Integer Programming
ISE 418

Lecture 11

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Reading for This Lecture

- "Computational Study of Search Strategies for Mixed Integer Programming," Linderoth and Savelsbergh.
- "Computational Issues for Branch-and-Cut Algorithms," Martin.
- "Constraint Integer Programming," Achterberg, Chapter II

Putting it All Together: Search Strategies

- In the last lecture, we discussed how to *branch*, i.e., divide the feasible region of a subproblem into two pieces.
- After branching, we still have to face the question of what node to process next.
- The strategy for deciding what node to work on next is called the *search strategy*.
- In other words, we are determining the priority function that will be used in the priority queue we use to keep track of the candidate nodes.
- In choosing a search strategy, we might consider our goal:
  - Minimize the time required to find a provably optimal solution.
  - Find the best possible solution in a limited amount of time.
- In practice, we may want some of each.

Basic Strategies: Best First

- A reasonable approach to minimizing overall solution time is to try to minimize the size of the search tree.
- In theory, we can do this by choosing the subproblem with the *best bound* (highest upper bound, if we are maximizing).
- A candidate node is said to be *critical* if its bound exceeds the value of an optimal solution solution to the IP.
- Every critical node will be processed no matter what the search order.
- Under mild conditions, best first is guaranteed to examine only critical nodes, thereby minimizing the size of the search tree (why?).
- However, it has some drawbacks:
  - Doesn't find feasible solutions quickly (why?).
  - Node setup costs.
  - Memory usage.
  - Fewer variables fixed by reduced cost (more about this later).

What Bound Do We Use?

- We have so far left out one detail: exactly what bound we assign initially to a new candidate subproblem?
- One option is to use the final bound of the parent node, but this does not allow us to distinguish between two children with the same parent.
- A better option is to simply use the same estimate of the bound we computed during branching.
  - If we used strong branching, then use the estimate computed during the pre-solve.
  - If we are using pseudo-cost branching, use that estimate.
- Below, we will also see some alternatives that use estimates of the optimal solution value of the subproblem itself (not the relaxation).

Basic Strategies: Depth First

- The depth first approach is to always choose the "deepest" node to process next.
- This avoids *most* of the problems with best first:
  - The number of candidate nodes is minimized (saving memory).
  - The node set-up costs are minimized.
  - Feasible solutions are found more quickly (why?).
- Unfortunately, if the initial lower bound is not very good, then we may end up processing lots of non-critical nodes.
- We want to avoid this extra expense if possible.

Estimate-based Strategies: Finding Feasible Solutions

- Let's focus on a strategy for finding feasible solutions quickly.
- One approach is to try to estimate the value of the optimal solution

$$z_i = \max_{x \in \mathcal{S}_i} c^\top$$

  to each subproblem itself (not the relaxation).

- For any subproblem S_i, let
  - $s_i = \sum_j \min(f_j, 1 - f_j)$ be the sum of the integer infeasibilities,
  - $U(i)$ be the upper bound, and
  - L the global lower bound.
- Also, let S_0 be the root subproblem.
- The *best projection* criterion is

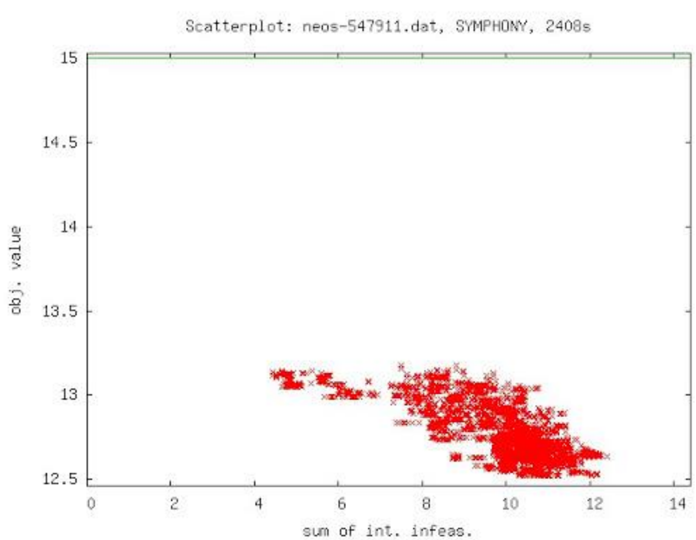
$$E_i = U(i) + \left(\frac{L - U(0)}{s_0} \right) s_i$$

- The *best estimate* criterion uses the pseudo-costs to obtain

$$E_i = U(i) + \sum_j \min \left(P_j^- f_j, P_j^+ (1 - f_j) \right)$$

Interpretation of Best Projection

- Best projection is based on the implicit assumption that there is a linear relationship between s_i and the gap $U(i) - z_i$.
- In order to solve the subproblem, we need to reduce the sum of the integer infeasibilities to zero by, e.g., further branching.
- Reducing the infeasibility reduces the upper bound.
- We try to figure out what the bound will be when the infeasibility is zero and this is our estimate.
- It is not always the case that our assumption about the linear relationship holds, but it seems to hold empirically in some cases.



Advanced Strategies: Proving Optimality

- For many combinatorial problems, we can find a "good" solution heuristically.
- In such cases, we are more concerned with minimizing the time to prove optimality.
- To retain the advantages of both best first and depth first search, we can use a combined strategy.
  - Proceed depth-first until the bound in the current node falls below the best bound by more than a given percentage.
  - Proceed depth-first until the difference between the current bound and the best bound is more than a given percentage of the "global gap."
- Note that if we actually knew the value of the optimal solution, then we could simply do pure depth-first search.
- Hence, another strategy is to estimate the optimal solution value and proceed depth-first until the bound falls below the estimate.

Hybrid Strategy

- In cases where we do not have a good feasible solution going, we might also try a *hybrid strategy*.
  - First try to find good feasible solutions.
  - Then switch to proving optimality.

Measuring Progress

- An important question is how we know if we're making progress?
- How much longer will it be until completion?
- The traditional measures of progress are
  - Optimality gap
  - Number of candidate nodes
- These measures are not ideal in many respects.
- Current research is being conducted into what measures are more appropriate.

Generalized Branch and Bound

- There are ways in which the basic framework of branch and bound presented here can be generalized.
- Note that we always bound and then immediately branch.
- This is because the effort in determining the branching disjunction is typically low relative to the effort of computing the bounds.
- This need not always be the case.
- We may interrupt the processing of a node at any point and return it to the queue with a different bound estimate.
- The basic algorithmic framework remains almost unchanged.
- We just need to allow for the option to return a node to the queue and to pick up where we left off the next time.
- As far as I know, this generalized version of branch and bound has only been used in research codes.
- It is also used in other research communities.

Interpreting Search as Dual Improvement

- Recall that branch and bound can be viewed as an algorithm for constructing a dual function.
- The choice of what node to process next affects how the construction algorithm progresses.
- By removing one node and replacing it with two others, we potentially improve the bound yielded by the tree.
- The best bound search strategy can be seen as a strategy aimed at improving the function at its current maximum point.
- The process can be seen as analogous to subgradient optimization in solving the Lagrangian dual.