Advanced Mathematical Programming IE417

Lecture 8

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Reading for This Lecture

- Chapter 4, Section 3-4

Optimality Conditions

Equality Constrained Problems

FJ with Equality Constraints

Theorem 1. *Consider* $S = \{x \in X : g_i(x) \leq 0, i \in [1,m], h_i(x) = 0, i \in [1,l]\}$ *where* X *is a nonempty open set in* $\mathbb{R}^n$ *and* $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i \in [1,m]$, $h_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i \in [1,l]$. *Given a feasible* $x^* \in S$, *set* $I = \{i : g_i(x^*) = 0\}$. *Assume that* f *and* g_i *are differentiable at* x^* *for* $i \in I, g_i$ *is continuous at* x^* *for* $i \notin I, h_i$ *is continuously differentiable. If* x^* *is a local minimum, then there exists* $\mu \in \mathbb{R}^m, v \in \mathbb{R}^l$ *such that*

$$\begin{aligned} \mu_0 \bigtriangledown f(x^*) + \sum \mu_i \bigtriangledown g_i(x^*) + \sum v_i \bigtriangledown h_i(x^*) &= 0 \\ \mu_i g_i(x^*) &= 0 \; \forall i \in [1,m] \\ \mu &\geq 0 \\ \mu &\neq 0 \end{aligned}$$

Constraint Qualification

- The same development applies here as with just inequality constraints.
- *Constraint qualification*: $\bigtriangledown g_i(x^*), i \in I$, and $\bigtriangledown h_i(x^*), i \in [1, l]$, are linearly independent.
- This CQ again implies $\mu_0 > 0$.
- We can hence derive similar KKT conditions for problems with equality constraints.

Convex Programs

- The KKT conditions are sufficient for *convex programs*:
  - f is convex
  - $g_1, ..., g_m$ is convex
  - $h_1, ..., h_l$ is linear
- The KKT conditions are necessary and sufficient for convex programs with all linear constraints.

Other Constraint Qualifications

- There are other (less restrictive) conditions that imply the necessity of the KKT conditions (Chapter 5).
- For convex programs, the *Slater condition* implies the necessity of the KKT conditions.
  - $\nabla h_i(x^*)$ are linearly independent.
  - there exists $x^{'} \in S$ such that $g_i(x^{'}) < 0, \forall i \in I$.

The Restricted Lagrangian

- Consider a given vector $x^* \in \mathbb{R}^n$.
- Define the *restricted Lagrangian* function with respect to x^*, u^*, and v^* as

$$L(x) \equiv \Phi(x, u^*, v^*) \equiv f(x) + \sum_{i \in I} u_i^* g_i(x) + \sum v_i^* h_i(x)$$

  where $I = \{i : g_i(x^*) = 0\}$.

- Note that dual feasibility is now equivalent to $\bigtriangledown L(x^*) = 0$.

KKT Sufficient Conditions (2^{nd} Order)

- **Theorem 2.** *Consider the problem to minimize* $f : \mathbb{R}^n \to \mathbb{R}$ *over* $S = \{x \in X : g_i(x) \leq 0, i \in [1, m], h_i(x) = 0, i \in [1, l]\}$ *where* X *is a nonempty open set in* $\mathbb{R}^n$ *and* $g_i : \mathbb{R}^n \to \mathbb{R}$ $, i \in [1, m]$, $h_i : \mathbb{R}^n \to \mathbb{R}$ $, i \in [1, l]$. *Assume* f *and all constraint functions are twice differentiable.*

  Suppose x^* *is a KKT point with restricted Lagrangian function* L.

  – *If* $\bigtriangledown^2 L(x)$ *is positive semi-definite* $\forall x \in S$, *then* x^* *is a global minimum.*
  – *If* $\bigtriangledown^2 L(x)$ *is positive semi-definite in a neighborhood of* x^*, *then* x^* *is a local minimum.*
  – *If* $\bigtriangledown^2 L(x^*)$ *is positive definite, then* x^* *is a strict local minimum.*

Strongly and Weakly Binding Constraints

- Consider a constrained optimization problem over a nonempty open set X in $\mathbb{R}^n$ where the objective function and all the constraints are twice differentiable.
- Let x^* be a KKT point with Lagrange multipliers u^* and $v*$.
- Inequality constraint i is called *weakly binding* if $u_i^* = 0$ and *strongly binding* otherwise.
- Note that we could delete the weakly binding constraints and we would still have a KKT point.
- However, this might change the optimality status of the solution.