

Integer Programming

ISE 418

Lecture 15

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Reading for This Lecture

- Nemhauser and Wolsey Sections II.1.1-II.1.3, II.1.6
- Wolsey Chapter 8
- CCZ Chapters 5 and 6
- “Valid Inequalities for Mixed Integer Linear Programs,” G. Cornuejols.
- “Corner Polyhedra and Intersection Cuts,” M. Conforti, G. Cornuejols, and G. Zambelli.
- “Generating Disjunctive Cuts for Mixed Integer Programs,” M. Perregaard.
- “On Optimizing Over Lift-And-Project Closures,” P. Bonami.
- “A Disjunctive Cutting Plane Procedure for General Mixed-integer Linear Programs,” J. Owens and S. Mehrotra.

Split Inequalities

- Let (α, β) be a split disjunction and define

$$\mathcal{P}_1 = \mathcal{P} \cap \{x \in \mathbb{R}^n \mid \alpha^\top x \leq \beta\}$$

$$\mathcal{P}_2 = \mathcal{P} \cap \{x \in \mathbb{R}^n \mid \alpha^\top x \geq \beta + 1\}$$

- Any inequality valid for $\text{conv}(\mathcal{P}_1 \cup \mathcal{P}_2)$ is valid for \mathcal{S} and is called a *split inequality*.
- Chvátal inequalities are exactly the split inequalities for which $\mathcal{P}_2 = \emptyset$.

Non-dominated Split Inequalities

- With respect to a given split disjunction (α, β) , all non-dominated split inequalities are of the form

$$\frac{u^+(b - Ax)}{ub - \beta} + \frac{u^-(b - Ax)}{\beta + 1 - ub} \geq 1$$

for some $u \in \mathbb{R}^m$ such that u is a solution to $uA = \alpha$ such that

- the rows of A corresponding to the non-zero values of u are linearly independent and
- $\beta < ub < \beta + 1$.

- Therefore, all non-dominated split inequalities are of the form

$$\frac{u^+(b - Ax)}{ub - \lfloor ub \rfloor} + \frac{u^-(b - Ax)}{\lceil ub \rceil - ub} \geq 1 \quad (\text{SPLIT})$$

for some $u \in \mathbb{R}^m$ such that $uA_I \in \mathbb{Z}$ and $uA_C = 0$ and the rows of A corresponding to the non-zero values of u are linearly independent.

GMI and MIR Cuts are Split Inequalities

- Consider a polyhedron $Q = \{Ax = b, x \geq 0\}$ in standard form and take $u \in \mathbb{R}^m$, $f_j = (uA)_j - \lfloor (uA)_j \rfloor$, $f_0 = ub - \lfloor ub \rfloor$.
- Then (GMIC) is a split inequality relative to the split disjunction defined by

$$\alpha_j = \begin{cases} \lfloor (uA)_j \rfloor & \text{if } 0 \leq i \leq p \text{ and } f_j \leq f_0 \\ \lceil (uA)_j \rceil & \text{if } 0 \leq i \leq p \text{ and } f_j > f_0 \\ 0 & \text{otherwise} \end{cases}$$

and $\beta = \lfloor ub \rfloor$.

- Alternatively, an inequality of the form (SPLIT) with respect to the polyhedron

$$\{x \in \mathbb{R}^m \mid Ax \leq b, -Ax \leq -b, -x \leq 0\}$$

is also equivalent to (GMIC).

- Our proof of validity for MIR cuts showed directly that they are split inequalities.

Split Inequalities are GMI and MIR Cuts

- Let $u \in \mathbb{R}^m$ be such that $uA_I \in \mathbb{Z}$ and $uA_C = 0$.
- The inequality (u^+A, u^+b) is valid for \mathcal{P} and thus

$$uAx - u^-(b - Ax) \leq ub \quad \forall x \in \mathcal{P}$$

- Also, since $u^- \in \mathbb{R}_+^m$, we have that (u^-A, u^-b) is valid for \mathcal{P} and hence $u^-(b - Ax) \geq 0 \quad \forall x \in \mathcal{P}$.
- Then, we can apply the same aggregation procedure we used to derive (MIR).
- This results in the inequality

$$uAx - \frac{u^-}{1-f}(b - Ax) \leq [ub]$$

which is equivalent to (SPLIT).

- Thus, the MIR, GMI, and split closures are all identical.

Lift and Project

- In lift and project, we formulate the separation problem with respect to a given point and variable disjunction.
- We assume the point to be separated is a basic solution x^B to the LP relaxation with respect to basis B .
- We have that $\text{conv}(\mathcal{S}) \subseteq \text{conv}(\mathcal{P}_{0j}^B \cup \mathcal{P}_{1j}^B)$ where $\mathcal{P}_{0j}^B = \mathcal{P} \cap \{x \in \mathbb{R}^n \mid x_j \leq \lfloor x_j^B \rfloor\}$ and $\mathcal{P}_{1j}^B = \mathcal{P} \cap \{x \in \mathbb{R}^n \mid x_j \geq \lceil x_j^B \rceil\}$.
- Applying Proposition 1 of the previous lecture, we see that the inequality (π, π_0) is valid for $\mathcal{P}_j^B = \text{conv}(\mathcal{P}_{0j}^B \cup \mathcal{P}_{1j}^B)$ if there exists $u, v \in \mathbb{R}_+^m$, and $u_0, v_0 \in \mathbb{R}_+$ for $i = 0, 1$ such that

$$\pi \leq uA + u_0 e_j,$$

$$\pi \leq vA - v_0 e_j,$$

$$\pi_0 \geq ub + u_0 \lfloor x_j^B \rfloor,$$

$$\pi_0 \geq vb - v_0 \lceil x_j^B \rceil,$$

- Notice that this is a set of linear constraints, i.e., we could write an LP to generate constraints based on this disjunction.

The Cut Generating LP

- This leads to the cut generating LP (CGLP_j^B), which generates the most violated inequality valid for \mathcal{P}_j^B .

$$\begin{array}{ll}
 \max & \pi x^* - \pi_0 \\
 \text{s.t.} & \pi \leq uA + u_0 e_j, \\
 & \pi \leq vA - v_0 e_j, \\
 & \pi_0 \geq ub + u_0 \lfloor x_j^* \rfloor, \quad (\text{CGLP}_j^B) \\
 & \pi_0 \geq vb - v_0 \lceil x_j^* \rceil, \\
 & \sum_{i=1}^m u_i + u_0 + \sum_{i=1}^m v_i + v_0 = 1 \\
 & u, u_0, v, v_0 \geq 0
 \end{array}$$

- The last constraint is for normalization.
- There are a number of alternatives for normalization and the choice does have an impact (see Perregaard).
- This shows that the separation problem for \mathcal{P}_j^B is polynomially solvable.

Separation Problem for Split Inequalities

- The LP (CGLP_j^B) can be generalized straightforwardly to produce the most violated split cut.

$$\begin{array}{ll}
 \max & \pi x^B - \pi_0 \\
 \text{s.t.} & \pi \leq uA + u_0\alpha, \\
 & \pi \leq vA - v_0\alpha, \\
 & \pi_0 \geq ub + u_0\beta, \\
 & \pi_0 \geq vb - v_0(\beta + 1), \quad (\text{SCGLP}_{(\alpha, \beta)}^B) \\
 & \sum_{i=1}^m u_i + u_0 + \sum_{i=1}^m v_i + v_0 = 1 \\
 & u, u_0, v, v_0 \geq 0 \\
 & \alpha \in \mathbb{Z}^n \\
 & \beta \in \mathbb{Z}
 \end{array}$$

- The separation problem is a mixed integer nonlinear optimization problem, however, and is not easy to solve (unless (α, β) is given).

Strengthening Lift-and-Project Cuts

- Note that (CGLP_j^B) only explicitly accounts for the integrality of a single variable.
- We can strengthen the generated cuts using the integrality of the other variables (we consider the pure binary case, but this can be generalized).
- To do this, we simply replace the original coefficients

$$\pi_k = \min\{uA_k, vA_k\} \text{ for } k \neq j$$

$$\pi_j = \min\{uA_j + u_0, vA_j - v_0\}$$

for the integer variables indexed $1 \leq k \leq p$ with

$$\pi_k = \max\{uA_k + u_0 \lfloor m_k \rfloor, vA_k - v_0 \lceil m_k \rceil\},$$

where

$$m_i = \frac{vA_i - uA_i}{u_0 + v_0}$$

- The proof is to fix the values of u, v, u_0, v_0 obtained by solving (CGLP_j^B) and then find an optimal (α, β) in $(\text{SCGLP}_{(\alpha, \beta)}^B)$.

GMI Cuts vs. Lift-and-Project Cuts

- There is a correspondence between GMI cuts generated from basic solutions of the LP relaxation and strengthened lift-and-project cuts.
 - We use the normalization $\pi_0 \in \{-1, 0, 1\}$ in (CGLP_j^B) .
 - Then each of the former can be derived as the latter from some basic solution to (CGLP_j^B) (and vice versa, though the relationship is not one-to-one).
- We may be able to get stronger GMI cuts from tableaus other than the one that is optimal to the current LP relaxation.
 - There are lift-and-project cuts that can only be obtained as GMI cuts from an *infeasible tableau*.
 - We may also be able to get stronger cuts from a basic solution that is suboptimal for the LP relaxation.
- By pivoting in the LP relaxation, we can implicitly solve the cut generating LP (see Balas and Perregaard).

Lift-and-Project Cut as GMI from Infeasible Basis

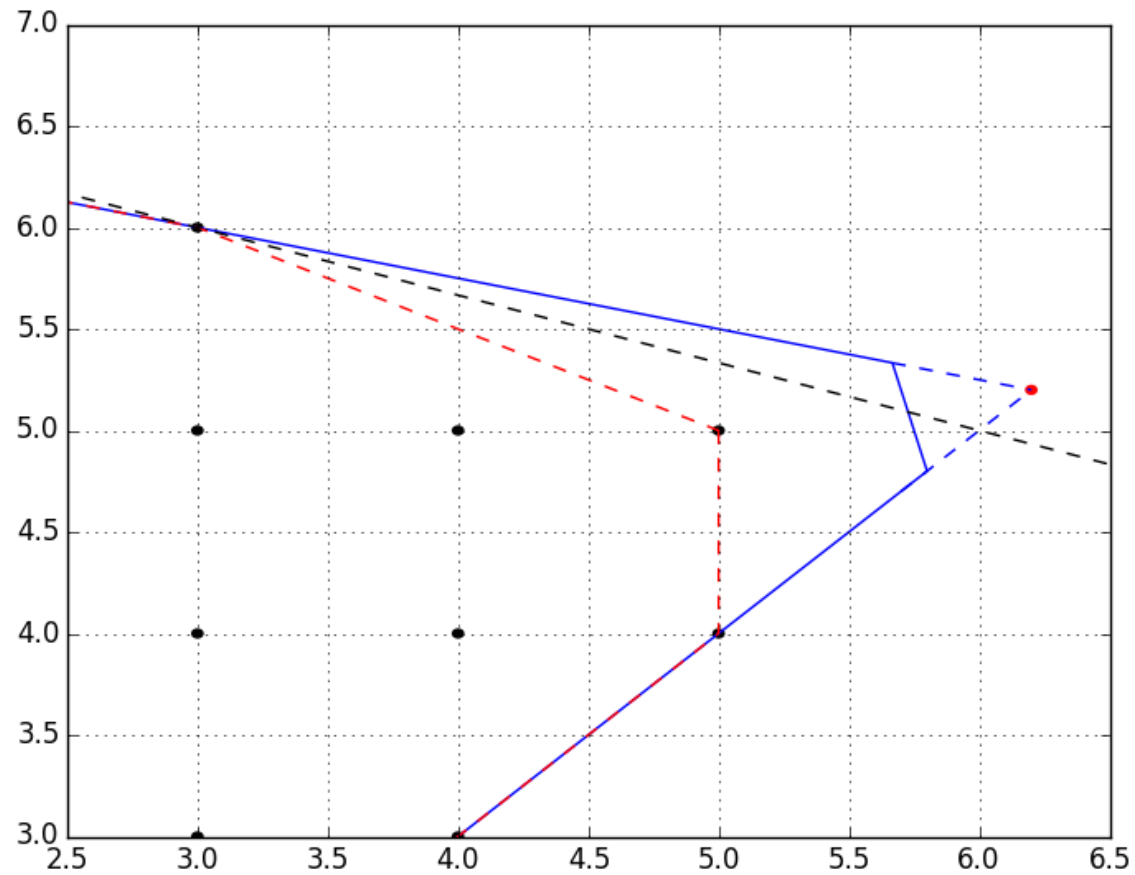


Figure 1: A GMI cut arising from an infeasible basis

Lift-and-Project Cut as GMI from Alternative Basis

- In our earlier example, the inequality $x_1 \leq 5$ dominates $3x_1 + 2x_2 \leq 26$, but the latter was generated from the current basis.
- With respect to the basic solution $(5.8, 4.8)$, we obtain the cut $x_1 \leq 5$ as a GMI cut.

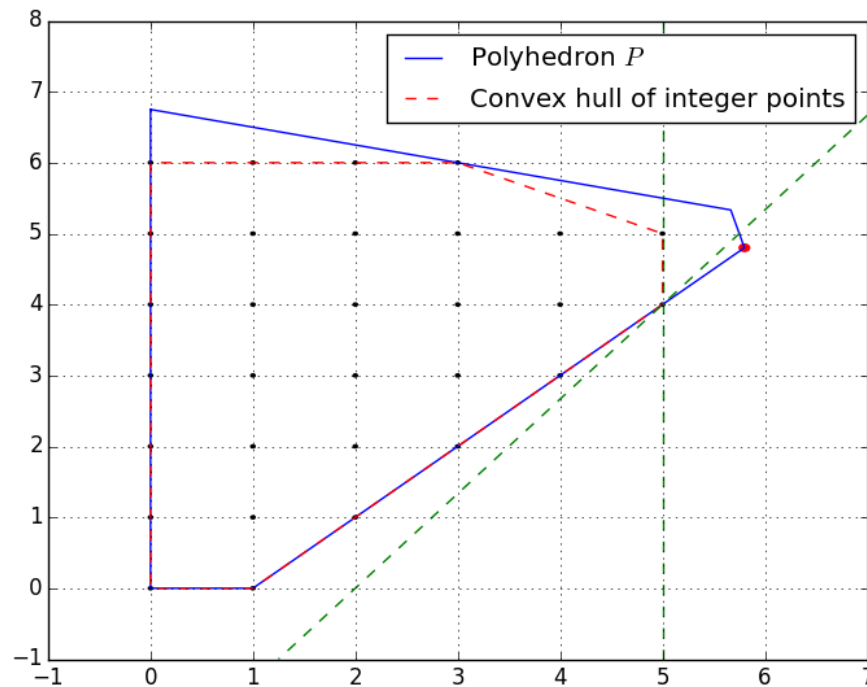


Figure 2: GMI cut arising from an alternative basis

The Lift-and-Project Closure

- The lift-and-project closure is comprised of all inequalities obtained by solving (CGLP $_j^B$) for each basis B of the LP relaxation and each variable j for which $x_j^B \notin \mathbb{Z}$.
- This is equivalent to generating all GMICs corresponding to all rows of all tableaux (feasible or infeasible) of the LP relaxation in which an integer variable is basic with fractional value.
- Note that in the binary case, there is only one possible variable disjunction per variable, so we need only one CGLP per variable.
- Thus, optimization over this closure can be accomplished in polynomial time in the binary case.
- Let \mathcal{P}^k be the lift-and-project closure of \mathcal{P}^{k-1} for $k > 1$.
- The lift-and-project rank of \mathcal{P} is the smallest number k such that $\mathcal{P}^k = \text{conv}(\mathcal{S})$.
- Surprisingly, the lift-and-project rank is bounded by n in the binary and mixed binary case.

Example: Lift and Project Closure

We consider the polyhedron \mathcal{P} in two dimensions defined by the constraints

$$-8x_1 + 30x_2 \leq 115$$

$$-2x_1 - 4x_2 \leq -5$$

$$-14x_1 + 8x_2 \leq 1$$

$$2x_1 - 36x_2 \leq -5$$

$$30x_1 - 8x_2 \leq 191$$

$$10x_1 + 10x_2 \leq 127$$

Lift-and-Project Closure for Example

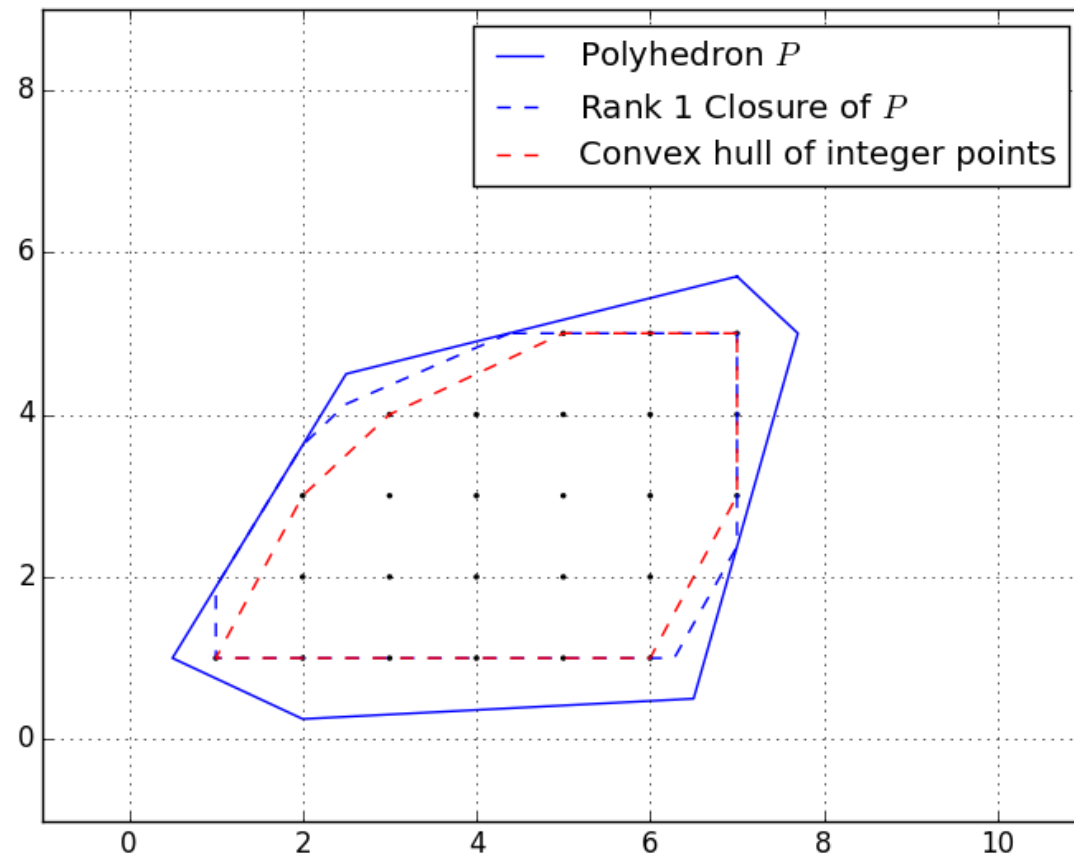


Figure 3: Lift-and-project closure for example

The GMI closure

- A GMI cut with respect to a polyhedron \mathcal{P} is any cut that can be derived using the GMI procedure starting from any inequality valid for \mathcal{P} .
- The GMI closure is obtained by adding all GMI cuts to the description of \mathcal{P} .
- The GMI closure is a polyhedron, but in contrast to the lift-and-project closure, optimizing over it is difficult (\mathcal{NP} -hard).
 - This seems like a paradox, since we have shown that most-violated GMI cuts are easy to generate.
 - This is only the case, however, for basic solutions to the LP relaxation—separating arbitrary points is difficult in general.
- The *GMI rank* of both valid inequalities and polyhedra can be defined in a fashion similar to that of the C-G rank (more on this later).

The GMI Closure and the Split Closure

- The *split closure* is the set of points satisfying all possible split cuts and is a polyhedron.
- Every split cut is also a GMI cut and vice versa.
- The split closure and the GMI closure are therefore *identical*.
- However, the GMI cut equivalent to a given split cut is not necessarily one that can be derived from a basic solution to the LP relaxation.
- We can define the *split rank* of an inequality and of a polyhedron as before.
- In the pure integer case, the split rank (and GMI rank) of \mathcal{P} is finite, but it may not be in the mixed case.
- In the mixed binary case, the split rank is bounded by n .

Aside: Selection Criteria

- The criteria by which we select cuts has a big impact on the overall effectiveness.
- We will see later that we in fact need two different kinds of selection criteria: one for generating cuts and one for choosing which cuts to add.
- We typically use bound improvement as a rough criteria when selecting disjunctions for branching, but we often use degree of violation with cuts.
- Why the difference?
- One simple answer is that degree of violation is a linear objective with respect to the cut generating LP.
- Generating cuts according to other criteria seems to be more difficult.
- See
<http://coral.ie.lehigh.edu/~ted/files/talks/DisjunctionINFORMS12.pdf>
<http://coral.ie.lehigh.edu/~jeff/mip-2006/posters/Fukasawa.pdf>