Graphs and Network Flows
ISE 411

Lecture 9

Dr. Ted Ralphs

References for Today's Lecture

- Required reading
  - Section 21.2
- References
  - AMO Sections 4.5–4.7
  - CLRS Section 24.3

Solving SPP with Non-Negative Arc Lengths

- When there are cycles, the situation is a bit more complex.
- Dijkstra's Algorithm generalizes the algorithm from Lecture 7 for the acyclic case.
- The difference is the order in which the nodes are examined.
- As before, nodes are divided into two groups
  - temporarily labeled
  - permanently labeled
- In order to produce the shortest paths tree, we keep track of the *predecessor node* each time a label is updated.
- Basic Idea: Fan out from source and permanently label nodes in order of distance from the source.

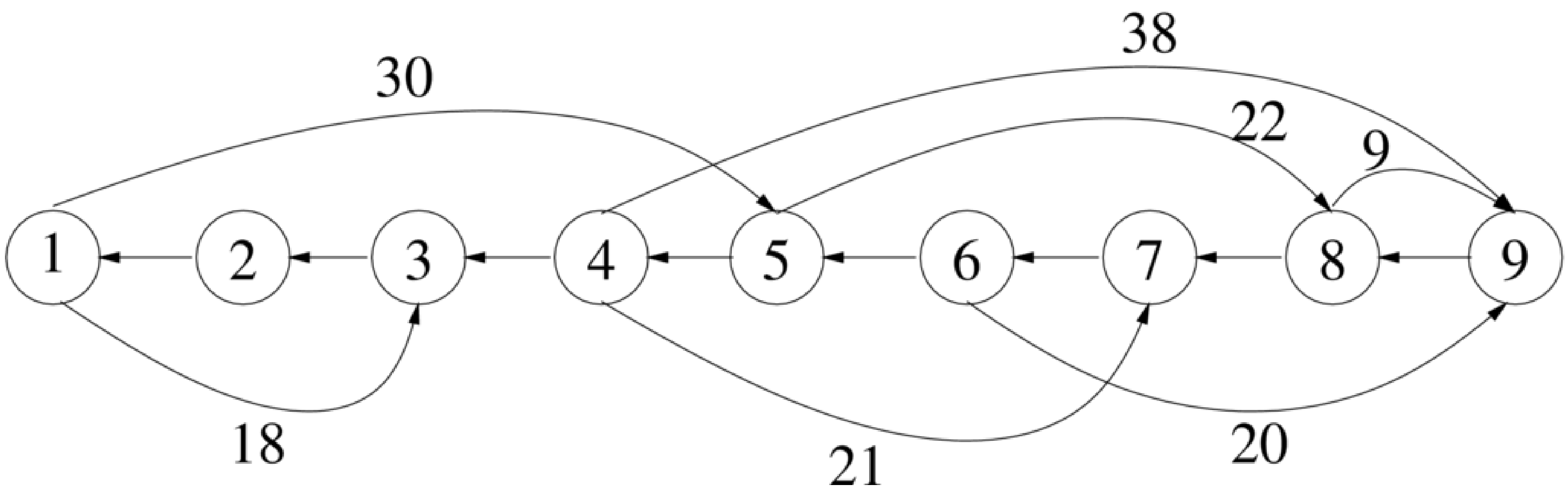
Dijkstra's Algorithm

Input: An network $G = (N, A)$ and a vector of arc lengths $c \in \mathbb{Z}_+^A$

Output: $d(i)$ is the length of a shortest path from node s to node i and pred(i) is the immediate predecessor of i in an associated shortest paths tree.

$S := \emptyset$
$\bar{S} := N$
$d(i) \leftarrow \infty \forall i \in N$
$d(s) \leftarrow 0$ and $pred(s) \leftarrow 0$
while $|S| < n$ **do**
  let $i \in \bar{S}$ be the node for which $d(i) = min\{d(j) : j \in \bar{S}\}$
  $S \leftarrow S \cup \{i\}$
  $\bar{S} \leftarrow \bar{S} \setminus \{i\}$
  for $(i, j) \in A(i)$ **do**
    if $d(j) > d(i) + c_{ij}$ **then**
      $d(j) \leftarrow d(i) + c_{ij}$ and $pred(j) \leftarrow i$
    end if
  end for
end while

Example of Dijkstra's Algorithm



Proof of Correctness

Claim 1. *At the end of any iteration the following inductive hypotheses hold:*

1. *The distance label* $d(i)$ *is optimal for any node* i *in the set* S*.*
2. *The distance label* $d(j)$ *for any node* $j \in \bar{S}$ *is the length of the shortest path from the source to* j *such that all internal path nodes are in* S*.*

Proof Strategy

- Show that statements 1 and 2 are true after the first iteration.
- Assume that they are true after iteration $i-1$ and prove that they hold after iteration i.
- (Assume iteration i moves node i from $\bar{S}$ to S.)

Running Time of Dijkstra's Algorithm

- Note that Dijkstra's Algorithm is a graph search procedure.
- It is very similar to Prim's Algorithm.
- At each step, we need to update some node labels and then be able to determine the node with the minimum label.
- What is the running time for a naive implementation?

Dial's Implementation

- Node selection is bottleneck operation
- Maintain distances in sorted fashion using following property

Property 1. [4.5] *The distance labels that Dijkstra's Algorithm designates as permanent are non-decreasing.*

- Create $nC + 1$ buckets numbered $0, 1, \cdots, nC + 1$ and store all nodes with temporary distance label k in bucket k
- Reduce number of buckets to $C + 1$ using following property

Property 2. [4.6] *If $d(i)$ is the distance label designated as permanent at the beginning of an iteration, then at the end of an iteration $d(j) \leq d(i) + C$ for each finitely labeled node $j \in \bar{S}$.*

- Algorithm runs in $O(m + nC)$ time

Implementation with Priority Queues

- To get a strongly polynomial time algorithm, we must use a more general data structure for maintaining a *priority queue*.
- For a given order set H, this data structure should support the operations
  - `push(item, value)` (to add and change value of an item)
  - `peek()`
  - `pop()`

Binary Heaps

- A *binary heap* is a balanced binary tree with additional structure that allows it to function efficiently as a priority queue.
- The additional structure needed to support these operations is that each node has a higher priority than either of its children.
- Balanced binary trees can be stored very efficiently in a single array.
  - The root is stored in position 0.
  - The children of the node in position i are stored in positions $2i + 1$ and $2i + 2$.
  - This determines a unique storage location for every node in the tree and makes it easy to find a node's parent and children.
  - Using an array, basic operations can be performed very efficiently.

Creating the Heap

- Any node whose priority is higher than either of its children is said to satisfy the *heap property*.
- Consider a tree in which all nodes except for the root have the heap property.
- We can easily transform this into a tree in which every node has the heap property (how?).
- This operation is called `heapify()`.
- By calling `heapify()` on each node, starting at the lowest level and working upward, we can transform an unordered binary tree into a heap.
- This is how we create the initial heap.
- Note that this step is unnecessary for implementing Dijkstra's. Why?

Operations on a Heap

- The node with the highest priority is always the root.
- To change the priority of a node
- To insert a node
- To delete a node
- What are the running times of these operations?

Analyzing Diskstra's with a Binary Heap

Running Times of Other Implementations

d-Heap: $O(m \log_d n + nd \log_d n)$ ($d = \max\{2, \lceil m/n \rceil\}$)

Fibonacci Heap: $O(m + n \log n)$ (best strongly polynomial time algorithm)

Johnson's: $O(m \log \log C)$

Radix Heap: $O(m + n \log(nC))$

Fibonacci Radix: $O(m + n\sqrt{logC})$