Advanced Mathematical Programming
IE417

Lecture 23

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Reading for This Lecture

- Chapter 10

Linear Programming

- We are given $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Assume $S = \{x \in \mathbb{R}^n$ s.t. $Ax = b, x \geq 0$ is bounded.
- We want to solve the LP

$$\begin{aligned} &\min c^\top x \\ &s.t.\ \ Ax = b \\ &\qquad\ \ x \geq 0 \end{aligned}$$

- We need only consider the extreme points.

Characterization of Extreme Points

- Arrange the columns of A such that $A = [B, N]$, where B is a non-singular $n \times n$ matrix.
- Then x is an extreme point of S if and only if $x = [x_B, 0]$ where $x_B = B^{-1}b$ for some arrangement such that $B^{-1}b \geq 0$
- This implies that the number of extreme points is finite (but still potentially very large).

The Simplex Algorithm

- Note that $x_B = B^{-1}b - B^{-1}Nx_N$
- Hence, $c^\top x = c_B^\top x_B + c - N^\top x_N = c_B^\top B^{-1}b + (c_N^\top - c_B^\top B^{-1}N)x_N$
- So if $c_N^\top - c_B^\top B^{-1}N \geq 0$, we have found the optimal solution (why?)
- Otherwise, suppose some component of $c_N^\top - c_B^\top B^{-1}N$ is negative.
- How can we interpret this in terms of our framework of improving feasible directions?

Steepest Edge Simplex

- The simplest rule for selecting an entering variables is to choose the one with the largest reduced cost.
- However, we can do something smarter.
- Choose the "steepest edge", i.e., the one that forms the smallest angle with $-\bigtriangledown f(x_k) = -c$.
- Let η_j be the direction derived earlier for the edge corresponding to a non-basic variable j entering the basis.
- Note that $c^\top \eta_j$ is the corresponding reduced cost.
- Instead of reduced costs, we look at $c^\top \eta_j / \|\eta_j\|$.

Implementing Steepest Edge Simplex Algorithm

- Steepest edge simplex algorithm
  - Compute $yB = c_B^\top$
  - Choose the column of a_j of N that maximizes $c_j - ya_j/\|\eta_j\|$.
  - Compute $Bd = a_j$
  - Find the largest t such that $x_B^* - td \geq 0$
  - Set the value of x_j to t and the values of the basic variables to $x_B^* - td$.
  - Update the basis and the steepest edge norms.

Implementing the Algorithm

- Because of the computational effort required to update the steepest edge norms, these methods were not widely used until the 1990s.
- Recall that last semester we discussed methods for updating the basis effeciently after each iteration.
- There are also methods for efficeintly updating the steepest edge norms.
- There are variations for the dual simplex.
- There are also variations in which the norms are approximated or where only a subset are considered.

Some Other Connections

- Notice that the reduced costs are also the values of the dual variables.
- This means that they are also equal to the KKT multipliers.
- Hence, each iteration of the simplex method essentially solves the equations given by KKT for the current basis.
- If the Lagrange multipliers/reduced costs/dual variables are positive, then we have a KKT point (optimal).

Reduced Gradient Methods

- We now consider the following problem

$$\begin{aligned} \min\ & f(x) \\ s.t.\ & Ax = b \\ & x \geq 0 \end{aligned}$$

  where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $f : \mathbb{R}^n \rightarrow \mathbb{R}$.

- We assume
  - f is continuously differentiable,
  - any m columns of A are linearly independent, and
  - all extreme points of the feasible region have m strictly positive components.

- We use a generalization of ideas from the simplex algorithm.

- The biggest difference is that we can now have nonbasic variables that are not at their bounds.

- The m basic variables are chosen to be the "most positive" ones.

- The search direction can change the value of more than one basic variable.
- This method is guaranteed to converge to a KKT point.

Convex Simplex Method

- Note that a major difference between the reduced gradient method and the simplex algorithm is that the search direction is not restricted.
- Restricting ourselves to changing only one nonbasic variable at a time yields a method that behaves much like the simplex algorithm.
- This method is know as the *convex simplex method*, because it was originally applied to solve problem with convex objective functions.
- This method is also guaranteed to converge to a KKT point.
- Recall that if f is convex, then the KKT conditions are necessary and sufficient.