Financial Optimization
ISE 347/447

Lecture 17

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Reading for This Lecture

- C&T Chapter 14

Option Pricing Revisited

- Recall the analysis from Lecture 6 in which we determined the fair price of an option.
- The price model for the underlying asset considered just one time period and two scenarios.
- Here, we extend this single period pricing model to consider an option on an asset whose price evolves over n periods.
- We will assume the option has a strike price of K.

Initial Conditions

- The price of the asset is S_0 at time 0.
- As before, the price at time 1 is a random variable S_1 over a probability space (Ω^1, P^1).
- We have a partition of Ω^1 into two disjoint events Ω^1_0 and Ω^1_1 such that $P^1(\Omega^1_0) = 1 - P^1(\Omega^1_1) = p$ for a given parameter $0 < p < 1$.
- Then

$$S_1(\omega) = \begin{cases} uS_0 & \text{if } \omega \in \Omega^1_0, \\ dS_0 & \text{if } \omega \in \Omega^1_1, \end{cases}$$

where parameters $u > 1$ and $d < 1$ are given.

The Binomial Lattice

- The price at time k is a random variable S_k over a probability space (Ω^k, P^k).
- Suppose we have already observed the price at time $k-1$ to be α.
- Then the price at time k will be such that the asset takes value $u\alpha$ with probability p or $d\alpha$ with probability $1-p$.
- We can visualize the evolution of the price as a "binomial lattice" in which the lattice points correspond to the possible prices in each period.

Price Scenarios

- After k periods, the price must be $u^j d^{k-j} S_0$ for some j.
- This corresponds to j periods in which the price went up and $k - j$ periods in which the price went down.
- Hence, the number of possible price scenarios for the asset in period k is $k + 1$.
- Recall that the probability of an "up move" is p and the probability of a "down move" is $1 - p$.
- Therefore, the probability of any path followed through the lattice that ends with price $u^j d^{k-j} S_0$ is $p^j (1-p)^{k-j}$.
- The number of such paths is the number of ways of choosing the j "up periods," i.e., the number of paths is $\binom{k}{j}$.

Distribution of S_k

- For the previous slide, S_k follows a standard binomial distribution with parameters k and p on the probability space (Ω^k, P^k).
- Formally, we assume that Ω^k is partitioned into disjoint events $\Omega_0^k, \ldots, \Omega_k^k$.
- Then we have

$$P^k(\Omega_j^k) = \binom{k}{j} p^j (1-p)^{k-j}$$

and

$$S_k(\omega) = u^j d^{k-j} S_0 \; \forall j \in 0, \ldots, k, \omega \in \Omega_j^k$$

Estimating the Parameters

- Before using the model, we need to choose values of u, d, and p.
- This is done so that the resulting price distribution has the same mean and variance as the asset itself.
- Because the model is multiplicative, it is convenient to use logarithms.
- Let μ and σ be the mean and standard deviation of $\ln(S_n/S_0)$, which we assume is known.
- Let $\Delta = 1/n$ be the length of time between periods.
- Then the mean and standard deviation of $\ln(S_1/S_0)$ is $\mu\Delta$ and $\sigma\sqrt{\Delta}$.
- It is easy to compute that the mean and variance of $\ln(S_1/S_0)$ are $p \ln u + (1-p) \ln d$ and $p(1-p)(\ln u - \ln d)^2$.

Estimating the Parameters (cont.)

- Matching these values, we get two equations

$$\begin{aligned} p \ln u + (1-p) \ln d &= \mu \Delta \\ p(1-p)(\ln u - \ln d)^2 &= \sigma^2 \Delta \end{aligned}$$

- Note that there are now two equations and three parameters.
- To get a solution, we further simplify by requiring $d = 1/u$.
- When Δ is small, the equations can be solved approximately to yield the values

$$\begin{aligned} u &= e^{\sigma\sqrt{\Delta}} \\ d &= e^{-\sigma\sqrt{\Delta}} \\ p &= \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta}\right) \end{aligned}$$

Pricing the Option

- Once we have the parameters, the option price can be determined using a dynamic programming approach.
- The stages are the time periods and the states correspond to the possible prices of the asset.
- The value function is defined such that $v(j, k)$ is the value of an option held in period k given that the current asset price is $u^j d^{k-j} S_0$.
- Although this is not really an optimization problem, we can still use the DP approach.

The Recursion

- We use a backwards recursion from the final period to compute the value $v(0,0)$.
- In the last period, there is no stochasticity and the value in state j for a call option is simply

$$v(N,j) = \max\{u^j d^{N-j} S_0 - K, 0\}$$

and a for a put option

$$v(N,j) = \max\{K - u^j d^{N-j} S_0, 0\}$$

- Using the technique from Lecture 6, we can compute $v(k,j)$ by knowing $v(k+1,j)$ and $v(k+1,j+1)$.
- As before, this is done using the risk-neutral probabilities.

$$p_u = \frac{R-d}{u-d} \text{ and } p_d = \frac{u-R}{u-d},$$

where $R = 1 + r$ and r is the one-period risk-free rate of return.

The Recursion (cont.)

- For a European option, we get

$$v(k,j) = \frac{1}{R}(p_u v(k+1, j+1) + p_d v(k+1, j))$$

- For an American call option, we get

$$v(k,j) = \max\{\frac{1}{R}(p_u v(k+1, j+1) + p_d v(k+1, j)), u^j d^{k-j} S_0 - K\}$$

- For an American put option, we get

$$v(k,j) = \max\{\frac{1}{R}(p_u v(k+1, j+1) + p_d v(k+1, j)), K - u^j d^{k-j} S_0\}$$

A Model for Optimal Exercise Decisions

- Let us now look at a more general price evolution model.
- We now let the price of a given asset in period k be a continuous random variable given by
$$S_k = S_{k-1} + X_k,$$
where X_k is a random variable with mean μ and distribution function F.
- The parameter μ can be interpreted as the mean return over one period.
- We assume that X_j and X_k are i.i.d. for any two periods j and k.
- We would like to know the optimal exercise policy for an American call option on this asset with strike price K.

A Dynamic Programming Model

- We will develop a continuous DP model for this situation.
- In such a DP, the number of states in each stage can be infinite.
- As before, the stages correspond to $N + 1$ time periods and the states correspond to the price of the asset.
- Now, however, the price distribution is continuous.
- The value function $v(k, S)$ will be the maximum expected profit given that the asset has price S and the option expires in k days.
- Note that this means stage 0 corresponds to period N and stage N corresponds to period 0.

A Dynamic Programming Model

- The decision set for each state is to either exercise the option or not.
- The immediate benefit from exercising is $S - K$.
- Otherwise, we are choosing to wait at least one more period.
- Given this formulation, the recursion is

$$v(k, S) = \max\{S - K, \int v(k-1, S+x) dF(x).$$

with the boundary condition $v(0, S = \max\{S - K, 0\}$.

- There is no closed form for $v(k, S)$.
- Using DP, we can obtain a solution numerically.

The Optimal Policy

Theorem 1. *The optimal policy for an American call option has the following form. There are nondecreasing numbers* $s_1 \leq s_2 \leq \cdots \leq s_k \leq \cdots \leq s_N$ *such that if the current stock price is* S *and there are* k *days until expiration, one should exercise the option if and only if* $S \geq s_k$.