j, t) can be computed:
  - In period k, a payment of c_{jt} times the remaining principal on the tranche is paid (as interest), and if $k \in [j, t]$, then the principal payment P_k is made.
  - The result is a total payment of C_k.
  - Then the present value is $T_{jt} = \sum_{k=1}^{t} C_k/(1 + r_k)^k$.

A Dynamic Programming Formulation

- To maximize earnings, the issuer now wants to structure the CMO into K sequential tranches so as to minimize the net present value of total payments to bond-holders.
- The stages will be the number of tranches and the states will be the years $1, \ldots, T$.
- We set the value function to be

$$v(k,t) = \quad \text{The minimum present value of total payments to bondholders in years 1 through } t \text{ when the CMO has } k \text{ tranches up to year } t.$$

- Then

$$\begin{aligned} v(1,t) &= T_{1,t} \\ v(k,t) &= \min_{j=k-1,\ldots,t-1} \{v(k-1,j) + T_{j+1,t}\}, k \le t. \end{aligned}$$

Finding the Optimal CMO Structure

- Using this recurrence, we compute $v(k,t)$ for $k = 1,\ldots,K$ and $t = 1,\ldots,T$.
- The optimal net present value of future payments to bondholders is then $\min_{k=1,\ldots,K}(k,T)$.