Integer Programming
ISE 418

Lecture 6

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Reading for This Lecture

- N&W Sections I.4.4 and I.4.6
- Wolsey Section 9.1
- CCZ Chapter 3

Describing Polyhedra

In Lecture 5, we derived the following fundamental results.

Theorem 1.

1. *Every full-dimensional polyhedron* $\mathcal{P}$ *has a unique (up to scalar multiplication) representation that consists of one inequality representing each facet of* $\mathcal{P}$*.*

2. *If* $dim(\mathcal{P}) = n - k$ *with* $k > 0$*, then* $\mathcal{P}$ *is described by a maximal set of linearly independent rows of* $(A^=, b^=)$*, as well as one inequality representing each facet of* $\mathcal{P}$*.*

Theorem 2. *If a facet* F *of* $\mathcal{P}$ *is represented by* (π, π_0)*, then the set of all representations of* F *is obtained by taking scalar multiples of* (π, π_0) *plus linear combinations of the equality set of* $\mathcal{P}$*.*

Extreme Points

Definition 1. *x is an extreme point of $\mathcal{P}$ if there do not exist $x^1, x^2 \in \mathcal{P}$ such that $x = \frac{1}{2}x^1 + \frac{1}{2}x^2$.*

Proposition 1. *x is an extreme point of $\mathcal{P}$ if and only if x is a zero-dimensional face of $\mathcal{P}$.*

Proposition 2. *If $\mathcal{P} \neq \emptyset$ and $rank(A) = n - k$, then $\mathcal{P}$ has a face of dimension k and no proper face of lower dimension.*

- These three results together imply that $\mathcal{P}$ has an extreme point if and only if $rank(A) = n$.
- This is the case for any polytope or any polyhedron lying in the non-negative orthant.
- Recall that in 406, we showed that a polyhedron has an extreme point if and only if it does not contain a line.
- Don't confuse $rank(A) = n$ with $\mathcal{P}$ being full-dimensional!

Extreme Rays

Definition 2. *The* recession cone $rec(\mathcal{P})$ *associated with* $\mathcal{P}$ *is* $\{r \in \mathbb{R}^n | Ar \leq 0\}$. *Members of the recession cone are called* rays *of* $\mathcal{P}$.

Definition 3. r *is an* extreme ray *of* $\mathcal{P}$ *if there do not exist rays* r^1 *and* r^2 *of* $\mathcal{P}$ *such that* $r = \frac{1}{2}r^1 + \frac{1}{2}r^2$.

Proposition 3. *If* $\mathcal{P} \neq \emptyset$, *then* r *is an extreme ray of* $\mathcal{P}$ *if and only if* $\{\lambda r \mid \lambda \in \mathbb{R}_+\}$ *is a one-dimensional face of the recession cone.*

Definition 4. *The* lineality space *of* $\mathcal{P}$ *is* $\mathrm{lin}(\mathcal{P}) = \{r \in \mathbb{R}^n | Ar = 0\} = \mathrm{rec}(\mathcal{P}) \cap (-\mathrm{rec}(\mathcal{P}))$.

- We have that $\mathrm{lin}(\mathcal{P}) = \{0\}$ if and only if $rank(A) = n$ and in that case, we call the polyhedron *pointed*.
- Note that if r is an extreme ray, then so is λr for $\lambda > 0$.
- We need only consider one "representative" of each one-dimensional face of the recession cone.
- We can do this by choosing extreme rays r with $\|r\| = 1$.
- The last two results together imply that a polyhedron has a finite number of extreme points and extreme rays.

Polarity

Definition 5. *The* polar *of a set* S *is* $S^* = \{y \in \mathbb{R}^n \mid yx \leq 1 \ \forall x \in S\}$.

Theorem 3. *Given* $a^1, \ldots, a^m \in \mathcal{Q}^n$ *and* $0 \leq k \leq m$*, let*

$$\mathcal{Q}_1 = \{x \in \mathbb{R}^n \mid a^i x \leq 1, i = 1, \ldots, k; a^i x \leq 0, i = k+1, \ldots, m\}$$

$$\mathcal{Q}_2 = \mathrm{conv}(\{0, a^1, \ldots, a^k\}) + \mathrm{cone}(\{a^{k+1}, \ldots a^n\})$$

Then $\mathcal{Q}_1^* = \mathcal{Q}_2$ *and* $\mathcal{Q}_2^* = \mathcal{Q}_1$

- From this definition, we can see that if $\mathcal{Q}$ is a polyhedron containing the origin, then have that
  1. $\mathcal{Q}^*$ is also a polyhedron containing the origin;
  2. $\mathcal{Q}^{**} = \mathcal{Q}$;
  3. $\mathcal{Q}^*$ is bounded if and only if $\mathcal{Q}$ contains the origin in its interior;
  4. $\mathrm{aff}(\mathcal{Q}^*)$ is the orthogonal complement of $\mathrm{lin}(\mathcal{Q})$ and $\dim(\mathcal{Q}^*) + \dim(\mathrm{lin}(\mathcal{Q})) = n$.
- $\mathcal{Q}^*$ can be roughly interpreted as the set of all valid inequalities of $\mathcal{Q}$.
- When $\mathcal{Q}$ is full-dimensional and has 0 in its interior, then the extreme of $\mathcal{Q}^*$ are in one-to-one correspondence with the facets of $\mathcal{Q}$.

Intuition

- Theorem 3 may seem mysterious at firat, but it's actually quite intuitive.
- There are two perspectives we can take to under stand it.
- First recall that inequalities valid for for non-empty polyhedron $\mathcal{P} = \{x \in \mathbb{R}^m \mid Ax \leq b\}$ can be generated by solving the system

$$\begin{aligned} uA &= \pi \\ ub &\leq \pi_0 \\ u &\geq 0 \end{aligned}$$

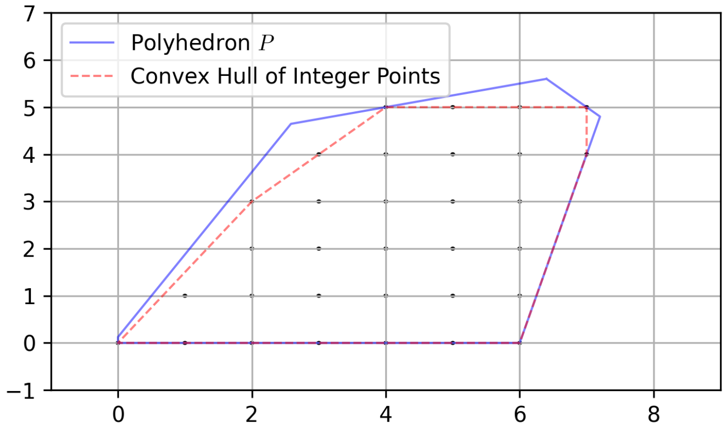
- Alternatively, the conic hull of the rows of A includes all non-dominated valid inequalities.
- Assuming A has full rank, then the rows of A are the extreme rays of this conic hull.
- To get to the polar, we normalize, which truncates the cone so that the extreme rays of the conic hull become extreme points of the polar.
- Inequalities through the origin have right-hand side zero and can't be normalized, so if the origin is not in the interior, the polar is unbounded.

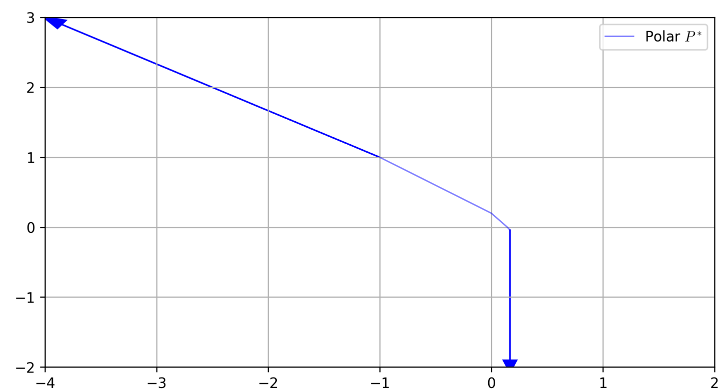
Example

Let $\mathcal{P} = \{x \in \mathbb{R}_+ \mid Ax \leq b\}$, where

$$A = \begin{bmatrix} -1 & 4 \\ -14 & 8 \\ 4 & -1 \\ 1 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 16 \\ 1 \\ 24 \\ 12 \end{bmatrix}$$

and let $\mathcal{S} = \mathcal{P} \cap \mathbb{Z}^2$. Then according to Theorem 3, the polyhedron and its polar are





Some Results from Linear Optimization

Theorem 4. *If* $\mathcal{P} \neq \emptyset$, $rank(A) = n$, *and* $\max\{cx \mid x \in \mathcal{P}\}$ *is finite, then there is an optimal solution that is an extreme point.*

Theorem 5. *For a given extreme point* x^*, *there exists a* $c \in \mathbb{Z}^n$ *such that* x^* *is the optimal solution to* $\max\{cx \mid x \in \mathcal{P}\}$

Theorem 6. *If* $\mathcal{P} \neq \emptyset$, $rank(A) = n$, *and* $\max\{cx \mid x \in \mathcal{P}\}$ *is unbounded, then there is an extreme ray* r^* *with* $cr^* > 0$.

- Note again that the set of all optimal solutions to a linear optimization problem is a face of the associated polyhedron.
- We call this the *optimal face*.
- Combining these results, we get Minkowski's Theorem.

Minkowski's Theorem

Theorem 7. *If* $\mathcal{P} \neq \emptyset$ *and* $rank(A) = n$, *then*

$$\mathcal{P} = \left\{ \sum_{k \in \mathcal{E}} \lambda_k x^k + \sum_{j \in \mathcal{R}} \mu_j r^j \;\middle|\; \lambda_k \geq 0 \text{ for } k \in \mathcal{E}, \mu_j \geq 0 \text{ for } j \in \mathcal{R}, \sum_{k \in \mathcal{E}} \lambda_i = 1 \right\},$$

where $\{x^k\}_{k \in \mathcal{E}}$ *are the extreme points and* $\{r^j\}_{j \in \mathcal{R}}$ *are the (representative) extreme rays.*

Corollary 1. *A nonempty polyhedron is bounded if and only if it has no extreme rays.*

Corollary 2. *A polytope is the convex hull of its extreme points.*

- A set of the form given above is called *finitely generated* when $\mathcal{R}$ and $\mathcal{E}$ are finite sets.
- If $\mathcal{R}$ or $\mathcal{E}$ were not finite, then the feasible region would be that of a *semi-infinite optimization problem*.
- This result is often stated as "every polyhedron is finitely generated."

More Results from Linear Optimization

Define the following:

- $\mathcal{P}^+ = \{x \in \mathbb{R}^n_+ \mid Ax \leq b\}$, $z = \max\{cx \mid x \in \mathcal{P}^+\}$
- $\mathcal{Q}^+ = \{u \in \mathbb{R}^m_+ \mid uA \geq c\}$, $w = \min\{ub \mid u \in \mathcal{Q}^+\}$
- $\{x^k\}_{k \in \mathcal{E}}$, $\{u^i\}_{i \in \mathcal{I}}$ are the extreme points of $\mathcal{P}^+$ and $\mathcal{Q}^+$ respectively.
- $\{r^j\}_{j \in \mathcal{R}}$, $\{v^t\}_{t \in \mathcal{T}}$ are the extreme rays of $\mathcal{P}^+$ and $\mathcal{Q}^+$ respectively.

Theorem 8 (Farkas)**.** $\mathcal{P}^+ \neq \emptyset \Leftrightarrow v^t b \geq 0 \; \forall t \in \mathcal{T}$

Theorem 9. *The following are equivalent when $\mathcal{P}^+ \neq \emptyset$:*

1. z is unbounded from above;

2. there exists $j \in \mathcal{R}$ with $cr^j > 0$; and

3. $\mathcal{Q}^+ = \emptyset$.

Theorem 10. *If $\mathcal{P}^+ \neq \emptyset$ and z is bounded, then*

$$z = \max_{k \in \mathcal{E}} cx^k = w = \min_{i \in \mathcal{I}} u^i b$$

The Projection of a Polyhedron

- Let $\mathcal{Q} = \{(x, y) \in \mathbb{R}^r \times \mathbb{R}^{n-r} \mid Gx + Hy \leq b\}$
- The projection of $\mathcal{P}$ into the space of just the x variables is

$$\begin{aligned} \operatorname{proj}_x(\mathcal{Q}) &= \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^p, (x, y) \in \mathcal{P}\} \\ &= \{x \in \mathbb{R}^n \mid v^t(b - Ax) \geq 0 \ \forall t \in \mathcal{T}\} \end{aligned}$$

  where $\{v^t\}_{t \in \mathcal{T}}$ are the extreme rays of $Q = \{v \in \mathbb{R}^m_+ \mid vH = 0\}$.

- This immediately implies that the projection of a polyhedron is a polyhedron.
- Note that this notion of projection does not technically coincide with the one from Lecture 4, as projection usually takes place in the same space.
- In other words, the ordinary notion would be to project $\mathcal{Q}$ into the subspace $\{(x, y) \in \mathbb{R}^r \times \mathbb{R}^{n-r} \mid y = 0\}$.
- The projection of a point (x, y) into this subspace is the point $(x, 0)$, where $x \in \operatorname{proj}_x(\mathcal{Q})$.

Weyl's Theorem

Theorem 11. *If*

$$\mathcal{Q} = \left\{ \sum_{k \in \mathcal{E}} \lambda_k x^k + \sum_{j \in \mathcal{R}} \mu_j r^j \;\middle|\; \lambda_k \geq 0 \textit{ for } k \in \mathcal{E}, \mu_j \geq 0 \textit{ for } j \in \mathcal{R}, \sum_{k \in \mathcal{E}} \lambda_i = 1 \right\},$$

where $\{x^k\}_{k \in \mathcal{E}}$ *and* $\{r^j\}_{j \in \mathcal{R}}$ *are given sets of rational vectors, then* $\mathcal{Q}$ *is a rational polyhedron.*

- This is the converse of Minkowski's Theorem.
- This says roughly "every finitely generated set is a polyhedron" (remember the rationality assumption).
- The proof is easy using projection.

The Fundamental Theorem

- We have already discussed informally the fact that an integer optimization problem can, in theory, be reduced to a linear optimization problem.
- We now make these ideas more formal.
- To do so, we would now like to show the following:

  Theorem 12. *(The Fundamental Theorem of Integer Optimization) If* $\mathcal{P} = \{x \in \mathbb{R}^n \mid Ax \leq b\}$*, where* $A \in \mathbb{Q}^{m \times n}$*,* $b \in \mathbb{Q}^m$*, and* $\mathcal{S} = \mathcal{P} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$*, then* $\text{conv}(\mathcal{S})$ *is a rational polyhedron with the same recession cone as* $\mathcal{P}$*.*

Proving $\mathcal{S}$ Is Finitely Generated

- This result is easily proven if $\mathcal{S}$ is bounded (how?).
- If $\mathcal{S}$ is not bounded, then it is not so obvious.
- Our approach will be to show that $\mathcal{S}$ itself can be finitely generated.
- It then follows that $\mathrm{conv}(\mathcal{S})$ is finitely generated.

Proving $\mathcal{S}$ Is Finitely Generated (cont.)

- Consider $\mathcal{P}$ and $\mathcal{S}$ from Theorem 12.
- By Minkowski's Theorem, we can write

$$\mathcal{P} = \left\{ \sum_{k \in \mathcal{E}} \lambda_k x^k + \sum_{j \in \mathcal{R}} \mu_j r^j \;\middle|\; \lambda_k \geq 0 \text{ for } k \in \mathcal{E}, \mu_j \geq 0 \text{ for } j \in \mathcal{R}, \sum_{k \in \mathcal{E}} \lambda_i = 1 \right\},$$

with $\{x^k\}_{k \in \mathcal{E}}$ the extreme points and $\{r^j\}_{j \in \mathcal{R}}$ the extreme rays.

- We can assume wlog that the extreme rays are integral.
- Then $\mathcal{S}$ is finitely generated by $\mathcal{Q} \cap \mathcal{S}$ and the extreme rays of $\mathcal{P}$, where

$$\mathcal{Q} = \left\{ \sum_{k \in \mathcal{E}} \lambda_k x^k + \sum_{j \in \mathcal{R}} \mu_j r^j \;\middle|\; \lambda_k \geq 0 \text{ for } k \in \mathcal{E}, 0 \leq \mu_j < 1 \text{ for } j \in \mathcal{R}, \sum_{k \in \mathcal{E}} \lambda_i = 1 \right\},$$

Example

- Let's find a finite set of generators for the set $\mathcal{S} = \mathcal{P} \cap \mathbb{Z}^2$, where

$$\mathcal{P} = \left\{x \in \mathbb{R}^2_+ \;\middle|\; 5x_1 + 3x_2 \geq 10, 5x_1 - 5x_2 \geq -1, -x_1 + 2x_2 \geq -2\right\}$$

- The generators for $\mathcal{S}$ are the set of integer points inside the set $\mathcal{Q}$ defined previously.
- Set $\mathcal{P}$ and its generator are shown in Figure 1 on the next slide.
- The set $\mathcal{Q}$ is defined as

$$\mathcal{Q} = \{\lambda_1 e_1 + \lambda_2 e_2 + \mu_1 r_1 + \mu_2 r_2 \mid \lambda_1, \lambda_2 \in \mathbb{R}_+, \lambda_1 + \lambda_2 = 1, \mu_1, \mu_2 \in [0, 1)\}$$

- The generators for $\mathcal{S}$ itself are then the points

$$\{(2,0), (2,1), (2,2), (3,1), (3,2), \text{ and } (4,1)\},$$

along with the extreme rays $(1,1)$ and $(2,1)$ of the recession cone.

- In this case, just the points $(2,0), (2,1), (2,2)$ are a minimal set of generators, since the other points above can be generated by those.

Example

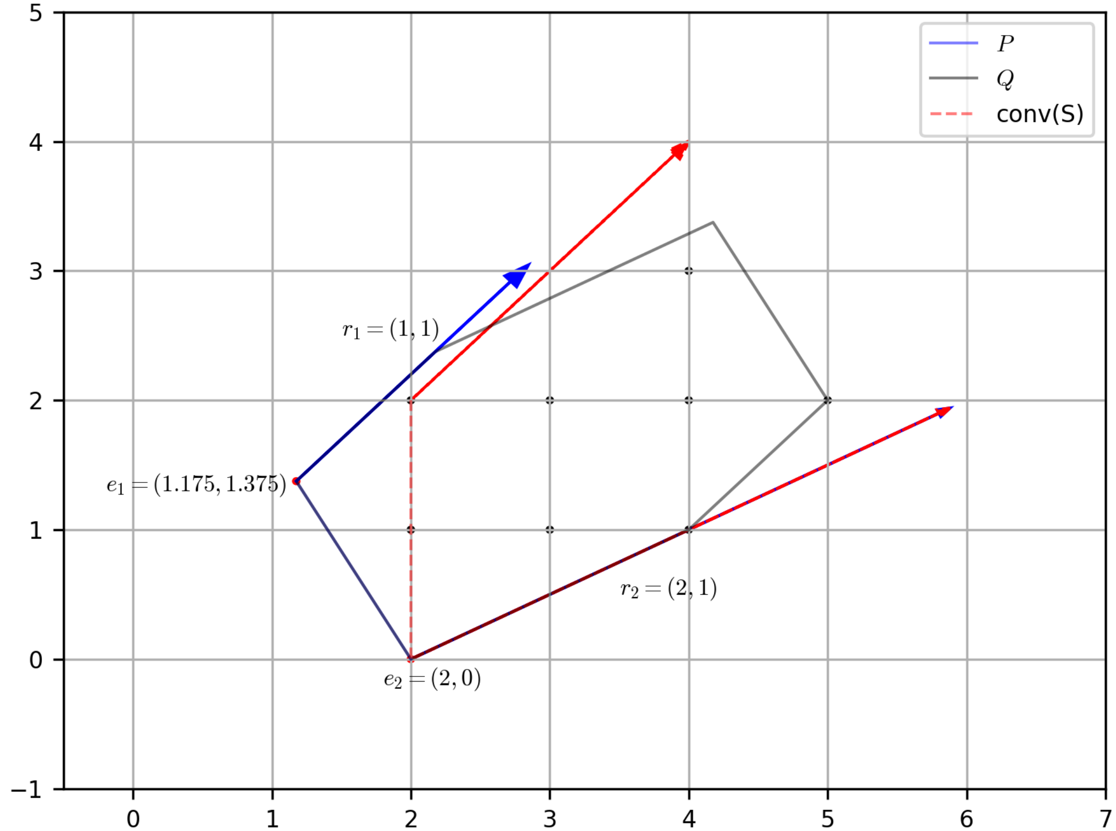


Figure 1: Generators for $\mathcal{P}$, the convex hull of $\mathcal{S}$, and $\mathcal{Q}$.

Consequences

- Once we have that $\mathcal{S}$ is finitely generated then we can easily show that $conv(S)$ is a rational polyhedron.
- Note that this result extends easily to the *mixed* case with rational data.
- Note also that if $\mathcal{P} \cap \mathcal{S} \neq \emptyset$, then the extreme rays of $\mathcal{P}$ and $conv(S)$ coincide.
- This also shows that solving the IP $\max\{cx \mid x \in S\}$ is essentially equivalent to solving the LP $\max\{cx \mid x \in conv(S)\}$.
  - The objective function of the IP is unbounded if and only if the objective function of the LP is unbounded.
  - If the LP has a bounded optimal value, then it has an optimal solution that is an optimal solution to the IP (an extreme point of $conv(S)$).
  - if $\hat{x}$ is an optimal solution to IP, then it is an optimal solution to the LP.
- We can also show that an IP is either infeasible, unbounded, or has an optimal solution.

Implicitly Described Polyhedra

- $\mathrm{conv}(S)$ is an "implicitly defined" polyhedron in the sense that we do not generally have a description of it in terms of half-spaces or generators.
- Knowing that $\mathrm{conv}(S)$ is a polyhedron does not help much in obtaining an explicit description of it.
- It will, however, help in proving convergence of solution methods and in other important ways.
- In some case, we will try to generate parts of the description of this polyhedron.
- Not all the inequalities appearing in the formulation will be facet-defining for it.
- Using the properties of polyhedra that we know, we will try to determine which inequalities from the formulation are the facet-defining ones.
- We will also try to generate new valid inequalities that are facet-defining.
- Adding these to the formulation will necessarily increase its "strength."