Advanced Mathematical Programming IE417

Lecture 11

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Reading for This Lecture

- Chapter 6, Section 4

Formulating the Lagrangian Dual

Formulating the Lagrangian Dual

- For each primal problem, there are a number of possible duals.
- The primal constraints can either
  - be included implicitly in the description of the set X, or
  - be "dualized" in the Lagrangian objective function.
- Usually, the "difficult" constraints are dualized to make solving the dual tractable.
- There is a tradeoff between the ease of evaluating $\Theta(\mu, v)$ and the resulting duality gap.
- Loosely speaking, the easier it is to evaluate $\Theta(\mu, v)$, the larger the duality gap will be.

Lagrangian Duality for Integer Linear Programming

- In ILP, the integrality constraints are the "tough" constraints.
- However, these cannot be dualized.
- Instead, dualize constraints that make the ILP hard to solve.
- Suppose we have an IP defined by

$$\begin{aligned} \min\ & cx \\ \text{s.t. } A^1x &\leq b^1 \text{ (the "complicating" constraints)} \\ A^2x &\leq b^2 \text{ (the "nice" constraints)} \\ x &\in \mathbf{Z}^n \end{aligned}$$

  where optimizing over $X = \{x \in \mathbf{Z}^n \mid A^2x \leq b^2\}$ is "easy."

- Lagrangian subproblem (for $\mu \geq 0$):

$$\Theta(\mu) = \min_{x \in X}\{(c - \mu A^1)x + \mu b^1\}.$$

Alternative Lagrangian Duals in Integer Linear Programming

- Choosing the set of constraints that define the set X affects the ease of solution and the size of the gap.
- Example: Traveling Salesman Problem.
- The Lagrangian dual can be rewritten as the following linear program

$$\max_{z,\mu}\{z : z \leq \Phi(x^i, \mu), i \in 1, \ldots, T, \mu \geq 0\}$$

where $\{x^i\}_{i=1}^{T}$ are the members of the set X (in this case finite).

- Based on the linear programming form of the Lagrangian dual, we can characterize the optimal value to the dual explicitly in the following way:

$$\min\{cx \mid A^1 x \leq b^1, x \in conv(Q)\}$$

Illustrating the Strength of the Lagrangian Dual