Advanced Mathematical Programming
IE417

Lecture 1

Dr. Ted Ralphs

Reading for This Lecture

- Suggested references
  - Solow, *How to Read and Do Proofs*
  - Bittinger, *Logic and Proof*
  - Velleman, *How to Prove It*

Mathematical Proof Techniques

Mathematical Systems

- Elements of a mathematical system
  - A universal set
  - A set of relations
  - A set of operations
  - A set of axioms
- To these given elements, we can add
  - Definitions
  - Theorems

Example: The Natural Numbers

Counting Axioms (Peano):

- 1 is a natural number.
- For each natural number x, there exists exactly one natural number, called the successor of x, and denoted x'
- For all natural numbers x, $x' \neq 1$
- If x and y are natural numbers such that $x' = y'$, then $x = y$
- Axiom of Induction: If S is a set of natural numbers, then if
  - $1 \in S$, and
  - $x \in S$, then $x' \in S$

  then S contains all of the natural numbers.
- Question: Does 0 belong to the natural numbers?
- Bonus Question: What is a "naturaaal" property of the integers that cannot be proved using only the Peano axioms?

Definitions, Theorems, Etc.

- A *definition* is simply an abbreviation or shortcut for a longer phrase.
- Example: "A set is said to be convex if . . . "
- From then on, we can simply use the term **convex set** instead of spelling out the property itself.
- *Theorems* are statements of the logical implications of the given axioms and definitions for the system.
- Other statements include *corollaries* and *propositions*.

Mathematical Proofs

- A mathematical proof shows the correctness of a given statement based on known definitions, axioms, and previously proven statements.
- Most proofs are for statements of the form $A \Rightarrow B$ where A and B are both statements.
- Example: "If $x > 2$ is a real number, then there exists a real number $y < 0$ such that $x = \frac{2y}{1+y}$".
- Proof:
- What are A and B in this example?

Quantifying Variables

- *Quantifying* is specifying from which set and for which values of a variable a statement is true.
- Example: "For all real numbers x and y, $(x+y)^2 = x^2 + 2xy + y^2$."
- This specifies that x and y can have any real value.
- Example: "For all real numbers $x \geq 0$, $x = |x|$.
- This specifies that the statement is true for nonnegative values of x.

Types of Quantifiers

- Universal Quantifiers
  - Statements that include "for all" or "for every."
  - Means that the statement is true when any element from the given set is substituted for the variable.
  - Example: "For all real numbers x, $cos^2x + sin^2x = 1$."
- Existential Quantifiers
  - Statements that include "there exists" or "there is."
  - Means that there is at least one element from the given set that makes the statement true.
  - Example: "For every real number $0 \leq x \leq 1$, there exists a real number $0 \leq y \leq \frac{\pi}{2}$ such that $sin(y) = x$."
- Notation: $\forall$ means "for all" and $\exists$ means "there exists".
- Example: "$\forall x \in \mathbb{R}$ such that $0 \leq x \leq 1$, $\exists y \in \mathbb{R}$ such that $0 \leq y \leq \frac{\pi}{2}$ and $sin(y) = x$."

Overview of Proofs

- All proofs use a sequence of logical implications to show either $A \Rightarrow B$ or something equivalent.
- In a proof, you can work either forward or backward.
- Working forward means deriving a sequence such as $A \Rightarrow A_1 \Rightarrow A_2 \Rightarrow A_3 \Rightarrow \ldots$
- Working backward means deriving a sequence such as $B \Leftarrow B_1 \Leftarrow B_2 \Leftarrow B_3 \Leftarrow \ldots$ (usually $\Leftrightarrow$)
- Once you can prove that $A_i \Rightarrow B_j$ for some i and j, then you're done!

Proofs with Universal Quantifiers

- To prove something about a universally quantified statement, first let an arbitrary set element *be given*.
- Example: "If $C \in \mathbb{R}^{n \times n}$ and $det(C) \neq 0$, then $\exists C^{-1} \in \mathbb{R}^{n \times n}$ such that $CC^{-1} = I$."
- Start of Proof: "Let an arbitrary matrix $C \in \mathbb{R}^{n \times n}$ be given and assume $det(C) \neq 0$..."
- Now prove that statement is true for the given element.
- Since the element was arbitrary, this proves the original statement.

Proofs with Existential Quantifiers

- If you are trying to prove something about an existentially quantified variable, the proof is usually *constructive*.
- The proof gives a technique for constructing an element of the set with the given property.
- Example: "If $C \in \mathbb{R}^{n \times n}$ and $det(C) \neq 0$, then $\exists C^{-1}$ such that $CC^{-1} = I$."
- Proof Technique: Construct C^{-1}.

Choosing an Element of a Set

- If you know from a previous theorem that an element of a set with a particular property exists, then you may "*choose*" it.
- Example: "Let r, a positive rational number be given. Then we may choose natural numbers p and q such that $r = \frac{p}{q}$."
- This can be especially useful in constructive proofs.

Proving the Contrapositive

- The *contrapositive* of the statement $A \Rightarrow B$ is $\sim B \Rightarrow \sim A$.
- Work forward from the statement $\sim B$ until reaching a statement equivalent to $\sim A$.
- You could also work backward from $\sim A$, but this is not usually done.
- If you work forward from A also, then the method is usually called *contradiction*.

Proof by Induction

- Induction is a proof technique for proving statements that are universally quantified on the natural numbers.
- Recall the fifth axiom of the natural numbers.
- Technique
  - Let $S(i)$ denote that the statement is true for the number i
  - Base Case: Prove $S(1)$.
  - Induction Step: Prove that $S(k) \Rightarrow S(k+1)$ for every $k \in \mathbb{N}$.

Special Proof Techniques

- Proof by cases
- Uniqueness proofs
- Either/or proofs
- If and only if proofs

Proof Style

- To make proofs easy to read, always summarize your proof technique before starting out.
- Example: "This will be a proof by contradiction. We therefore first assume that B is not true and try to derive a contradiction. Since B is not true, . . . "
- Divide the proof up in to logical parts and clearly label each part so that the reader can easily see the outline of the approach.
- The approach should be clear without reading the details.
- Lemmas can also be used for clarity.