Advanced Mathematical Programming IE417

Lecture 6

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Reading for This Lecture

- Chapter 4, Sections 1-2

Optimality Conditions

Unconstrained Problems

First-order Necessary Conditions

Theorem 1. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be differentiable at* x^*. *If there is a vector* d *such that* $\nabla f(x^*)^T d < 0$, *then there exists a* $\delta > 0$ *such that* $f(x^* + \lambda d) < f(x^*)$ *for each* $\lambda \in (0, \delta)$.

Corollary 1. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be differentiable at* x^*. *If* x^* *is a local minimum, then* $\nabla f(x^*) = 0$.

- The direction d is called a *descent direction*.

Second-order Necessary Conditions

Theorem 2. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be twice differentiable at* x^**. If* x^* *is a local minimum, then* $\nabla f(x^*) = 0$ *and* $H(x^*)$ *is positive semi-definite.*

Sufficient Conditions

Theorem 3. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be twice differentiable at* x^**. If* $\nabla f(x^*) = 0$ *and* $H(x^*)$ *is positive definite, then* x^* *is a local minimum.*

Theorem 4. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be pseudoconvex at* x^**. Then* x^* *is a global minimum, if and only if* $\nabla f(x^*) = 0$*.*

Theorem 5. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be infinitely differentiable. Then* x^* *is a local minimum if and only if either* $f^{(j)}(x^*) = 0 \ \ \forall j$*, or else there exists an even* n *such that* $f^{(n)}(x^*) > 0$ *while* $f^{(j)}(x^*) = 0 \ \ \forall j < n$*.*

Optimality Conditions

Inequality Constrained Problems

Feasible and Improving Directions

Definition 1. *Let S be a nonempty set in $\mathbb{R}^n$ and let $x^* \in cl(S)$. The* cone of feasible directions *of S at x^* is given by*

$$D = \{d : d \neq 0 \text{ and } x^* + \lambda d \in S, \forall \lambda \in (0, \delta), \exists \delta > 0\}$$

Definition 2. *Let S be a nonempty set in $\mathbb{R}^n$ and let $x^* \in cl(S)$. Given a function $f : \mathbb{R}^n \to \mathbb{R}$, the* cone of improving directions *of f at x^* is given by*

$$F = \{d : f(x^* + \lambda d) < f(x^*), \forall \lambda \in (0, \delta), \exists \delta > 0\}$$

Characterizing Set F

Define $F_0 = \{d : \bigtriangledown f(x^*)^T d < 0\}$ and $F_0^{'} = \{d : d \neq 0, \bigtriangledown f(x^*)^T d \leq 0\}$. Then $F_0 \subseteq F \subseteq F_0^{'}$.

Theorem 6. *Let S be a nonempty set in $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be given. Consider the constrained optimization problem*

$$\min f(x)$$
$$\textit{s.t. } x \in S$$

If x^* is a local optimal solution and f is differentiable at x^*, then $F_0 \cap D$ is empty. Conversely,...

Characterizing Set D

- Consider the feasible region $S = \{x \in X : g_i(x) \leq 0, i \in [1, m]\}$ where X is a nonempty open set in $\mathbb{R}^n$ and $g_i : \mathbb{R}^n \rightarrow \mathbb{R}, i \in [1, m]$.
- Given a feasible $x^* \in \mathbb{R}^n$, set $I = \{i : g_i(x^*) = 0\}$.
- Assume that g_i is differentiable at x^* for $i \in I$ and g_i is continuous at x^* for $i \notin I$ and define $G_0 = \{d : \nabla g_i(x^*)^T d < 0 \ \forall i \in I\}$ and $G_0^{'} = \{d : d \neq 0, \nabla g_i(x^*)^T d \leq 0 \ \forall i \in I\}$. Then $G_0 \subseteq D \subseteq G_0^{'}$.

More Optimality Conditions

Theorem 7. *Let* X *be a nonempty open set in* $\mathbb{R}^n$ *and let* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *and* $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ *be given. Consider the constrained optimization problem*

$$\begin{aligned} &\min f(x) \\ &\textit{s.t.}\;\; g_i(x) \leq 0 \\ &\qquad x \in X \end{aligned}$$

If x^* *is a local optimal solution, then* $F_0 \cap G_0$ *is empty. Conversely, ...*

Fritz-John Necessary Conditions

Theorem 8. *Consider the feasible region* $S = \{x \in X : g_i(x) \leq 0, i \in [1, m]\}$ *where* X *is a nonempty open set in* $\mathbb{R}^n$ *and* $g_i : \mathbb{R}^n \to \mathbb{R}, i \in [1, m]$. *Given a feasible* $x^* \in S$, *set* $I = \{i : g_i(x^*) = 0\}$. *Assume that* f *and* g_i *are differentiable at* x^* *for* $i \in I$ *and* g_i *is continuous at* x^* *for* $i \notin I$. *If* x^* *is a local minimum, then there exists* $\mu \in \mathbb{R}^m$ *such that*

$$\begin{aligned} \mu_0 \bigtriangledown f(x^*) + \sum \mu_i \bigtriangledown g_i(x^*) &= 0 \\ \mu_i g_i(x^*) &= 0 \, \forall i \in [1, m] \\ \mu &\geq 0 \\ \mu &\neq 0 \end{aligned}$$