Integer Programming
ISE 418

Lecture 13

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Reading for This Lecture

- Nemhauser and Wolsey Sections II.1.1-II.1.3, II.1.6
- Wolsey Chapter 8
- CCZ Chapters 5 and 6
- “Valid Inequalities for Mixed Integer Linear Programs,” G. Cornuejols.
- “Generating Disjunctive Cuts for Mixed Integer Programs,” M. Perregaard.

Generating Cutting Planes: Two Basic Viewpoints

- There are a number of different points of view from which one can derive the standard methods used to generate cutting planes for general MILPs.
- As we have seen before, there is an *algebraic* point of view and a *geometric* point of view.
- Algebraic:
  - Take combinations of the known valid inequalities.
  - Use rounding to produce stronger ones.
- Geometric:
  - Use a disjunction (as in branching) to generate several disjoint polyhedra whose union contains $\mathcal{S}$.
  - Generate inequalities valid for the convex hull of this union.
- Although these seem like very different approaches, they turn out to be very closely related.

Generating Valid Inequalities: Algebraic Viewpoint

- Recall that valid inequalities for $\mathcal{P}$ can be obtained by taking non-negative linear combinations of the rows of (A, b).
- Except for one pathological case[1], all valid inequalities for $\mathcal{P}$ are either equivalent to or dominated by an inequality of the form

$$uAx \leq ub, u \in \mathbb{R}^m_+.$$

- We are taking combinations of inequalities existing in the description, so any such inequalities will be redundant for $\mathcal{P}$ itself.
- Nevertheless, such redundant inequalities can be strengthened by a simple procedure that ensures they remain valid for $\text{conv}(\mathcal{S})$.

[1] the pathological case is when both the primal and dual problems are infeasible.

Generating Valid Inequalities for $\text{conv}(\mathcal{S})$

As usual, we consider the MILP

$$z_{IP} = \max\{c^\top x \mid x \in \mathcal{S}\}, \tag{MILP}$$

where

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid Ax \leq b\} \tag{FEAS-LP}$$

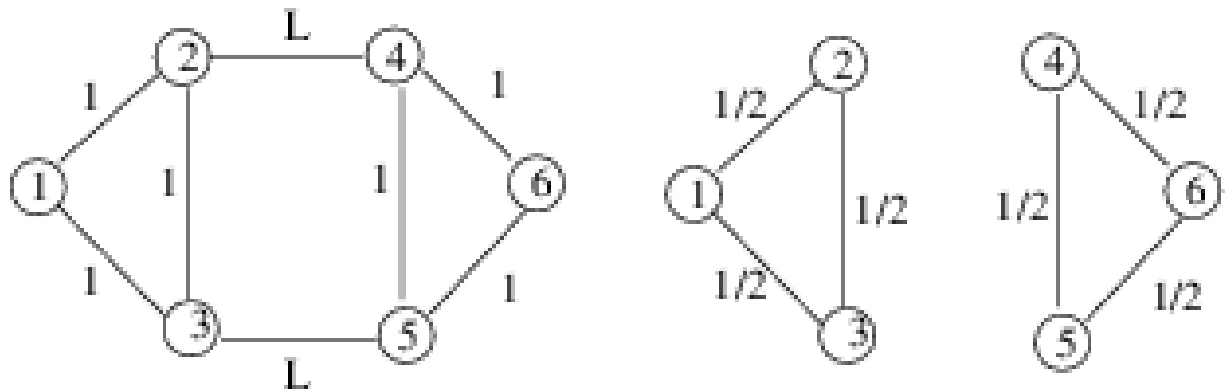
$$\mathcal{S} = \mathcal{P} \cap (\mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+) \tag{FEAS-MIP}$$

- All inequalities valid for $\mathcal{P}$ are also valid for $\text{conv}(\mathcal{S})$, but they are not cutting planes.
- We can do better.
- We need the following simple principle: if $a \leq b$ and a is an integer, then $a \leq \lfloor b \rfloor$.
- Believe it or not, this simple fact is all we need to generate all valid inequalities for $\text{conv}(\mathcal{S})$!

Simple Example

- Suppose $4x_1 + 2x_2 \leq 3$ is an inequality in the formulation $\mathcal{P}$ for a given MILP.
- Dividing through by 2, we get that $2x_1 + x_2 \leq 3/2$ is also valid for $\mathcal{P}$.
- Using the rounding principle, we can easily derive that $2x_1 + x_2 \leq 1$ is valid for $\text{conv}(\mathcal{S})$.

Back to the Matching Problem



Recall again the matching problem.

$$\begin{aligned} \min \quad & \sum_{e=\{i,j\}\in E} c_e x_e \\ s.t. \quad & \sum_{\{j \mid \{i,j\}\in E\}} x_{ij} = 1, \ \forall i \in N \\ & x_e \in \{0,1\}, \qquad \forall e = \{i,j\} \in E. \end{aligned}$$

Generating the Odd Cut Inequalities

- Recall that each odd cutset induces a possible valid inequality.

$$\sum_{e \in \delta(S)} x_e \geq 1, S \subset N, |S| \text{ odd.}$$

- Let's derive these another way.
  - Consider an odd set of nodes U.
  - Sum the (relaxed) constraints $\sum_{\{j | \{i,j\} \in E\}} x_{ij} \leq 1$ for $i \in U$.
  - This results in the inequality $2 \sum_{e \in E(U)} x_e + \sum_{e \in \delta(U)} x_e \leq |U|$.
  - Dividing through by 2, we obtain $\sum_{e \in E(U)} x_e + \frac{1}{2} \sum_{e \in \delta(u)} x_e \leq \frac{1}{2}|U|$.
  - We can drop the second term of the sum to obtain

$$\sum_{e \in E(U)} x_e \leq \frac{1}{2}|U|.$$

  - What's the last step?

Chvátal Inequalities

- Suppose we can find a $u \in \mathbb{R}^m_+$ such that $\pi = uA$ is *integer* ($uA_I \in \mathbb{Z}^p$, $uA_C = 0$) and $\pi_0 = ub \notin \mathbb{Z}$.
- In this case, we have $\pi^\top x \in \mathbb{Z}$ for all $x \in \mathcal{S}$, and so $\pi^\top x \leq \lfloor \pi_0 \rfloor$ for all $x \in \mathcal{S}$.
- In other words, $(\pi, \lfloor \pi_0 \rfloor)$ is both a valid inequality *and* a split disjunction for which
$$\{x \in \mathcal{P} \mid \pi^\top x \geq \lfloor \pi_0 \rfloor + 1\} = \emptyset \quad (1)$$
- Such an inequality is called a *Chvátal inequality*.
- Note that we have not used the non-negativity constraints in deriving this inequality.

Chvátal-Gomory Inequalities

- Now let's assume that $\mathcal{P} \subseteq \mathbb{R}^n_+$ and let $u \in \mathbb{R}^n_+$ be such that $uA_C \geq 0$.
- First, observe that (uA, ub) is valid for $\mathcal{P}$.
- Since the variables are non-negative, we have that $uA_C x_C \geq 0$, so

$$\sum_{i=1}^{p} (uA_i)x_i \leq ub \; \forall x \in \mathcal{P}$$

- Again because the variables are non-negative, we have that

$$\sum_{i=1}^{p} \lfloor uA_i \rfloor x_i \leq ub \; \forall x \in \mathcal{P}$$

- Finally, we have that

$$\sum_{i=1}^{p} \lfloor uA_i \rfloor x_i \leq \lfloor ub \rfloor \; \forall x \in \mathcal{S},$$

which is a Chvátal inequality known as a *Chvátal-Gomory Inequality*.

Chvátal-Gomory Inequalities

- Chvátal-Gomory (C-G) inequalities can also be derived in another way.
- We explicitly add the non-negativity constraints to the formulation along with the other linear constraints with associated multipliers $v \in \mathbb{R}^n_+$.
- We cannot round the coefficients to make them integral, so we require π integral.

$$\begin{aligned} \pi_i = uA_i - v_i &\in \mathbb{Z} \text{ for } 1 \leq i \leq p \\ \pi_i = uA_i - v_i &= 0 \text{ for } p+1 \leq i \leq n. \end{aligned}$$

- v_i will be non-negative as as long as we have

$$\begin{aligned} v_i &\geq uA_i - \lfloor uA_i \rfloor && \text{for } 1 \leq i \leq p \\ v_i &= uA_i \geq 0 && \text{for } p+1 \leq i \leq n \end{aligned}$$

- Taking $v_i = uA_i - \lfloor uA_i \rfloor$ for $1 \leq i \leq p$, we then obtain that

$$\sum_{i=1}^{p} \pi_i x_i = \sum_{i=1}^{p} \lfloor uA_i \rfloor x_i \leq \lfloor ub \rfloor = \pi_0 \tag{C-G}$$

is a C-G inequality for all $u \in \mathbb{R}^m_+$ such that $uA_C \geq 0$.

The Chvátal-Gomory Procedure

1. Choose a weight vector $u \in \mathbb{R}^m_+$ such that $uA_C \geq 0$.
2. Obtain the valid inequality $\sum_{i=1}^{p} (uA_i)x_i \leq ub$.
3. Round the coefficients down to obtain $\sum_{i=1}^{p} (\lfloor uA_i \rfloor)x_i \leq ub$.
4. Finally, round the right-hand side down to obtain the valid inequality

$$\sum_{i=1}^{p} (\lfloor uA_i \rfloor)x_i \leq \lfloor ub \rfloor$$

- This procedure is called the *Chvátal-Gomory* rounding procedure, or simply the *C-G procedure*.
- Surprisingly, for pure ILPs ($p = n$), any inequality valid for $\text{conv}(\mathcal{S})$ can be produced by a finite number of applications of this procedure!
- Note that this procedure is recursive and requires exploiting inequalities derived in previous rounds to get new inequalities.

Assessing the Procedure

- Although it is *theoretically* possible to generate any valid inequality using the C-G procedure, this is not true in practice.
- The two biggest challenges are numerical errors and slow convergence.
- The slow convergence is because the inequalities produced are not very strong in general.
- Typically, we do not even obtain an inequality supporting $\mathrm{conv}(\mathcal{S})$.
- This is is because the rounding only "pushes" the inequality until it meets some point in $\mathbb{Z}^n$, which may or may not even be in $\mathcal{S}$.
- We cannot do better than this without taking additional structural information into account.
- We have to be careful to ensure the generated hyperplane even includes an integer point!
- We illustrate with an example next.

Example: C-G Cuts

Consider the polyhedron $\mathcal{P}$ described by the constraints

$$4x_1 + x_2 \leq 28 \quad (2)$$

$$x_1 + 4x_2 \leq 27 \quad (3)$$

$$x_1 - x_2 \leq 1 \quad (4)$$

$$x_1, x_2 \geq 0 \quad (5)$$

Graphically, it can be easily determined that the facet-inducing valid inequalities describing $\mathrm{conv}(\mathcal{S}) = \mathrm{conv}(\mathcal{P} \cap \mathbb{Z}^2)$ are

$$x_1 + 2x_2 \leq 15 \quad (6)$$

$$x_1 - x_2 \leq 1 \quad (7)$$

$$x_1 \leq 5 \quad (8)$$

$$x_2 \leq 6 \quad (9)$$

$$x_1 \geq 0 \quad (10)$$

$$x_2 \geq 0 \quad (11)$$

Example: C-G Cuts (cont.)

Consider the LP relaxation of the ILP

$$\max\{2x_1 + 5x_2 \mid x \in \mathcal{S}\},$$

with optimal basic feasible solution indicated below.

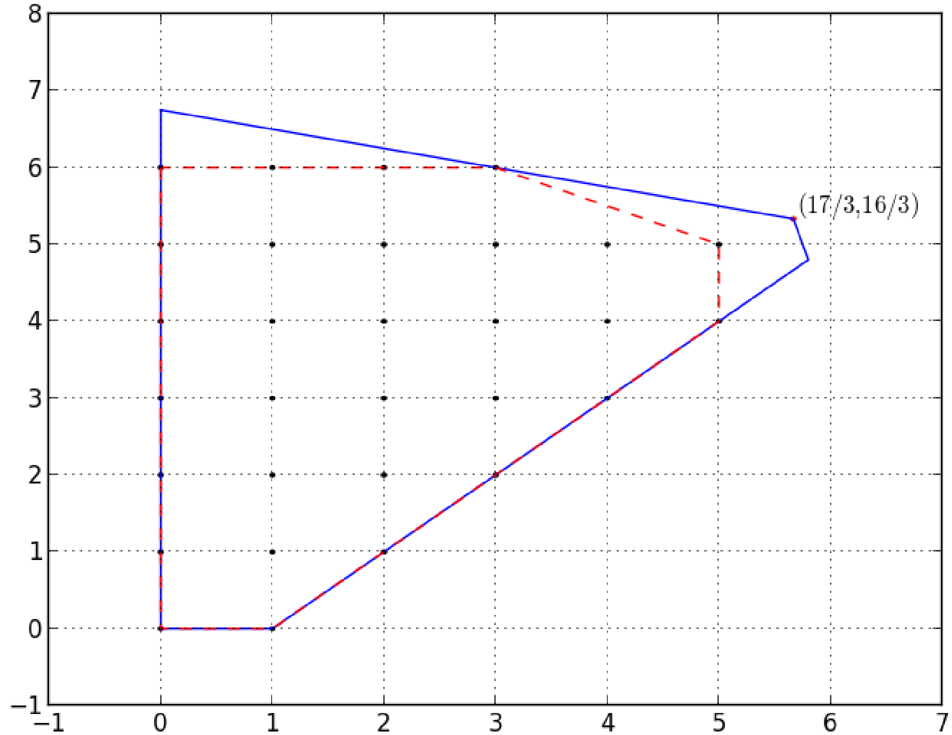
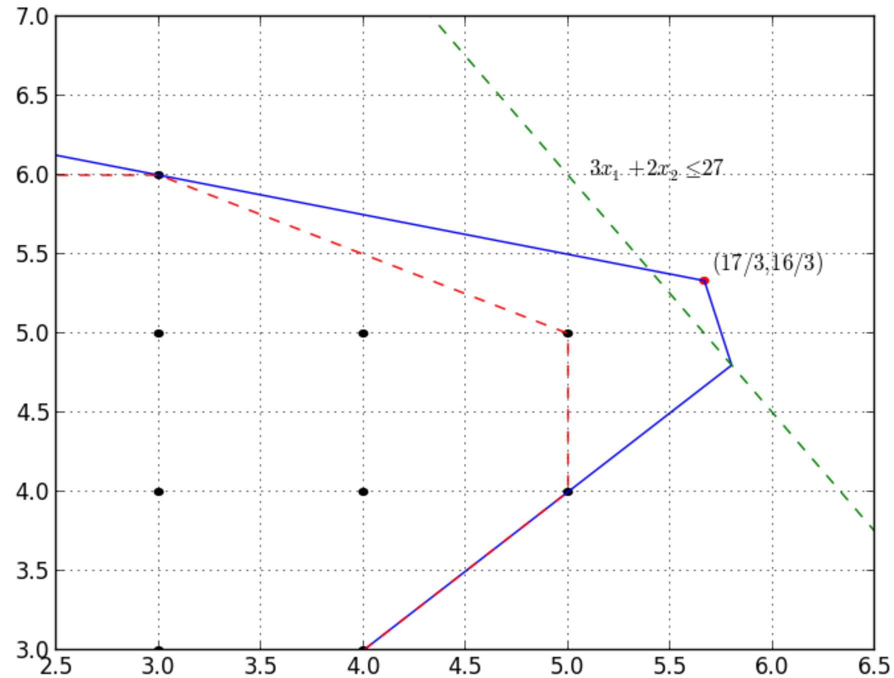


Figure 1: Convex hull of $\mathcal{S}$

Example: C-G Cuts (cont.)

- Suppose we combine the inequalities from the formulation that are binding at optimality with weights $2/3$ and $1/3$.
- We get the inequality
$$3x_1 + 2x_2 \leq 83/3.$$
- Rounding, we obtain
$$3x_1 + 2x_2 \leq 27, \qquad \text{(C-G)}$$



Gomory Inequalities

- For the derivation of Gomory inequalities, we consider pure integer programs for simplicity (we'll address the general case next lecture).
- Let's consider T, the set of solutions to a pure ILP with one equation:

$$T = \left\{ x \in \mathbb{Z}_+^n \;\middle|\; \sum_{j=1}^{n} a_j x_j = a_0 \right\}$$

- For each j, let $f_j = a_j - \lfloor a_j \rfloor$ and let $f_0 = a_0 - \lfloor a_0 \rfloor$. Then equivalently

$$T = \left\{ x \in \mathbb{Z}_+^n \;\middle|\; \sum_{j=1}^{n} f_j x_j = f_0 + \lfloor a_0 \rfloor - \sum_{j=1}^{n} \lfloor a_j \rfloor x_j \right\}$$

- Since $\sum_{j=1}^{n} f_j x_j \geq 0$ and $f_0 < 1$, then $\lfloor a_0 \rfloor \geq \sum_{j=1}^{n} \lfloor a_j \rfloor x_j$ so

$$\sum_{j=1}^{n} f_j x_j \geq f_0$$

  is a valid inequality for $\mathcal{S}$ called a *Gomory inequality*.

Gomory Cuts from the Tableau

- Gomory cutting planes can also be derived directly from the tableau while solving an LP relaxation.
- We assume for now that A and b are integral so that the slack variables also have integer values implicitly (this is wlog if $\mathcal{P}$ is rational).
- Consider the set

$$\left\{(x, s) \in \mathbb{Z}_+^{n+m} \mid Ax + Is = b\right\}$$

in which the LP relaxation of an ILP is put in standard form.

- The tableau corresponding to basis matrix B is

$$B^{-1}Ax + B^{-1}s = B^{-1}b$$

- Each row of this tableau corresponds to a weighted combination of the original constraints.
- The weight vectors are the rows of B^{-1}.

Gomory Cuts from the Tableau (cont.)

- The k^{th} row of the tableau is obtained by combining the equations in the standard form with weight vector $\lambda = B_k^{-1}$ to obtain

$$\lambda Ax + \lambda s = \lambda b,$$

where A_j is the j^{th} column of A and λ is the k^{th} row of B^{-1}.

- Applying the previous procedure, we can obtain the valid inequality

$$(\lambda A - \lfloor \lambda A \rfloor)x + (\lambda - \lfloor \lambda \rfloor)s \geq \lambda b - \lfloor \lambda b \rfloor.$$

- We then typically substitute out the slack variables by using the equation $s = b - Ax$ to obtain this cut in the original space.

$$(\lfloor \lambda A \rfloor - \lfloor \lambda \rfloor A)\, x \leq \lfloor \lambda b \rfloor - \lfloor \lambda \rfloor b. \qquad \text{(GF)}$$

Gomory Versus C-G

- The Gomory cut (GF) is equivalent to the C-G inequality with weights $u_i = \lambda_i - \lfloor \lambda_i \rfloor$, as we show next.
- To see this, let $u_i = \lambda_i - \lfloor \lambda_i \rfloor$, so that

$$uAx = \lambda Ax - \lfloor \lambda \rfloor Ax \leq \lambda b - \lfloor \lambda \rfloor b = ub.$$

- Since A and b are integral by assumption, rounding then yields

$$(\lfloor \lambda A \rfloor - \lfloor \lambda \rfloor A)\, x \leq \lfloor \lambda b \rfloor - \lfloor \lambda \rfloor b,$$

which is exactly the inequality (GF).

Strength of Gomory Cuts from the Tableau

- Consider a row of the tableau in which the value of the basic variable is not an integer.
- Applying the procedure from the last slide, the resulting inequality will only involve nonbasic variables and will be of the form

$$\sum_{j \in NB} f_j x_j \geq f_0$$

where $0 \leq f_j < 1$ and $0 < f_0 < 1$.

- The left-hand side of this cut has value zero with respect to the solution to the current LP relaxation.
- We can conclude that the generated inequality will be violated by the current solution to the LP relaxation.

Example: Gomory Cuts

Consider the optimal tableau of the LP relaxation of the ILP

$$\max\{2x_1 + 5x_2 \mid x \in \mathbb{Z}^2 \text{ satisfying (2)–(5)}\},$$

shown in Table 1.

Basic var.	x_1	x_2	s_1	s_2	s_3	RHS
x_2	0	1	-1/15	4/15	0	16/3
s_3	0	0	-1/3	1/3	1	2/3
x_1	1	0	4/15	-1/15	0	17/3

Table 1: Optimal tableau of the LP relaxation

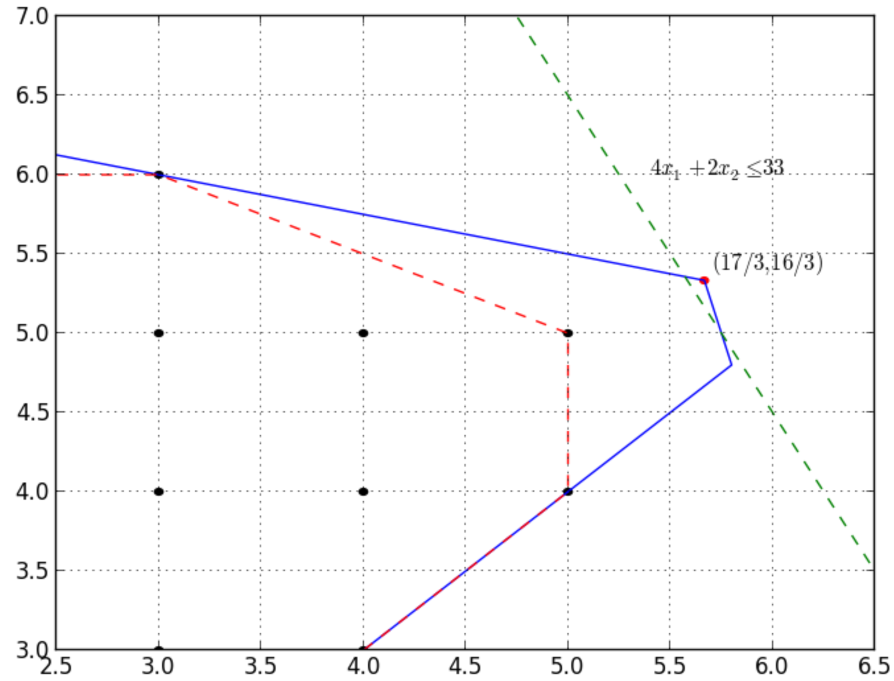
Example: Gomory Cuts (cont.)

The Gomory cut from the first row is

$$\frac{14}{15}s_1 + \frac{4}{15}s_2 \geq \frac{1}{3},$$

In terms of x_1 and x_2, we have

$$4x_1 + 2x_2 \leq 33, \qquad \text{(G-C1)}$$



Example: Equivalent C-G Inequality (cont.)

- Let's derive the same inequality as a C-G inequality.
- We combine the first two inequalities from the original formulation with weights $-1/15 - (-1) = 14/15$ and $4/15$ to get

$$4x_1 + 2x_2 \leq 100/3.$$

- After rounding, this is the Gomory inequality from the previous slide.
- A Gomory inequality is always a C-G cut obtained by combining inequalities that are binding at the optimal basic feasible solution.
  - Binding constraints correspond to non-basic slack variables.
  - Columns in the tableau associated with basic slack variables are unit columns.
  - This means the slack constraints get zero weight.
- Combining the binding constraint yields an inequality that is satisfied at equality by the optimal basic feasible solution.
- We then round to get an inequality violated by that basic feasible solution.

Trivial Strengthening

- Note the inquality can be trivially strengthened by dividing by 2.
- Since the gcd of the coefficients is 2, there are no integer points satisfying $4x_1 + 2x_2 = 33$.
- Thus, the right-hand side can be strengthened further without removing any integer point.
- Dividing by 2 and rounding, we get

$$2x_1 + x_2 \leq 16,$$

- The following proposition states formally what is necessary to ensure the strongest possible C-G inequality.

Proposition 1. *Let* $\mathcal{S} = \{x \in \mathbb{Z}^n \mid \sum_{j \in N} a_j x_j \leq b\}$, *where* $a_j \in \mathbb{Z}$ *for* $j \in N$, *and let* $k = gcd\{a_1, \ldots, a_n\}$. *Then* $\mathrm{conv}(\mathcal{S}) = \{x \in \mathbb{R}^n \mid \sum_{j \in N} (a_j/k) x_j \leq \lfloor b/k \rfloor\}$.

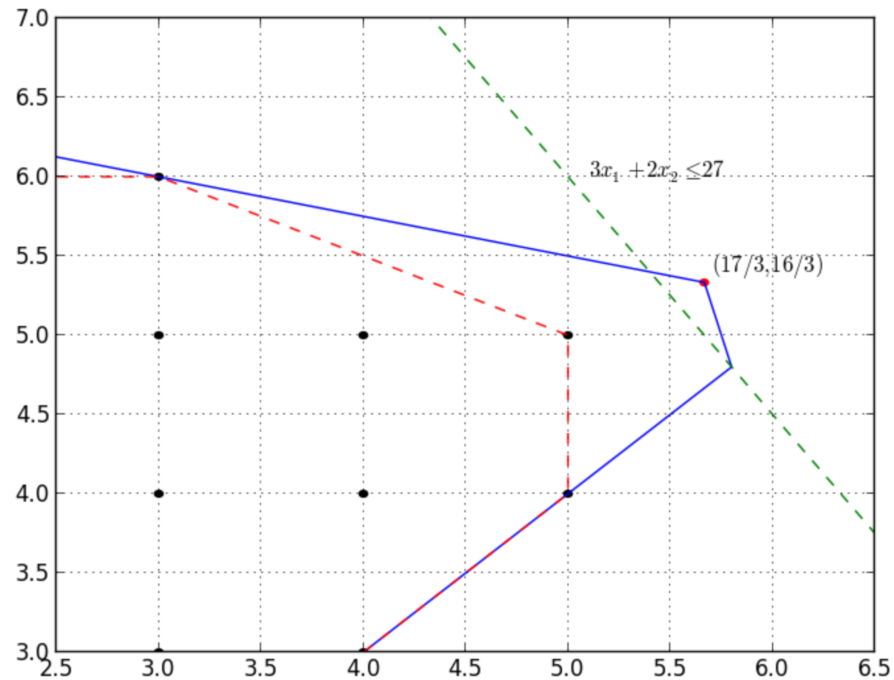
Example: Gomory Cuts (cont.)

The Gomory cut from the second row is

$$\frac{2}{3}s_1 + \frac{1}{3}s_2 \geq \frac{2}{3},$$

In terms of x_1 and x_2, we have

$$3x_1 + 2x_2 \leq 27, \qquad \text{(G-C2)}$$



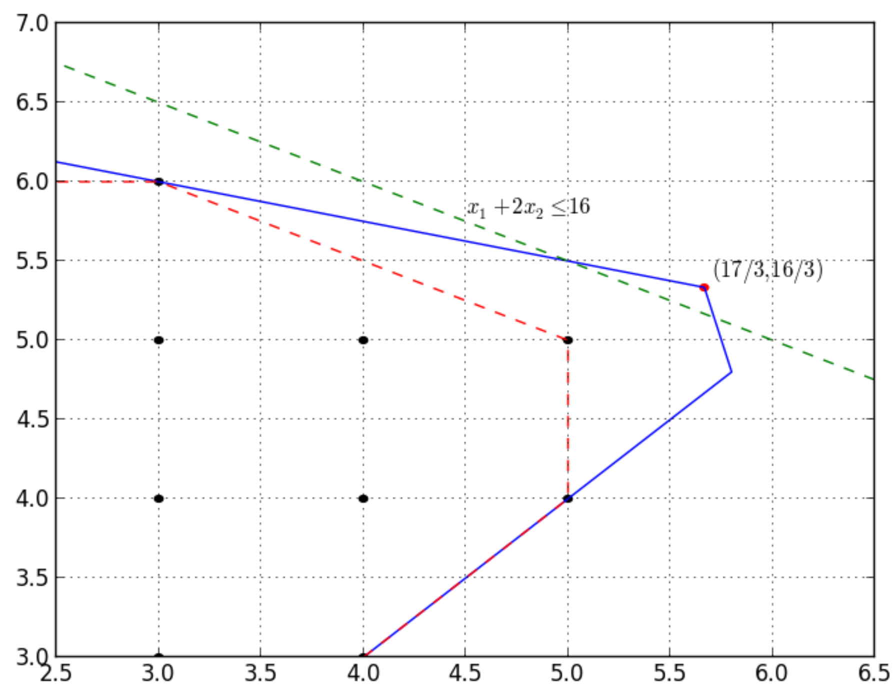
Example: Gomory Cuts (cont.)

The Gomory cut from the third row is

$$\frac{4}{15}s_1 + \frac{14}{15}s_2 \geq \frac{2}{3},$$

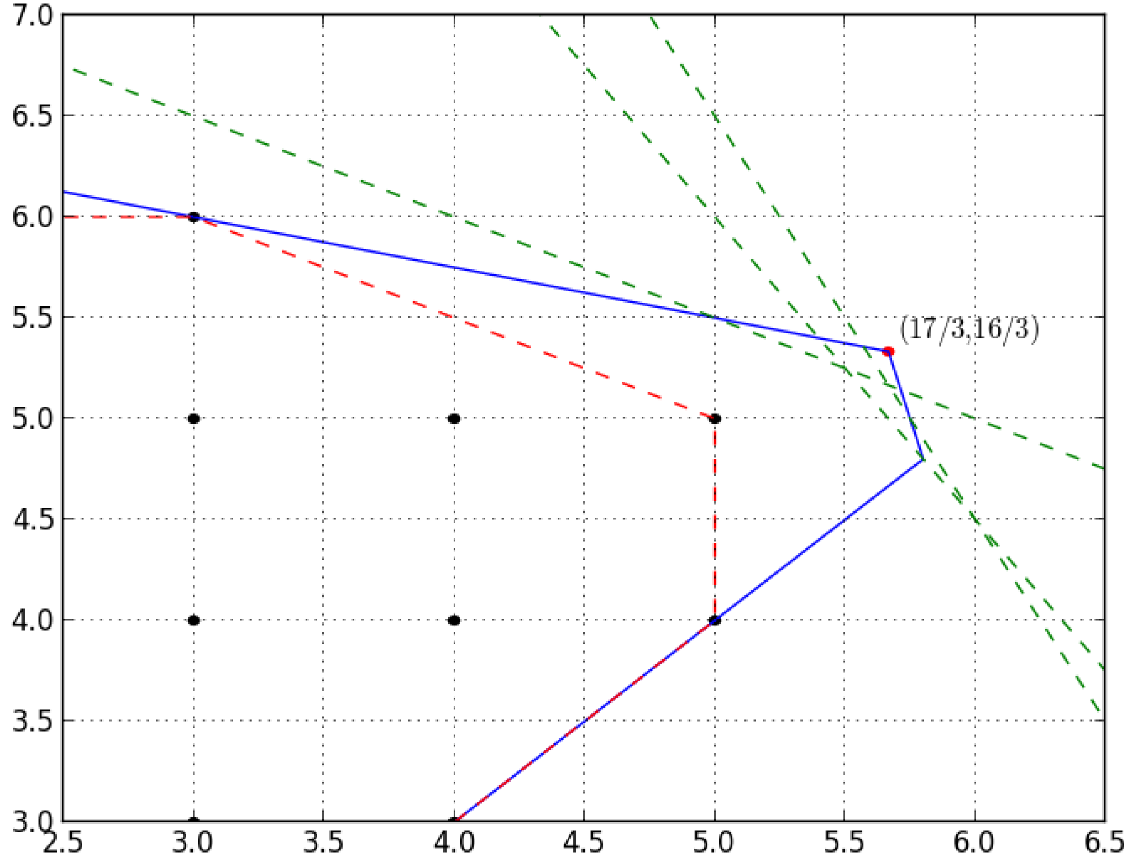
In terms of x_1 and x_2, we have

$$x_1 + 2x_2 \leq 16, \qquad \text{(G-C3)}$$



Example: Gomory Cuts (cont.)

This picture shows the effect of adding all Gomory cuts in the first round.



Connection with Dual Functions

- Recall that an inequality (π, π_0) is valid for $\text{conv}(\mathcal{S})$ if

$$\pi_0 \geq F(b),$$

where F is a dual function with respect to the optimization problem

$$\max_{x \in \mathcal{S}} \pi^\top x$$

- When $uA_I \in \mathbb{Z}^p$, $uA_C \geq 0$, then $F(b) = \lfloor ub \rfloor$ is a dual function for

$$\max_{x \in \mathcal{S}} \pi^\top x,$$

where $\pi = uA$.

- Thus, Chvátal inequalities can be derived directly using an argument based on duality.

Applying the Procedure Recursively

- This procedure can be applied recursively by adding the generated inequalities to the formulation and performing the same steps again.
- Any inequality that can be obtained by recursive application of the C-G procedure (or is dominated by such an inequality) is a *C-G inequality*.
- For pure ILPs, all valid inequalities are C-G inequalities.

  Theorem 1. *Let* $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ *be a valid inequality for* $\mathcal{S} = \{x \in \mathbb{Z}^n_+ \mid Ax \leq b\} \neq \emptyset$. *Then* (π, π_0) *is a C-G inequality for* $\mathcal{S}$.

- In the next few slides, we will make these ideas more precise.

Elementary Closure

- The *elementary closure*, or *C-G closure*, of a polyhedron $\mathcal{P} \subseteq \mathbb{R}^n_+$ is the intersection of half-spaces defined by C-G inequalities, e.g.,

$$e(\mathcal{P}) = \{x \in \mathcal{P} \mid \pi^\top x \leq \pi_0, \pi_j = \lfloor ua_j \rfloor \text{ for } 1 \leq j \leq p, \\ \pi_j = 0 \text{ for } p+1 \leq j \leq n, \pi_0 = \lfloor ub \rfloor, u \in \mathbb{R}^m_+\}$$

- Although it is not obvious, one can show that the elementary closure *is a polyhedron*.
- Optimizing over this polyhedron is difficult (NP-hard) in general.
- For a general polyhedron $\mathcal{P}$, not necessarily contained in the non-negative orthant, we can similarly define the *Chvátal closure*.

$$\mathcal{P}^{\mathrm{CH}} = \{x \in \mathcal{P} \mid \pi^\top x \leq \pi_0, \pi = uA, \pi_0 = \lfloor ub \rfloor, uA_I \in \mathbb{Z}^p, uA_C = 0\}$$

Rank of C-G Inequalities

- The *rank k C-G closure* $\mathcal{P}^k$ of $\mathcal{P}$ is defined recursively as follows.
  - The rank 1 closure of $\mathcal{P}$ is $\mathcal{P}^1 = e(\mathcal{P})$.
  - The rank k closure $\mathcal{P}^k = e(\mathcal{P}^{k-1})$ is the elementary closure of the $\mathcal{P}^{k-1}$.
  - An inequality is rank k with respect to $\mathcal{P}$ if it is valid for the rank k closure $\mathcal{P}^k$ and not for $\mathcal{P}^{k-1}$.
- The *C-G rank* of $\mathcal{P}$ is the maximum rank of any facet-defining inequality of $\mathrm{conv}(\mathcal{S})$ with respect to $\mathcal{P}$.
- We can define a similar notion of rank with respect to the Chvátal closure.

A Finite Cutting Plane Procedure

- Under mild assumptions on the algorithm used to solve the LP, this yields a general algorithm for solving (pure) ILPs.
- The details are contained in Section 5.2.5 of CCZ.

Determining the C-G Rank

- By solving an LP, it can be determined whether a given inequality has maximum rank 1.

  Proposition 2. *If* $(\pi, \pi_0) \in e(\mathcal{P})$*, then* $\pi_0 \geq \lfloor \pi_0^{LP} \rfloor$*, where* $\pi_0^{LP} = \max_{x \in \mathcal{P}} \pi^\top x$

- Alternatively, if $\pi \in \mathbb{Z}^n$, the inequality $(\pi, \lfloor \pi_0^{LP} \rfloor)$ is rank 1.
- Further, any valid inequality (π, π_0) for which $\pi_0 < \lfloor \pi_0^{LP} \rfloor$ has rank at least 2.
- This tells us that the effectiveness of the C-G procedure is strongly tied to the strength of our original formulation.
- In general it is difficult to determine the rank of any inequality that is not rank 1.

Example: C-G Rank

- Let's consider the C-G rank of the inequality

$$x_1 + 2x_2 \leq 15,$$

which is facet-defining for $\text{conv}(\mathcal{S})$ in our example.

- We have

$$\max_{x \in \mathcal{P}} x_1 + 2x_2 = 49/3. \tag{12}$$

- Since $\lfloor 49/3 \rfloor = 16$, we conclude that this is not a rank 1 cut.
- Note that the dual solution to the LP (12) gives us weights with which to combine the original inequalities to get a C-G cut.
- This is the strongest possible C-G cut of rank 1 with those coefficients.

Bounding The C-G Rank of a Polyhedron

- For most classes of MILPs, the rank of the associated polyhedron is an unbounded function of the dimension.
- Example:
  - $\mathcal{P} = \{x \in \mathbb{R}^n_+ \mid x_i + x_j \leq 1 \text{ for } i, j \in N, i \neq j\}$ and $S = \mathcal{P}^n \cap \mathbb{Z}^n$
  - $\text{conv}(\mathcal{S}) = \{x \in \mathbb{R}^n_+ \mid \sum_{j \in N} x_j \leq 1\}$.
  - $\text{rank}(\mathcal{P}) = O(\log n)$.
- For a family of polyhedra with bounded rank, there is a certificate for the validity of any given inequality.
- This leads to a certificate of optimality for the associated optimization problem.
- Hence, it is unlikely that the problem of optimizing over any family of MILPs formulated by polyhedra with bounded rank is in NP-hard[2].
- Conversely, for any family of MILPs that is in NP-hard, the associated family of polyhedra is likely to have unbounded rank.

[2]More on what this means later