Financial Optimization
ISE 347/447

Lecture 1

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Reading for This Lecture

- C&T Chapter 1

What will this class be about?

- Modeling financial optimization problems.
- Interpreting those models as mathematical programs.
- Analyzing those mathematical programs using optimization methodology and software.
- Using the analysis to gain insight and make decisions.

What is the purpose of modeling?

- The exercise of building a model can provide insight.
- It's possible to do things with models that we can't do with "the real thing."
- Analyzing models can help us decide on a course of action.

The Modeling Process

- The modeling process consists generally of the following steps.
  - Determine the "real-world" state variables, system constraints, and goal(s) or objective(s) for operating the system.
  - Translate these variables and constraints into the form of a mathematical optimization problem (the "formulation").
  - Solve the mathematical optimization problem.
  - Interpret the solution in terms of the real-world system.
- This process presents many challenges.
  - Simplifications may be required in order to ensure the eventual mathematical program is "tractable".
  - The mappings from the real-world system to the model and back are sometimes not very obvious.
  - There may be more than one valid "formulation".
- All in all, an intimate knowledge of mathematical optimization definitely helps during the modeling process.

Types of (Mathematical) Models

- There are many different classes of mathematical models that may be used to analyze financial decision-making and prediction problems.
  - Simulation Models
  - Probability Models
  - Stochastic Models
  - Behavior Models
  - Mathematical Programming/Optimization Models
- Mathematical optimization models are widely used in practice and we will focus only on these.
- This will provide a rich class of methods for analysis.

Mathematical Programming/Optimization Models

- What does *mathematical programming* mean?
- Programming here means "planning."
- Literally, these are "mathematical models for planning."
- Also called *optimization models*.
- A *mathematical optimization problem* consists of
  - a set of *variables* that describe the state of the system,
  - a set of *constraints* that determine the states that are allowable,
  - external input *parameters* and *data*, and
  - an *objective function* that provides an assessment of how well the system is functioning.
- The variables represent operating decisions that must be made.
- The constraints represent operating specifications.
- The goal is to determine the best operating state consistent with specifications.

Forming a Mathematical Optimization Model

The general form of a mathematical optimization model is:

$$
\begin{array}{lrcl}
\min \ \text{or} \ \max & f(x_1, \ldots, x_n) & & \\
\text{s.t.} & g_i(x_1, \ldots, x_n) & \left\{ \begin{array}{c} \leq \\ = \\ \geq \end{array} \right\} & b_i, \ i \in M \\
& (x_1, \ldots, x_n) & \in & X
\end{array}
$$

X may be a discrete set, such as $\mathbb{Z}^n$.

Notes:

- There is an important assumption here that all input data are known and fixed.
- Such a mathematical program is called *deterministic*.
- Is this realistic?

Types of Mathematical Optimization Problema

- Unless otherwise specified, optimization problems are usually assumed to be deterministic.
- The type of a (deterministic) optimization problem is determined primarily by
  - The form of the objective and the constraints.
  - The set X.
- A wide range of mathematical optimization model types are described at
  - the NEOS Guide, and
  - the on-line version of Operations Research Models and Methods.
- We will review the various classes in more detail in Lecture 2.

Solutions

- What is the result of analyzing an optimization problem?
- A *solution* is an assignment of values to variables.
- A solution can be thought of as a vector.
- A *feasible solution* is an assignment of values to variables such that all the constraints are satisfied.
- The *objective function value* of a solution is obtained by evaluating the objective function at the given solution.
- An *optimal solution* (assuming minimization) is one whose objective function value is less than or equal to that of all other feasible solutions.
- We may also be interested in some additional qualities of the solution.
  - Sensitivity
  - Robustness
  - Risk

Stochastic Optimization

- In the real world, little is known ahead of time with certainty.
- A *risky investment* is one whose return is not known ahead of time.
- a *risk-free* investment is one whose return is fixed.
- To make decision involving risky investments, we need to incorporate some degree of *stochasticity* into our models.

Types of Uncertainty

- Where does uncertainty come from?
  - Weather
  - Financial Uncertainty
  - Market Uncertainty
  - Competition
  - Technology
- In decision analysis, we must proceed through this list and identify items that might affect the outcome of a decision.

The Scenario Approach to Uncertainty

- The scenario approach assumes that there are a finite number of possible future outcomes of uncertainty.
- Each of these possible outcomes is called a *scenario*.
  - Demand for a product is "low, medium, or high."
  - Weather is "dry or wet."
  - The market will go "up or down."
- Even if this is not reality, often a discrete approximation is useful.

Multi-period Optimization Models

- When we introduce time as an element of a stochastic optimization model, we also have to address the following questions.
  - When do we have to make a given decision?
  - What will we know at the time we are making the decision?
  - How far into the future are we looking?
- Multi-stage models generally assume that decision are made in stages and that in each stage, some amount of uncertainty is resolved.
- Example
  - Fred decides to rebalance his investment portfolio once a quarter.
  - At the outset, he only knows current prices and perhaps some predictions about future prices.
  - At the beginning of each quarter, prices have been revealed and Fred gets a chance to make a "recourse decision."

Example 1: Short Term Financing

A company needs to make provisions for the following cash flows over the coming five months: $-150K$, $-100K$, $200K$, $-200K$, $300K$.

- The following sources of funds are available,
  - Up to $100K$ credit at 1% interest per month,
  - The company can issue a 2-month zero-coupon bond yielding 2% interest over the two months,
  - Excess funds can be invested at 0.3% monthly interest.
- How should the company finance these cash flows if no payment obligations are to remain at the end of the period?

Example 1 (cont.)

- Note that all investments are risk-free.
- What are the decision variables?
  - x_i, the amount drawn from the line of credit in month i,
  - y_i, the number of bonds issued in month i,
  - z_i, the amount invested in month i,
  - v, the wealth of the company at the end of Month 5.

Example 1 (cont.)

The problem can then be modelled as the following linear program:

$$
\begin{aligned}
\max_{(x,y,z,v)\in\mathbb{R}^{12}} \; & f(x,y,z,v) = v \\
\text{s.t. } & x_1 + y_1 - z_1 = 150 \\
& x_2 - 1.01x_1 + y_2 - z_2 + 1.003z_1 = 100 \\
& x_3 - 1.01x_2 + y_3 - 1.02y_1 - z_3 + 1.003z_2 = -200 \\
& x_4 - 1.01x_3 - 1.02y_2 - z_4 + 1.003z_3 = 200 \\
& -1.01x_4 - 1.02y_3 - v + 1.003z_4 = -300 \\
& 100 - x_i \geq 0 \quad (i = 1,\dots,4) \\
& x_i \geq 0 \quad (i = 1,\dots,4) \\
& y_i \geq 0 \quad (i = 1,\dots,3) \\
& z_i \geq 0 \quad (i = 1,\dots,4) \\
& v \geq 0.
\end{aligned}
$$

Example 2: Portfolio Optimization

- We have the choice to invest in either Intel, Wal-Mart, or G.E..
- All stocks currently cost $\$100$/share.
- In one year's time, the market will be either "up" or "down."
- the Following are the possible outcomes

	Intel	Pepsico	Wal-Mart
Up	130	112	115
Down	90	108	103

- What would you do?

Example 2 (cont.)

- If the probability that the market is up is p, then the value at year 1 from investing $\$1000$ are as follows:
  - Intel: $900 + 400p$
  - Pepsico: $1080 + 40p$
  - Wal-Mart: $1030 + 120p$
- What is your objective?

Convex Sets

A set S is *convex*

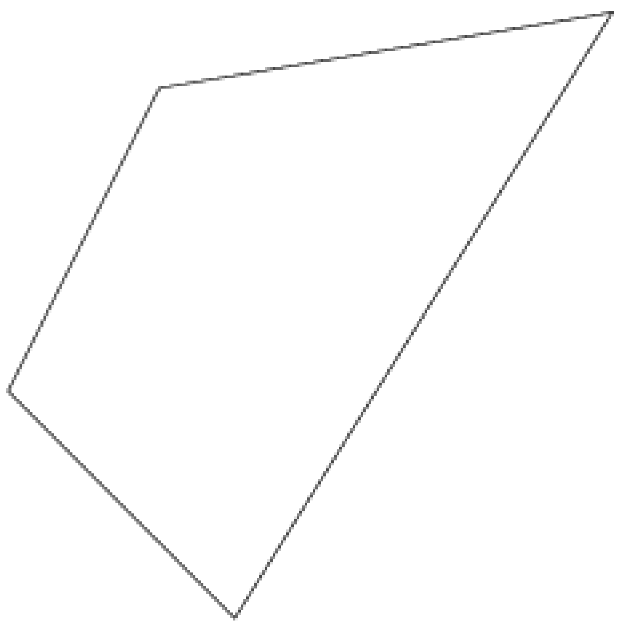
$$\Leftrightarrow$$

$$x_1, x_2 \in S, \lambda \in [0,1] \Rightarrow \lambda x_1 + (1-\lambda) x_2 \in S$$

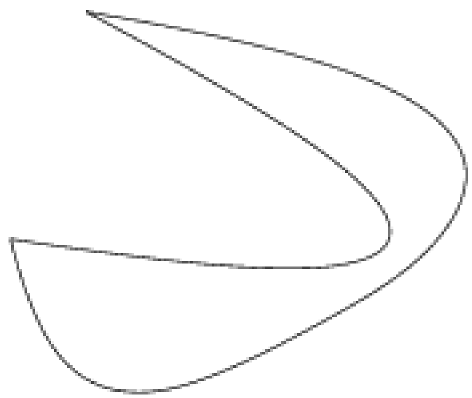
- If $y = \sum \lambda_i x_i$, where $\lambda_i \geq 0$ and $\sum \lambda_i = 1$, then y is a *convex combination* of the x_i's.
- If the positivity restriction on λ is removed, then y is an *affine combination* of the x_i's.
- If we further remove the restriction that $\sum \lambda_i = 1$, then we have a *linear combination*.

Example: Convex and Nonconvex Sets

CONVEX



NONCONVEX



Review: Convex Functions

Definition 1. *Let S be a nonempty convex set on $\mathbb{R}^n$. Then the function $f : S \to \mathbb{R}$ is said to be convex on S if*

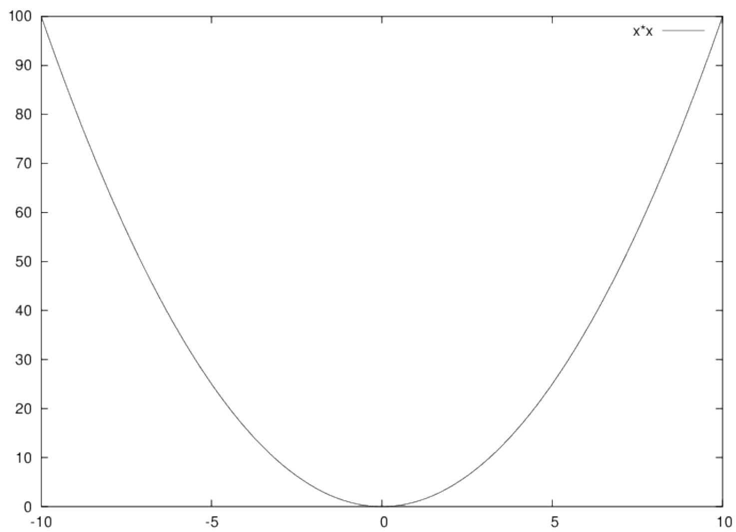
$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$$

for each $x_1, x_2 \in S$ and $\lambda \in (0,1)$.

- *Strictly convex* means the inequality is strict.
- *(Strictly) concave* is defined analogously.
- A function that is neither convex nor concave is called *nonconvex*.
- Can a function be both concave and convex?

Example: Convex and Nonconvex Functions

CONVEX



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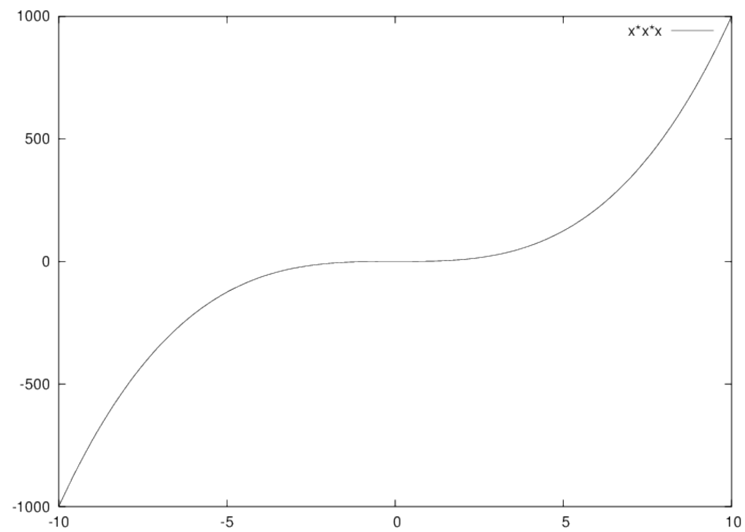

The Epigraph

Convex sets and convex functions are related by the following result:

Definition 2. *Let S be a nonempty set on $\mathbb{R}^n$ and let $f : S \to \mathbb{R}$. The* epigraph *of f is a subset of $\mathbb{R}^{n+1}$ defined by*

$$\text{epi} f = \{(x, y) : x \in S, y \in \mathbb{R}, y \geq f(x)\}$$

Theorem 1. *Let S be a nonempty convex set on $\mathbb{R}^n$ and let $f : S \to \mathbb{R}$. f is convex if and only if $\text{epi} f$ is a convex set.*

Review: Local versus Global Optimization

- For a function $f : \mathbb{R}^n \to \mathbb{R}$, a *local minimizer* is an $\hat{x} \in \mathbb{R}^n$ such that $f(\hat{x}) \leq f(x)$ for all x in a *neighborhood* of $\hat{x}$.
- In general, it is "easy" to find local minimizers.
- A *global minimizer* is $x^* \in \mathbb{R}^n$ such that $f(x^*) \leq f(x)$ for all $x \in \mathbb{R}$.
- The importance of convexity is the following:

  Theorem 2. *For a convex function $f : \mathbb{R}^n \to \mathbb{R}$, if $x^* \in \mathbb{R}^n$ is a local optimal solution to $min_{x \in \mathbb{R}^n} f(x)$, then $x^* \in \mathbb{R}^n$ is also a global optimal solution.*

Review: Vectors and Matrices

- An $m \times n$ *matrix* is an array of mn real numbers:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- A is said to have n *columns* and m *rows*.
- An n*-dimensional column vector* is a matrix with one column.
- An n*-dimensional row vector* is a matrix with one row .
- By default, a *vector* will be considered a column vector.
- The set of all n*-dimensional vectors* will be denoted $\mathbb{R}^n$.
- The set of all $m \times n$ *matrices* will be denoted $\mathbb{R}^{m \times n}$.

Review: Matrices

- The *transpose* of a matrix A is

$$A^\top = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

- If $x, y \in \mathbb{R}^n$, then $x^\top y = \sum_{i=1}^n x_i y_i$.
- This is called the *inner product* of x and y.
- If $A \in \mathbb{R}^{m \times n}$, then A_j is the j^{th} column, and $a_j^\top$ is the j^{th} row.
- If $A \in \mathbb{R}^{m \times k}, B \in \mathbb{R}^{k \times n}$, then $[AB]_{ij} = a_i^\top B_j$.

Review: Linear Functions

- A *linear function* $f : \mathbb{R}^n \to \mathbb{R}$ is a weighted sum, written as

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} c_i x_i$$

  for given coefficients $c_1, \ldots, c_n$.

- We can write $x_1, \ldots, x_n$ and $c_1, \ldots, c_n$ as vectors $x, c \in \mathbb{R}^n$ to obtain:

$$f(x) = c^\top x$$

- In this way, a linear function can be represented simply as a vector.