Advanced Mathematical Programming
IE417

Lecture 3

Dr. Ted Ralphs

Reading for This Lecture

- Primary Reading
  - Chapter 2, Sections 4-7
- Secondary Reading
  - Chapter 1
  - Appendix A

Hyperplanes and Half-spaces

- A *hyperplane* is a set of the form $H = \{x : p^T x = \alpha\}$ where p is a nonzero vector in $\mathbb{R}^n$ and α is a scalar.
- A hyperplane defines two *closed half-spaces* $H^- = \{x : p^T x \leq \alpha\}$ and $H^+ = \{x : p^T x \geq \alpha\}$.
- There are also corresponding *open half-spaces*.
- A hyperplane $H = \{x : p^T x = \alpha\}$ is said to *separate* two nonempty sets S_1 and S_2 if
  - $S_1 \subseteq \{x : p^T x \leq \alpha\}$, and
  - $S_2 \subseteq \{x : p^T x \geq \alpha\}$

Separation Theorem I

Theorem 1. *Let S be a nonempty, closed convex set in $\mathbb{R}^n$ and $y \notin S$. Then there exists a unique point $x^* \in S$ with minimum distance from y. Furthermore, x is the minimizing point if and only if $(y - x^*)^T(x - x^*) \leq 0 \ \forall x \in S$.*

- What is this really saying?
- Idea of Proof:

Separation Theorem II

Theorem 2. *Let* S *be a nonempty, closed convex set in* $\mathbb{R}^n$ *and* $y \notin S$*. Then there exists a nonzero vector* p *and a scalar* α *such that* $p^T y > \alpha$ *and* $p^T x \leq \alpha$ *for every* $x \in S$*.*

- What is this really saying?
- Idea of Proof:

Corollaries

Corollary 1. *Let S be a closed, convex set in $\mathbb{R}^n$. Then S is the intersection of all half-spaces containing S.*

Corollary 2. *Let S be a nonempty set in $\mathbb{R}^n$, and let $y \notin \text{cl}(\text{conv}(S))$. Then there exists a hyperplane strongly separating y and S.*

Farkas Theorem

Theorem 3. *Let $A \in \mathbb{R}^{m \times n}$ and $c \in \mathbb{R}^m$. Then exactly one of the following systems has a solution:*

I. $Ax \leq 0$ and $c^T x > 0$.

II. $A^T y = c$ and $y \geq 0$.

- This can be seen as a consequence of the preceding theorem (how?).
- Also see Gordon's Theorem.

Supporting Hyperplane Theorem

Definition 1. *Let S be a nonempty set in $\mathbb{R}^n$ and let x^* be in the boundary of S. A hyperplane $H = \{x \in \mathbb{R}^n : p^T(x - x^*) = 0\}$ is called* supporting of S at x^* *if either $S \subseteq H^+$ or $S \subseteq H^-$.*

Theorem 4. *Let S be a nonempty, convex set in $\mathbb{R}^n$ and let x^* be in the boundary of S. Then there exists a hyperplane that supports S at x^*.*

Idea of Proof:

Separation of Convex Sets

Theorem 5. *Let S_1 and S_2 be nonempty convex sets in $\mathbb{R}^n$. If $S_1 \cap S_2$ is empty, then there exists a hyperplane that separates S_1 and S_2.*

Theorem 6. *Let S_1 and S_2 be closed convex sets in $\mathbb{R}^n$ and suppose that S_1 is bounded. If $S_1 \cap S_2$ is empty, then there exists a hyperplane that strongly separates S_1 and S_2.*

Cones and Polar Cones

- A nonempty set C in $\mathbb{R}^n$ is called *cone* with vertex 0 if $x \in C \Rightarrow \lambda x \in C \ \forall \lambda \geq 0$.
- Let S be a nonempty set in $\mathbb{R}^n$. The *polar cone* of S, is $S^* = \{p : p^T x \leq 0 \forall x \in S\}$.

  Theorem 7. *Let C be a nonempty closed convex cone. Then $C = C^{**}$.*

- Examples: