Advanced Mathematical Programming
IE417

Lecture 2

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Reading for This Lecture

- Primary Reading
  - Chapter 2, Sections 1-3
- Secondary Reading
  - Chapter 1
  - Appendix A

Preliminaries

Real Vector Spaces

- A real vector space is a set V, along with
  - an addition operation that is closed, commutative, and associative.
  - an element $0 \in V$ such that $a + 0 = a,\ \forall a \in V$.
  - an additive inverse operation such that $\forall a \in V, \exists a' \in V$ such that $a + a' = 0$.
  - a closed, scalar multiplication operation such that $\forall \lambda, \mu \in \mathbb{R}, a, b \in V$
    * $\lambda(a + b) = \lambda a + \lambda b$
    * $(\lambda + \mu)a = \lambda a + \mu a$
    * $\lambda(\mu a) = (\lambda\mu)a$
    * $1a = a$

Norms on Vector Spaces

- A *norm* on a vector space is a function $\|\cdot\| : V \to \mathbb{R}$ satisfying
  - $\|v\| \geq 0 \; \forall v \in V$
  - $\|v\| = 0$ if and only if $v = 0$
  - $\|v + w\| \leq \|v\| + \|w\|, \; \forall v, w \in V$
  - $\|\lambda v\| = |\lambda| \cdot \|v\|$
- Norms are used for measuring the "size" of an object or the "distance" between two objects in a vector space.
- These are the normal properties you would expect such a measure to have.

Examples of Vector Spaces

- $\mathbb{R}^n$
- $\mathbb{Z}^n$
- $\mathbb{R}^{n\times n}$
- $\{y \in \mathbb{R}^m : Ax = y,\ \exists x \in \mathbb{R}^n\}$
- Unless otherwise noted, we will be dealing with $\mathbb{R}^n$

Matrix and Vector Norms

- Unless otherwise indicated, we will use the L_2 norm for vectors and the corresponding norm for matrices.
- We will denote this by $\| \cdot \|$.
- The L_2 norm for matrices is defined as follows:

$$\|A\| = \max\{\|Ax\|/\|x\|, x \neq 0\}$$

- Note the following properties:
  - $|x^T y| \leq \|x\| \cdot \|y\|$
  - $\|Ax\| \leq \|A\| \cdot \|x\|$
  - $\|AB\| \leq \|A\| \cdot \|B\|$

Types of Optimization Problems

- Unconstrained Optimization

$$\begin{aligned} &\min f(x) \\ &\text{s.t. } x \in X, \end{aligned}$$

- Constrained Optimization

$$\begin{aligned} \min\ & f(x) \\ \text{s.t. }\ & g_i(x) \leq 0, \ i = 1, \dots, l, \\ & h_i(x) = 0, \ i = 1, \dots, m, \\ & x \in X \end{aligned}$$

Constrained Optimization Problems

- If f, g_i, and h_i are linear functions, then we have a *linear program* (by convention, X is $\mathbb{R}^n$ in this case).
- If some of f, g_i, and h_i are nonlinear functions, then we have a *nonlinear program* (again, X is $\mathbb{R}^n$ by convention).
- If X is a discrete set, then we have a *discrete optimization problem* (DOP).
- If $X = \mathbb{Z}^n$, then we have an *integer program* (this terminology usually refers only to linear models).

Some Terms

- Feasible point
- Ball of radius ε, denoted $N_\varepsilon(x)$
- (Strict) local minimum
- (Strict) local maximum
- (Strict) global minimum/maximum

Where We're Going

- Given an optimization problem, the end goal is to determine a globally optimal solution.
- For nonlinear problems, we will sometimes have to settle for local optima.
- First, we will look at theoretical conditions that help us determine whether a given point is a local/global optimum.
- Then, we will look at algorithms which help us get there.
- We start by studying *convexity*.

Convex Analysis

Convex Sets

A set S is *convex*

$$\Leftrightarrow$$

$$x_1, x_2 \in S, \lambda \in [0,1] \Rightarrow \lambda x_1 + (1-\lambda)x_2 \in S$$

- If $y = \sum \lambda_i x_i$, where $\lambda_i \geq 0$ and $\sum \lambda_i = 1$, then y is a *convex combination* of the x_i's.
- If the positivity restriction on λ is removed, then y is an *affine combination* of the x_i's.
- If we further remove the restriction that $\sum \lambda_i = 1$, then we have a *linear combination*.

Convex Hull

- The *convex hull* of a set S, *conv*(S), is the set of all convex combinations of the members of S.
- The convex hull of S is
  - The smallest convex set containing S
  - The intersection of all convex sets containing S
- If S_1 and S_2 are convex sets, then so are the following:
  - $S_1 \cap S_2$
  - $S_1 + S_2$
  - $S_1 - S_2$

Some More Terms

- We can similarly define the *affine hull* of a set S, *aff*(S).
- A set of points $x_1, \ldots, x_k$ in $\mathbb{R}^n$ are *affinely independent* if $x_i \notin$ *aff*$(\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k\})\ \forall i \in 1 \ldots k$.
- Note that $x_1, \ldots, x_k$ are *affinely independent* if and only if $x_2 - x_1, \ldots, x_n - x_1$ are *linearly independent*.
- A *polytope* is the convex hull of a set S containing a finite number of points.
- If the set of points in S are affinely independent, then the polytope is called a *simplex* and the points in S are its *vertices*.

Carathéodory's Theorem

Theorem 1. *If* S *is an arbitrary set in* $\mathbb{R}^n$ *and* $x \in \text{conv}(S)$, *then* x *is the convex combination of at most* $n+1$ *points.*

Idea of Proof:

Closure and Interior

- A point x is in the *closure* of a set S, denoted $\mathit{cl}(S)$, if $S \cap N_\varepsilon(x) \neq \emptyset, \forall \varepsilon > 0$.
- A set is *closed* if $S = \mathit{cl}(S)$.
- A point x is in the *interior* of a set S, denoted $\mathit{int}(S)$, if $\exists \varepsilon > 0 \text{ such that } N_\varepsilon(x) \subset S$
- A set is *open* if $S = \mathit{int}(S)$
- A point x is on the *boundary* of a set S if . . .
- A set S is *bounded* if . . .
- A set S is *compact* if it is closed and bounded.

Weierstrass's Theorem

Theorem 2. *Let S be a nonempty, compact set, and let $f : S \to \mathbb{R}$ be continuous on S. Then there exists a solution to the optimization problem*

$$\min f(x)$$
$$\textit{s.t. } x \in S,$$

Idea of Proof: