Integer Programming
ISE 418

Lecture 11

Dr. Ted Ralphs
Reading for This Lecture

- “Constraint Integer Programming,” Achterberg, Chapter II
Putting it All Together: Search Strategies

• In the last lecture, we discussed how to branch, i.e., divide the feasible region of a subproblem into two pieces.

• After branching, we still have to face the question of what node to process next.

• The strategy for deciding what node to work on next is called the search strategy.

• In other words, we are determining the priority function that will be used in the priority queue we use to keep track of the candidate nodes.

• In choosing a search strategy, we might consider our goal:
  – Minimize the time required to find a provably optimal solution.
  – Find the best possible solution in a limited amount of time.

• In practice, we may want some of each.
Basic Strategies: Best First

• A reasonable approach to minimizing overall solution time is to try to minimize the size of the search tree.

• In theory, we can do this by choosing the subproblem with the best bound (highest upper bound, if we are maximizing).

• A candidate node is said to be critical if its bound exceeds the value of an optimal solution solution to the IP.

• Every critical node will be processed no matter what the search order.

• Under mild conditions, best first is guaranteed to examine only critical nodes, thereby minimizing the size of the search tree (why?).

• However, it has some drawbacks:
  – Doesn’t find feasible solutions quickly (why?).
  – Node setup costs.
  – Memory usage.
  – Fewer variables fixed by reduced cost (more about this later).
What Bound Do We Use?

• We have so far left out one detail: exactly what bound we assign initially to a new candidate subproblem?

• One option is to use the final bound of the parent node, but this does not allow us to distinguish between two children with the same parent.

• A better option is to simply use the same estimate of the bound we computed during branching.
  – If we used strong branching, then use the estimate computed during the pre-solve.
  – If we are using pseudo-cost branching, use that estimate.

• Below, we will also see some alternatives that use estimates of the optimal solution value of the subproblem itself (not the relaxation).
Basic Strategies: Depth First

• The depth first approach is to always choose the “deepest” node to process next.

• This avoids *most* of the problems with best first:
  – The number of candidate nodes is minimized (saving memory).
  – The node set-up costs are minimized.
  – Feasible solutions are found more quickly (*why?*).

• Unfortunately, if the initial lower bound is not very good, then we may end up processing lots of *non-critical nodes*.

• We want to avoid this extra expense if possible.
Estimate-based Strategies: Finding Feasible Solutions

• Let’s focus on a strategy for finding feasible solutions quickly.
• One approach is to try to estimate the value of the optimal solution

\[ z_i = \max_{x \in S_i} c^\top x \]

to each subproblem itself (not the relaxation).
• For any subproblem \( S_i \), let

- \( s_i = \sum_j \min(f_j, 1 - f_j) \) be the sum of the integer infeasibilities,
- \( U(i) \) be the upper bound, and
- \( L \) the global lower bound.
• Also, let \( S_0 \) be the root subproblem.
• The \textit{best projection} criterion is

\[ E_i = U(i) + \left( \frac{L - U(0)}{s_0} \right) s_i \]

• The \textit{best estimate} criterion uses the pseudo-costs to obtain

\[ E_i = U(i) + \sum_j \min \left( P_j^- f_j, P_j^+(1 - f_j) \right) \]
Interpretation of Best Projection

- Best projection is based on the implicit assumption that there is a linear relationship between $s_i$ and the gap $U(i) - z_i$.

- In order to solve the subproblem, we need to reduce the sum of the integer infeasibilities to zero by, e.g., further branching.

- Reducing the infeasibility reduces the upper bound.

- We try to figure out what the bound will be when the infeasibility is zero and this is our estimate.

- It is not always the case that our assumption about the linear relationship holds, but it seems to hold empirically in some cases.
Advanced Strategies: Proving Optimality

• For many combinatorial problems, we can find a “good” solution heuristically.

• In such cases, we are more concerned with minimizing the time to prove optimality.

• To retain the advantages of both best first and depth first search, we can use a combined strategy.
  – Proceed depth-first until the bound in the current node falls below the best bound by more than a given percentage.
  – Proceed depth-first until the difference between the current bound and the best bound is more than a given percentage of the “global gap.”

• Note that if we actually knew the value of the optimal solution, then we could simply do pure depth-first search.

• Hence, another strategy is to estimate the optimal solution value and proceed depth-first until the bound falls below the estimate.
Hybrid Strategy

- In cases where we do not have a good feasible solution going, we might also try a *hybrid strategy*.
  - First try to find good feasible solutions.
  - Then switch to proving optimality.
Measuring Progress

• An important question is how we know if we’re making progress?
• How much longer will it be until completion?
• The traditional measures of progress are
  – Optimality gap
  – Number of candidate nodes
• These measures are not ideal in many respects.
• Current research is being conducted into what measures are more appropriate.
Generalized Branch and Bound

• There are ways in which the basic framework of branch and bound presented here can be generalized.

• Note that we always bound and then immediately branch.

• This is because the effort in determining the branching disjunction is typically low relative to the effort of computing the bounds.

• This need not always be the case.

• We may interrupt the processing of a node at any point and return it to the queue with a different bound estimate.

• The basic algorithmic framework remains almost unchanged.

• We just need to allow for the option to return a node to the queue and to pick up where we left off the next time.

• As far as I know, this generalized version of branch and bound has only been used in research codes.

• It is also used in other research communities.
Interpreting Search as Dual Improvement

- Recall that branch and bound can be viewed as an algorithm for constructing a dual function.
- The choice of what node to process next affects how the construction algorithm progresses.
- By removing one node and replacing it with two others, we potentially improve the bound yielded by the tree.
- The best bound search strategy can be seen as a strategy aimed at improving the function at its current maximum point.
- The process can be seen as analogous to subgradient optimization in solving the Lagrangian dual.