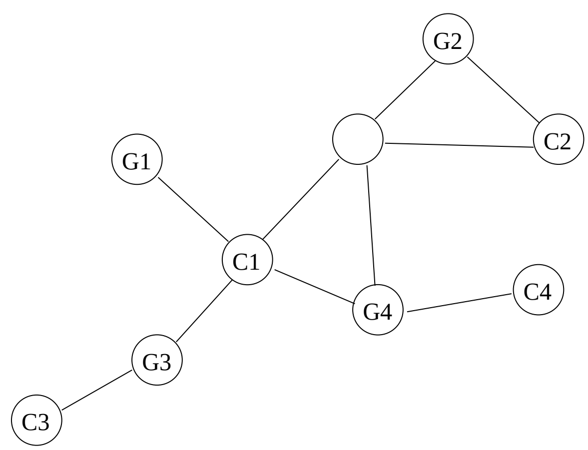
Final Exam Sample Questions
IE411: Networks and Graphs
Dr. Ralphs

1. (a) Suppose you are given a graph in the form of a node-node adjacency matrix. What is the running time of a procedure to convert this into an adjacency list representation?
   (b) The implementation of Dijkstra's Algorithm we discussed in class assumes that the graph is given in adjacency list format. If it was given in node-node adjacency matrix format, how would the running time change?
   (c) If you were given a graph in node-node-adjacency matrix format and needed to execute Dijkstra's Algorithm on it, would it make sense to convert the node-node adjacency matrix to an adjacency list format and then execute the standard Dijkstra's Algorithm?

2. The following represents an electrical power distribution network connecting power generating points G_1 to G_4 with power consuming points C_1 through C_4. The arcs are undirected, i.e., power may flow in either direction. The generating capacities and unit costs are given by the following table.

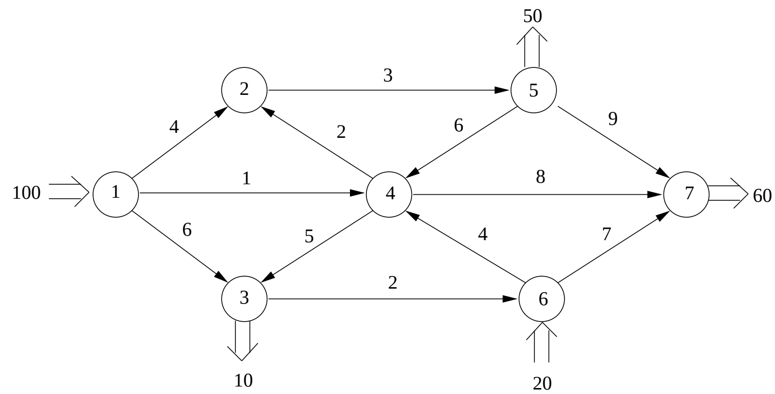
	Generator			
	G_1	G_2	G_3	G_4
Capacity (thousands of KWH)	100	60	80	150
Unit Cost ($ per thousand KWH)	15.0	13.5	21.0	23.5

The power consumption for points C_1 through C_4 is 35K, 50K, 60K, and 40K respectively. There is no capacity limitation on the transmission lines and the unit cost of transmission is \$11 per 1000 KWH on all lines. Note that supply exceeds demand.



   (a) Formulate the problem of finding the optimal power allocation as a minimum cost network flow problem (Hint: you may have to introduce additional arcs and/or nodes).
   (b) Find the optimal power allocation. You may use any method at all to find the optimal solution, as long as you prove rigorously that your solution is optimal. Please explain how you arrived at your answer.

3. Consider the following representation of a minimum cost network flow problem.



The numbers inside each circle represent the node numbers, the numbers next to each arc represent the cost of a unit of flow on that arc, and the remaining numbers represent supply/demand at each node (those without numbers are transshipment nodes). All arcs have zero lower bound and infinite capacity.

(a) Explain why an optimal set of node potentials can be obtained by solving a single-source shortest path problem in this case and compute such potentials (this can be done by inspection).

(b) Using complementary slackness optimality conditions, determine an optimal flow. Is there more than one optimal flow? How many are there? How can you tell?

(c) Does any shortest paths tree correspond to a feasible tree solution? Explain why or give a counterexample.

4. (a) A matching M is *maximal* for $G = (N, A)$ if $M \cup \{(i, j)\}$ is not a matching for every $(i, j) \notin M$. Give a simple algorithm for finding a maximal matching and state its running time.

(b) Explain why the cardinality of any maximal matching must be at least $\lfloor |M^*|/2 \rfloor$, where M^* is a maximum cardinality matching of G.

(c) Modify your algorithm from 4a for the weighted case by describing an algorithm like Kruskal's in which the edges are added into the matching in weighted order. How does the running time change? Can you say anything about the weight of this matching as compared to the optimal maximum weight matching?

5. Suppose that a network $G = (N, A)$ contains no negative cycle. In this network, let f_{ij} denote the maximum amount by which we can decrease the length of arc (i, j) without creating any negative cycle, assuming all other arc lengths are held constant. Describe an efficient algorithm for determining f_{ij} for each arc (i, j).

6. Let $F \in \mathbb{Z}^{n \times n}$ symmetric nonnegative matrix with (integral) elements f_{ij}, where $f_{ii} = \infty$ for all $i \in N = \{1, \ldots, n\}$. F is said to be *max flow realizable* if there exists an (undirected) network in which the maximum flow between nodes i and j is f_{ij}. In the first two parts below, we will show that F is max flow realizable if and only if $f_{ij} \geq \min\{f_{ik}, f_{kj}\}$ for all $k \in N \setminus \{i, j\}$.

(a) Show that if $f_{ij} < \min\{f_{ik}, f_{kj}\}$ for some $k \in N$, then F is not max flow realizable.

(b) Show that if $f_{ij} \geq \min\{f_ik, f_{kj}\} \forall k = N \setminus \{i, j\}$, then F is max flow realizable. (Hint: show that the maximum weight spanning tree with respect to the complete graph $G = (N, N \times N)$, where the weight of edge (i, j) is taken to be f_{ij}, is a graph in which the max flow between i and j is f_{ij} for all $(i, j) \in N \times N$).

(c) Can you draw any conclusion about the maximum number of maximum flow values between any pair of nodes in an undirected network?

7. (a) Consider the problem of determining the shortest path from node 1 to node n in a directed network having no negative cost circuits. Explain how to formulate this problem as an assignment problem.

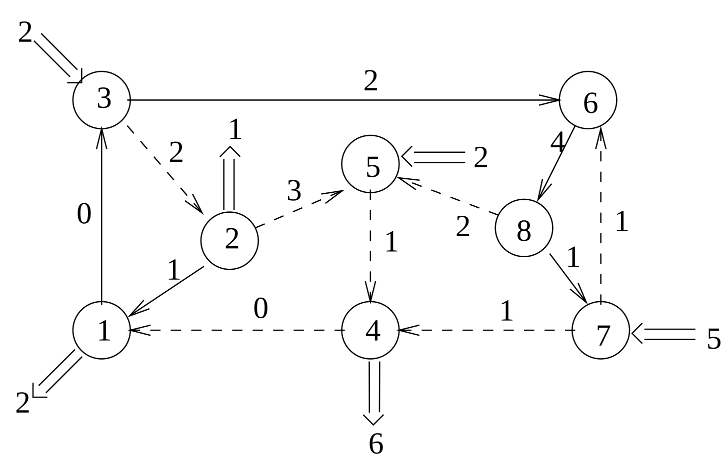
(b) Consider applying the successive shortest paths algorithm to an assignment problem instance arising from the formulation in part (a) in order to solve a given shortest path problem. Since the successive shortest paths problem involves solving a shortest path problem in each iteration, you might expect that it would solve this assignment problem in one iteration. Is this true or false? Explain.

8. Consider the problem of finding the shortest simple path (path without repeated nodes) from node s to node t in a given network that may contain negative cost cycles. Suppose you have obtained the optimal solution to a minimum cost flow problem on the given network in which the supply at node s is 1, the demand at node t is 1, the supply/demand at all other nodes is 0, and all arcs have a capacity of 1.

(a) What would a solution to this flow problem be like? How would you identify a simple path from s to t with a flow of one in the solution to the above-described minimum cost flow problem?

(b) Would this path necessarily be the shortest simple path? If yes, give a proof. If no, give a counterexample.

9. Consider the minimum cost network flow problem shown below. In this figure, the numbers next to the arcs are the costs and the numbers next to the arrows are the supplies/demands, as on the homework. All arc flows have zero lower bound and infinite upper bound.



(a) Determine both the flow and the node potentials corresponding to the spanning tree indicated by the dashed arcs.

(b) Determine whether the tree is strongly feasible. If not, find a strongly feasible spanning tree.
(c) Solve the problem using the network simplex algorithm. Start with the tree indicated by the dashed arcs in the figure.

10. Consider the following *production-inventory-transportation* problem. A firm produces a single product in two locations in each of two time periods. The unit costs and production limits vary by time period, as shown below:

Consumer Location	Time Period 1	Time Period 2	
1	\$25/6	\$35/2	Unit cost/
2	\$30/10	\$42/9	Production limit

The product will be shipped (instantaneously) to each of two locations to satisfy demands over the two periods, as specified below:

Consumer Location	Time Period 1	Time Period 2
1	3	1
2	5	4

The unit shipping cost varies over time and is given below:

Production Location	Period 1 Consumer Location 1	Period 1 Consumer Location 2	Production Location	Period 2 Consumer Location 1	Period 2 Consumer Location 2
1	\$50	\$60	1	\$60	\$80
2	\$40	\$70	2	\$70	\$90

Finally, the product may be held in inventory at the production and consumer locations to satisfy later period needs, as per the costs shown here:

Production Location 1	Production Location 2	Consumer Location 1	Consumer Location 2	
\$1/2	\$2/3	\$3/1	\$4/3	Unit cost/ Inventory limit

Set up a network flow problem that can be used to solve the problem of minimizing the total cost to satisfy the demand over the two periods.

11. Consider the maximum flow in a given network between two designated nodes s and t. For each of the following statements, either explain why it is true or provide a counterexample.

(a) If the capacity of every arc is even, then the value of the maximum flow must be even.
(b) If the capacity of every arc is even, then there is a maximum flow in which the flow on each arc is even.
(c) If the capacity of every arc is odd, then the value of the maximum flow must be odd.
(d) If the capacity of every arc is odd, then there is a maximum flow in which the flow on each arc is odd.