

Integer Programming

ISE 418

Lecture 4

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Reading for This Lecture

- N&W Sections I.4.1-I.4.3

Linear Algebra Review: Linear Independence

Definition 1. A finite collection of vectors $x^1, \dots, x^k \in \mathbb{R}^n$ is **linearly independent** if the unique solution to $\sum_{i=1}^k \lambda_i x^i = 0$ is $\lambda_i = 0, i \in [1..k]$. Otherwise, the vectors are **linearly dependent**.

Let A be a square matrix. Then, the following statements are equivalent:

- The matrix A is invertible.
- The matrix A^T is invertible.
- The determinant of A is nonzero.
- The rows of A are linearly independent.
- The columns of A are linearly independent.
- For every vector b , the system $Ax = b$ has a unique solution.
- There exists some vector b for which the system $Ax = b$ has a unique solution.

Linear Algebra Review: Affine Independence

Definition 2. A finite collection of vectors $x^1, \dots, x^k \in \mathbb{R}^n$ is **affinely independent** if the vectors $x^2 - x^1, \dots, x^k - x^1 \in \mathbb{R}^n$ are linearly independent.

- Linear independence implies affine independence, but not vice versa.
- The property of linear independence is with respect to a given origin.
- Affine independence is essentially a “coordinate-free” version of linear independence.

Proposition 1. *The following statements are equivalent:*

1. $x_1, \dots, x_k \in \mathbb{R}^n$ are affinely independent.
2. $x_2 - x_1, \dots, x_k - x_1$ are linearly independent.
3. $(x_1, 1), \dots, (x_k, 1) \in \mathbb{R}^{n+1}$ are linearly independent.

Linear Algebra Review: Subspaces

Definition 3. A nonempty subset $H \subseteq \mathbb{R}^n$ is called a **subspace** if $\alpha x + \gamma y \in H \forall x, y \in H$ and $\forall \alpha, \gamma \in \mathbb{R}$.

Definition 4. A **linear combination** of a collection of vectors $x^1, \dots, x^k \in \mathbb{R}^n$ is any vector $y \in \mathbb{R}^n$ such that $y = \sum_{i=1}^k \lambda_i x^i$ for some $\lambda \in \mathbb{R}^k$.

Definition 5. The **span** of a collection of vectors $x^1, \dots, x^k \in \mathbb{R}^n$ is the set of all linear combinations of those vectors.

Definition 6. Given a subspace $H \subseteq \mathbb{R}^n$, a collection of linearly independent vectors whose span is H is called a **basis** of H . The number of vectors in the basis is the **dimension** of the subspace.

Linear Algebra Review: Subspaces and Bases

- A given subspace has an infinite number of bases.
- Each basis has the same number of vectors in it (the dimension).
- If S and T are subspaces such that $S \subseteq T \subseteq \mathbb{R}^n$, then a basis of S can be extended to a basis of T .
- The span of the columns of a matrix A is a subspace called the *column space* or the *range*, denoted $\text{range}(A)$.
- The span of the rows of a matrix A is a subspace called the *row space*.
- The dimensions of the column space and row space are always equal. We call this number $\text{rank}(A)$.
- Clearly, $\text{rank}(A) \leq \min\{m, n\}$. If $\text{rank}(A) = \min\{m, n\}$, then A is said to have *full rank*.
- The set $\{x \in \mathbb{R}^n \mid Ax = 0\}$ is called the *nullspace* of A (denoted $\text{null}(A)$) and has dimension $n - \text{rank}(A)$.

Some Properties of Subspaces

Proposition 2. *The following are equivalent:*

1. $H \subseteq \mathbb{R}^n$ is a subspace.
2. There is an $m \times n$ matrix A such that $H = \{x \in \mathbb{R}^n \mid Ax = 0\}$.
3. There is a $k \times n$ matrix B such that $H = \{x \in \mathbb{R}^n \mid x^\top = uB, u \in \mathbb{R}^k\}$.

Proposition 3. *If $\{x \in \mathbb{R}^n \mid Ax = b\} \neq \emptyset$, the maximum number of affinely independent solutions of $Ax = b$ is $n + 1 - \text{rank}(A)$.*

Proposition 4. *If $H \subseteq \mathbb{R}^n$ is a subspace, the subspace $\{x \in \mathbb{R}^n \mid x^\top y = 0 \forall y \in H\}$ is a subspace called the **orthogonal subspace** or **orthogonal complement** and denoted H^\perp .*

Proposition 5. *If $H = \{x \in \mathbb{R}^n \mid Ax = 0\}$, with A being an $m \times n$ matrix, then $H^\perp = \{x \in \mathbb{R}^n \mid x = A^\top u, u \in \mathbb{R}^m\}$ and $\dim(H) + \dim(H^\perp) = n$.*

Affine Spaces

Definition 7. An **affine combination** of a collection of vectors $x^1, \dots, x^k \in \mathbb{R}^n$ is any vector $y \in \mathbb{R}^n$ such that $y = \sum_{i=1}^k \lambda_i x^i$ for some $\lambda \in \mathbb{R}^k$ with $\sum_{j=1}^k \lambda_j = 1$.

Definition 8. A nonempty subset $\mathcal{A} \subseteq \mathbb{R}^n$ is called an **affine space** if \mathcal{A} is closed with respect to affine combination.

Definition 9. A **basis** of an affine space $\mathcal{A} \subseteq \mathbb{R}^n$ is a maximal set of affinely independent points of \mathcal{A} .

Definition 10. The inclusionwise minimal affine space containing a set \mathcal{S} is called the **affine hull** of \mathcal{S} , denoted $\text{aff}(\mathcal{S})$.

Definition 11. All bases of an affine space \mathcal{A} have the same cardinality and this cardinality exceeds the **dimension** of the affine space by one.

Projections

Definition 12. If $p \in \mathbb{R}^n$ and H is a subspace, the projection of p onto H is the vector $q \in H$ such that $p - q \in H^\perp$.

- Note that this is a decomposition of a vector p into the sum of a vector in H and a vector in H^\perp .
- The projection of a set is the union of the projections of all its members.
- Projections play a very important role in discrete optimization, as we will see later in the course.

Polyhedra, Hyperplanes, and Half-spaces

Definition 13. A **polyhedron** is a set of the form $\{x \in \mathbb{R}^n \mid Ax \leq b\}$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

Definition 14. A polyhedron $\mathcal{P} \subset \mathbb{R}^n$ is **bounded** if there exists a constant K such that $|x_i| < K \forall x \in \mathcal{P}, \forall i \in [1, n]$.

Definition 15. A bounded polyhedron is called a **polytope**.

Definition 16. Let $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ be given.

- The set $\{x \in \mathbb{R}^n \mid a^\top x = b\}$ is called a **hyperplane**.
- The set $\{x \in \mathbb{R}^n \mid a^\top x \leq b\}$ is called a **half-space**.

Convex Sets

Definition 17. A set $S \subseteq \mathbb{R}^n$ is **convex** if $\forall x, y \in S, \lambda \in [0, 1]$, we have $\lambda x + (1 - \lambda)y \in S$.

Definition 18. Let $x^1, \dots, x^k \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}_+^k$ be given such that $\lambda^\top \mathbf{1} = 1$. Then

1. The vector $\sum_{i=1}^k \lambda_i x^i$ is said to be a **convex combination** of x^1, \dots, x^k .
2. The **convex hull** of x^1, \dots, x^k is the set of all convex combinations of these vectors.

Definition 19. Let $r^1, \dots, r^j \in \mathbb{R}^n$ and $\mu \in \mathbb{R}_+^j$ be given. Then

1. The vector $\sum_{i=1}^j \mu_i r^i$ is said to be a **conic combination** of r^1, \dots, r^j .
2. The **conic hull** of r^1, \dots, r^j is the set of all conic combinations of these vectors.

- The convex hull of two points is a line segment.
- A set is convex if and only if for any two points in the set, the line segment joining those two points lies entirely in the set.
- All polyhedra are convex.