Advanced Mathematical Programming
IE417

Lecture 24

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Reading for This Lecture

- Sections 11.2-11.2

The Linear Complementarity Problem

- Given $M \in \mathbb{R}^{p\times p}$ and $q \in \mathbb{R}^p$, the *linear complementarity problem* is to find $w, z \in \mathbb{R}^p$ such that

$$\begin{aligned} Iw - Mz &= q \\ w, z &\geq 0 \\ w_j z_j &= 0 \,\forall j \end{aligned}$$

- This importance of this problem, for our purposes is in solving quadratic programming problems.

Terminology

- A cone spanned by p vectors, one from each pair e_j, $-m_j$ is called a *complementary cone*.
- Note that we are trying to show that q belongs to at least one complementary cone.
- Vectors w, z satisfying $w_j z_j = 0 \ \forall j$ are called *complementary*.
- A solution to the given system is called a *complementary feasible solution*.

Solving the LCP

- We can formulate the LCP as an optimization problem:

$$\begin{aligned} min \sum [y_j w_j + (1 - y_j) z_j] & \\ s.t. \quad Iw - Mz & = q \\ w, z & \geq 0 \\ y & \in \{0, 1\} \end{aligned}$$

- The optimal solution is zero if and only if the solution is complementary.
- The variable y_j indicates which of w_j and z_j is nonzero.

Another Approach

- A solution is called a *complementary basic feasible solution* (CBFS) if
  - it is a complementary feasible solution, and
  - it belongs to a complementary cone whose generators are linearly independent.
- This definition is similar to that used in the simplex algorithm.
- Note that either w_j or z_j is basic.
- Idea: Use a pivoting algorithm to find a solution.

Finding a Starting Solution

- Note that if q is nonnegative, then $w = q, z = 0$ is a complementary feasible solution.
- If $q_j < 0$ for some j, then consider the following problem,

$$\begin{aligned} Iw - Mz - \mathbf{1}z_0 &= q \\ w, z, z_0 &\geq 0 \\ w_j z_j &= 0 \, \forall j \end{aligned}$$

- If $z_0 = \max\{-q_i\}$, then $w = q + \mathbf{1}z_0$, $z = 0$ is a complementary solution satisfying the above system.
- Idea: Try to drive z_0 out of the basis.

Almost Complementary Basic Feasible Solutions

- A triplet (w, z, z_0) is called a *almost complementary basic feasible solution* (ACBFS) if
  - (w, z, z_0) is a basic feasible solution to

$$\begin{aligned} Iw - Mz - \mathbf{1}z_0 &= q \\ w, z, z_0 \qquad &\geq 0 \end{aligned}$$

  - There is exactly complementary pair (w_s, z_s) such that neither w_s nor z_s is basic.
  - z_0 is basic.
- All we need is pivot z_0 out of the basis, as in Phase I of the two-phase simplex algorithm.

Pivoting

- We will pivot from one ACBFS to another.
- Note that each ACBFS has at most two adjacent ACBFS's.
- Pivoting is done just as in the simplex algorithm.
- Remove one column from the basis and insert another in its place.
- Update the values of the basic variables to maintain feasibility.

Lemke's Algorithm

- Let s be such that $-q = \max\{-q_i\}$.
- The initial basis is z_0 and w_j, $j \neq s$. Set $y = z_s$.
- Iteration
  - Let d be the basic direction associated with variable y. If $d \leq 0$, STOP. The feasible region is unbounded.
  - Otherwise, the variable y enters the basis.
  - Perform a minimum ratio test to determine the leaving variable.
    * If w_1 just left the basis, set $y = z_1$.
    * If z_1 just left the basis, set $y = w_1$.
    * If z_0 left the basis, STOP.

Convergence of Lemke's Algorithm

Definition 1. *A matrix* M *is* $copositive$ *if* $z^\top M z \geq 0$ *for every* $z \geq 0$*.* M *is* $copositive$*-* $plus$ *if* $z \geq 0$ *and* $z^\top M z = 0$ *implies* $(M + M^\top)z = 0$*.*

Recall that a solution is called nondegenerate if all the basic variables have positive value.

Theorem 1. *If each CBFS is nondegenerate and* M *is copositive-plus, then Lemke's algorithm terminates in a finite number of steps.*

If M has nonnegative entries with positive diagonal elements, then M is copositive-plus.

Quadratic Programming

- For convex QP problems, there exists polynomial time algorithms (barrier methods, for instance).
- If the Hessian has even a single negative eigenvalue, these problems are NP-hard in general.
- However, we can attempt to find KKT points.
- Consider QP:

$$\begin{aligned} & min\, c^\top x + (x^\top Hx)/2 \\ & s.t. \quad Ax \leq b \\ & \qquad\quad x \geq 0 \end{aligned}$$

KKT Conditions for QP

- The KKT conditions for QP are

$$
\begin{aligned}
Ax + y &= b \\
-Hx - A^\top u + v &= c \\
x^\top v &= 0 \\
u^\top y &= 0 \\
x, y, u, v &\geq 0
\end{aligned}
$$

- This is a linear complementarity problem.
- Denote by M_{QP} the corresponding constraint matrix.

Properties of the KKT System

- If H is copositive, then so is M_{QP}.
- If additionally, $y \geq 0$ and $y^\top My \geq 0$ implies $Hy = 0$, then M_{QP} is copositive-plus.
- In particular, M_{QP} is copositive-plus if either
  - H is positive semi-definite, or
  - H has nonnegative entries with positive diagonal elements.

Convergence Result

- In the absence of degeneracy, Lemke's algorithm will find a KKT point in a finite number of iterations under any of the following conditions:
  - H is positive semi-definite and $c = 0$.
  - H is positive definite.
  - H has nonnegative elements with positive diagonal entries.