Advanced Mathematical Programming
IE417

Lecture 9

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Reading for This Lecture

- Chapter 6, Section 1-2

Lagrangian Duality

The Primal Problem

- Given functions $f : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{g} : \mathbb{R}^\mathbf{n} \to \mathbb{R}^\mathbf{m}$, and $\mathbf{h} : \mathbb{R}^\mathbf{n} \to \mathbb{R}^\mathbf{l}$, consider the constrained optimization problem P, which we will now call the *primal problem*:

$$\begin{aligned} \min\ & f(x) \\ \text{s.t.}\ & \mathbf{g}(x) \leq 0 \\ & \mathbf{h}(x) = 0 \\ & x \in X \end{aligned}$$

- Here, X is a set that implicitly enforces additional constraints.
- There is usually more than one way to define X and this choice can be important, as we will see.

The Dual Problem

- We can now formulate the following *dual problem* D:

$$\max \Theta(\mu, v)$$
$$\text{s.t.} \quad \mu \geq 0$$

where $\Theta(\mu, v) = \inf\{\Phi(x, \mu, v) : x \in X\}$.

- How do we interpret this?

Weak Duality

Theorem 1. *Let* x *be a feasible solution to the primal problem* P *and let* (μ, v) *be a solution to the dual problem* D*. Then* $f(x) \geq \Theta(\mu, v)$.

Let $S = \{x \in X, \mathbf{g}(x) \leq 0, \mathbf{h}(x) = 0\}$

Corollary 1. $\inf\{f(x) : x \in S\} \geq \sup\{\Theta(\mu, v), \mu \geq 0\}$.

Corollary 2. *If* $f(x^*) = \Theta(\mu^*, v^*)$ *for some* $x^* \in S, \mu \geq 0$*, then* x^* *solves* P *and* (μ^*, v^*) *solves* D.

Corollary 3. *If* $\sup\{\Theta(\mu, v), \mu \geq 0\} = \infty$*, then the primal problem has no feasible solution.*

Theorem of the Alternative

Theorem 2. *Let X be a given nonempty convex set in R^n, $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\mathbf{g} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ given convex functions, and $\mathbf{h} : \mathbb{R}^n \rightarrow \mathbb{R}^l$ a given affine function, i.e., $\mathbf{h}(x) = Ax - b$ for $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, x \in \mathbb{R}^n$. Then if I has no solution, II has a solution. The converse holds if $\mu_0 > 0$.*

$$I. \ \ \alpha(x) < 0, \mathbf{g}(x) \leq 0, \mathbf{h}(x) = 0, \exists x \in X.$$

$$II. \ \ \mu_0 \alpha(x) + \mu^T \mathbf{g}(x) + v^T \mathbf{h}(x), \forall x \in X, \mu \geq 0, (\mu, v) \neq 0.$$

Strong Duality

Theorem 3. *Let* X *be a given nonempty convex set in* $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$ *and* $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^m$ *given convex functions, and* $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^l$ *a given affine function. If there exists* $x' \in X$ *such that* $\mathbf{g}(x') < 0, \boldsymbol{h}(x') = 0$, *and* $0 \in int\ \boldsymbol{h}(X)$, *then*

$$\inf\{f(x) : x \in S\} = \sup\{\Theta(\mu, v)\mu \geq 0\}$$

If the $\inf$ *is finite, then* $\sup\{\Theta(\mu, v), \mu \geq 0\}$ *is achieved at* (μ^*, v^*) *with* $\mu^* \geq 0$. *If the* $\inf$ *is achieved at* x^*, *then* ${\mu^*}^T \mathbf{g}(x^*) = 0$.