Advanced Mathematical Programming
IE417

Lecture 5

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Reading for This Lecture

- Chapter 3, Sections 4-5
- Chapter 4, Section 1

Maxima and Minima of Convex Functions

Minimizing a Convex Function

Theorem 1. *Let* S *be a nonempty convex set on* $\mathbb{R}^n$ *and let* $f : S \rightarrow \mathbb{R}$ *be convex on* S*. Suppose that* x^* *is a local optimal solution to* $min_{x \in S} f(x)$*.*

- *Then* x^* *is a global optimal solution.*
- *If either* x^* *is a strict local optimum or* f *is strictly convex, then* x^* *is the unique global optimal solution.*

Necessary and Sufficient Conditions

Theorem 2. *Let S be a nonempty convex set on $\mathbb{R}^n$ and let $f : S \rightarrow \mathbb{R}$ be convex on S. The point $x^* \in S$ is an optimal solution to the problem $min_{x \in S} \; f(x)$ if and only if f has a subgradient ξ such that $\xi^T(x - x^*) \geq 0 \;\; \forall x \in S$.*

- This implies that if S is open, then x^* is an optimal solution if and only if there is a zero subgradient of f at x^*.

Alternative Optima

Theorem 3. *Let* S *be a nonempty convex set on* $\mathbb{R}^n$ *and let* $f : S \to \mathbb{R}$ *be convex on* S*. If the point* $x^* \in S$ *is an optimal solution to the problem* $min_{x \in S}\ f(x)$*, then the set of alternative optima are characterized by the set*

$$S^* = \{x \in S : \nabla f(x^*)^T(x - x^*) \leq 0 \text{ and } \nabla f(x^*) = \nabla f(x)$$

Corollaries:

Maximizing a Convex Function

Theorem 4. *Let S be a nonempty convex set on $\mathbb{R}^n$ and let $f : S \to \mathbb{R}$ be convex on S. If the point $x^* \in S$ is an optimal solution to the problem $max_{x \in S}\ f(x)$, then $\xi^T(x - x^*) \leq 0\ \ \forall x \in S$, where ξ is any subgradient of f.*

Theorem 5. *Let S be a nonempty compact polyhedral set on $\mathbb{R}^n$ and let $f : S \to \mathbb{R}$ be convex on S. An optimal solution x^* to the problem $max_{x \in S}\ f(x)$ exists where x^* is an extreme point of S.*

Importance of Convexity

- Allows easy approximation.
- Allows easy global optimization.
- Allows easy constrained optimization over polyhedral sets.

Generalizing Convexity

Quasiconvex Functions

Definition 1. *Let S be a nonempty convex set in $\mathbb{R}^n$ and let $f : S \to \mathbb{R}$. The function f is said to be* quasiconvex *if, for each $x_1, x_2 \in S$, we have*

$$f[\lambda x_1 + (1 - \lambda x_2)] \leq \max\{f(x_1), f(x_2)\}$$

Theorem 6. *Let S be a nonempty convex set in $\mathbb{R}^n$ and let $f : S \to \mathbb{R}$. The function f is quasiconvex if and only if $S_\alpha = \{x \in S : f(x) \leq \alpha\}$ is convex for each $\alpha \in \mathbb{R}$.*

Strictly Quasiconvex Functions

Definition 2. *Let S be a nonempty convex set in $\mathbb{R}^n$ and let $f : S \to \mathbb{R}$. The function f is said to be* strictly quasiconvex *if, for each $x_1, x_2 \in S$ with $f(x_1) \neq f(x_2)$, we have*

$$f[\lambda x_1 + (1 - \lambda x_2)] < \max\{f(x_1), f(x_2)\}$$

Theorem 7. *Let S be a nonempty convex set in $\mathbb{R}^n$ and let $f : S \to \mathbb{R}$ be strictly quasiconvex. Consider the problem of minimizing f over S. If $\hat{x}$ is a local optimal solution, then $\hat{x}$ is a global optimal solution.*

- Note that strict quasiconvexity does not imply quasiconvexity unless the function is lower semicontinuous.

Strongly Quasiconvex Functions

Definition 3. *Let S be a nonempty convex set in $\mathbb{R}^n$ and let $f : S \to \mathbb{R}$. The function f is said to be* strictly quasiconvex *if, for each $x_1, x_2 \in S$ with $x_1 \neq x_2$, we have*

$$f[\lambda x_1 + (1 - \lambda x_2)] < \max\{f(x_1), f(x_2)\}$$

Theorem 8. *Let S be a nonempty convex set in $\mathbb{R}^n$ and let $f : S \to \mathbb{R}$ be strictly quasiconvex. Consider the problem of minimizing f over S. If $\hat{x}$ is a local optimal solution, then $\hat{x}$ is the unique global optimal solution.*

Pseudoconvex Functions

Definition 4. *Let S be a nonempty convex set on $\mathbb{R}^n$ and let $f : S \to \mathbb{R}$ be differentiable on S. The function f is said to be* pseudoconvex *if $\forall x_1, x_2 \in S$, with $\nabla f(x_1)^T(x_2 - x_1) \geq 0$, we have $f(x_2) \geq f(x_1)$.*

- *Strictly pseudoconvex* can be defined analogously.
- *Notice that all convex functions are pseudoconvex*.
- Also, for a pseudoconvex function f, if $\nabla f(x^*) = 0$, then x^* is a local minimum.

Local Convexity

Definition 5. *Let S be a nonempty convex set on $\mathbb{R}^n$ and let $f : S \to \mathbb{R}$. Then the following are local properties of f.*

- *f is said to be convex at x^* if $f(\lambda x^* + (1-\lambda)x) \leq \lambda f(x^*) + (1-\lambda)f(x)$ for each $\lambda \in (0,1)$ and $x \in S$.*
- *f is said to be pseudoconvex at x^* if $\nabla f(x^*)^T(x - x^*) \geq 0$ implies $f(x) \geq f(x^*)$.*